Up until now we have discussed ESD in various systems. This chapter is devoted to the study of control procedures to suppress the entanglement decay and delay ESD. Given the obvious importance of ESD regarding the success of quantum tasks, it is thus a worthwhile exercise to investigate ways and means of controlling the rate of loss of entanglement. Error-correcting codes [40, 41, 154, 106] and error-avoiding codes [186, 59, 113] (which are also known as decoherence-free subspaces) are such attempts. Open loop decoherence control strategies [170, 10, 60, 171, 172, 169, 3, 5, 6] are another class of widely used strategies used to this effect, where the system of interest is subjected to external, suitably designed, time-dependent drivings that are independent of the system dynamics. The aim is to cause an effective dynamic decoupling of the system from the ambient environment. A comparative analysis of some of these methods has been made in [67]. Another mechanism known to slow down the process of decoherence is through manipulation of the density of states. This has been put to use in photonic band-gap materials, which is used to address questions related to the phenomenon of localization of light [102, 182, 103, 183, 104]. In [88] we have studied the effect of control procedures on the evolution of entanglement. This chapter is based on that study.

In this chapter, we analyze the evolution of entanglement in two-qubit systems connected to local baths (or reservoirs). A number of studies of entanglement in open quantum systems have been made [12, 13, 47]. Here we address the need to have a control on the resulting nonunitary evolution, as motivated by the above
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discussion, and study several methods of doing so. These include manipulation of the density of states in photonic crystals, modulation of the frequency of the system-bath coupling and modulation of external driving on two-qubit systems. A significant part of the chapter is devoted to the study of control methods in two-qubit systems undergoing non-Markovian evolution. The first of these is dynamic decoupling — which is an open-loop strategy — on a two-qubit system that is in contact with a harmonic oscillator bath. This system undergoes a quantum non-demolition interaction, where dephasing occurs without the system getting damped. The second is a Josephson-junction charge qubit subject to random telegraph $(1/f)$ noise due to charge impurities.

The surprising aspect of this study is that suppression of decoherence due to a control procedure need not necessarily mean preservation of entanglement. In fact, application of resonance fluorescence on a two-level atom exposed to a thermal bath or dynamical decoupling on the Josephson junction charge qubit, undergoing non-Markovian evolution, results in faster ESD even though decoherence gets suppressed.

10.1 Evolution of entanglement in the presence of photonic crystals

Let us consider a system of two level atoms interacting with a periodic dielectric crystal, this particular structure of which gives rise to the photonic band gap [102, 92, 183]. The effect of this on electromagnetic waves is analogous to the effect semiconductor crystals have on the propagation of electrons, and leads to interesting phenomena like strong localization of light [103], inhibition of spontaneous emission [183] and atom-photon bound states [104, 112, 184]. The origin of such phenomena can ultimately be traced to the photon density of states changing at a rate comparable to the spontaneous emission rates. The photon density of states are of course estimated from the local photon mode density which constitutes the reservoir. It is this photonic band that suppress decoherence [174].

Let us consider a two-qubit system, one qubit of which is locally coupled to a photonic crystal reservoir kept at zero temperature . In this case, entanglement dynamics can be obtained by studying any one of the qubits individually. We start
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with the following Hamiltonian:

\[ H = \frac{\omega_0}{2} \sigma_z + \sum_k \omega_k a_k^\dagger a_k + \sum_k (g_k a_k^\dagger \sigma_- + g_k^* a_k \sigma_+), \]

(10.1)

where \( \omega_0 \) is the natural frequency of the two level atom, \( \omega_k \) is the energy of the \( k \)-th mode and \( g_k \) is the frequency dependent coupling between the qubit and the photonic crystal, the latter acting as the reservoir here. Also, \( \sigma_z \) and \( \sigma_\pm = \sigma_x \pm i \sigma_y \) are the Pauli matrices, with \( a_k \) and \( a_k^\dagger \) being the annihilation and creation operators for \( k \)-th mode. If we restrict the total atom-reservoir system to the case of a single excitation [79], the evolution of a given state of the qubit is then given by [174]:

\[ \rho(t) = \begin{pmatrix} \rho_{11}(0)|c(t)|^2 & \rho_{01}(0)c(t) \\ \rho_{10}(0)c^*(t) & \rho_{00}(0) + \rho_{11}(0)(1 - |c(t)|^2) \end{pmatrix}, \]

(10.2)

where

\[ c(t) = \varepsilon \left( \lambda_+ e^{i\lambda_+^2 t}[1 + \Phi(\lambda_+ e^{i\pi/4}\sqrt{t})] - \lambda_- e^{i\lambda_-^2 t}[1 + \Phi(\lambda_- e^{i\pi/4}\sqrt{t})] \right), \]

\[ \Phi(x) = \sum_{k=0}^{\infty} \frac{2^{2k+1}}{\sqrt{\pi}(2k+1)!} \]

is the error function ,

\[ \varepsilon = \frac{e^{i\delta t}}{\sqrt{\alpha^2 - 4\delta}}, \]

\[ \lambda_\pm = -\alpha \pm \sqrt{\alpha^2 - 4\delta} \]

\[ \alpha \approx \frac{\omega_0^2 d^2}{8\omega_c \epsilon_0 (\pi A)^{3/2}}. \]

Here \( \delta = \omega_0 - \omega_c \) is the detuning of the atomic frequency and \( \omega_c \) is the upper band-edge frequency. We have made use of the following photon-dispersion relation near the band edge: \( \omega_k \approx \omega_c + A(k - k_0)^2 \), where \( A \approx \omega_c/k_0^2, d \) is the atomic dipole moment and \( \epsilon_0 \) is the vacuum dielectric constant.

The density matrix \( \rho(t) \) in Eq. (10.2) is related to the initial density matrix
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\( \rho(0) \) by a map \( \Lambda \), given by \( \rho(t) = \Lambda_{pc}[\rho(0)] \), whose matrix representation is

\[
V_{pc} = \begin{pmatrix}
|c(t)|^2 & 0 & 0 & 0 \\
0 & c(t) & 0 & 0 \\
0 & 0 & c^*(t) & 0 \\
1 - |c(t)|^2 & 0 & 0 & 1
\end{pmatrix},
\]

(10.3)

Channel-state duality, explained earlier in Chapter 5, ensures that there exists a two-qubit density matrix \( M_{pc} \) for every single-qubit channel \( V_{pc} \). This matrix \( M_{pc} \) can be written as

\[
M_{pc} = (I \otimes \Lambda_{pc})(|\Phi^+\rangle\langle\Phi^+|)
= \frac{1}{2} \begin{pmatrix}
|c(t)|^2 & 0 & 0 & c(t) \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 - |c(t)|^2 & 0 \\
c^*(t) & 0 & 0 & 1
\end{pmatrix},
\]

(10.4)

where \( |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \) is a two-qubit maximally entangled state. The concurrence of \( M_{pc} \) is \( |c(t)|^2 \), where \( c(t) \) is a complex-valued function of the detuning parameter \( \delta \) and time \( t \). Therefore, we need to see the effect of \( \alpha \) on entanglement in \( M_{pc} \). If we assume that \( \delta = \Delta \alpha^2 \), \( c(t) \) can then be written in the following simplified form:

\[
c(t) = \frac{e^{i\Delta \tau}}{\sqrt{1 - 4\Delta}} \frac{1}{2} \left( d_+ e^{i \frac{\pi}{4} \tau} \left[ 1 + \Phi(d_+ e^{i \frac{\pi}{4} \sqrt{\tau}}) \right] - d_- e^{i \frac{\pi}{4} \tau} \left[ 1 + \Phi(d_- e^{i \frac{\pi}{4} \sqrt{\tau}}) \right] \right),
\]

where \( d_\pm = -1 \pm \sqrt{1 - 4\Delta} \) and \( \tau = \alpha^2 t \). Since the entanglement in \( M_{pc} \) is \( |c(t)|^2 \), it is now a function of \( \delta \) and \( \tau \). Invoking the factorization law of entanglement decay, it is sufficient to study entanglement in \( M_{pc} \) in order to understand the nature of evolution of entanglement in the two-qubit system.

We show the evolution of entanglement in \( M_{pc} \) for different values of \( \Delta \) in Figs. (10.1(a)), (10.1(b)). The insets of the figures depict the evolution of entanglement – computed using concurrence (see appendix) – for the usual case of zero band gap, while the main panels show the evolution of entanglement for increasing influence.
of the band gap. In FIG. (10.1(a)), the system is within a gap in the photonic spectrum, indicated by the negative value of $\Delta$ and hence also $\delta$, as a result of which coherence is preserved and the decay of entanglement is arrested. This feature is further highlighted in FIG. (10.1(b)), which is also for the case of negative $\Delta$ of higher order of magnitude than that in FIG. (10.1(a)), and as a result there is a greater persistence of entanglement. Thus we find that with the increase of the influence of the photonic band gap on the evolution, entanglement is preserved longer. From Eq. (10.4), it can be seen that, following the arguments of the previous section, there is no ESD in this case, a feature corroborated by the FIGS. (10.1).

Apart from these, Fig. (10.1) shows another interesting phenomenon – the temporally damped oscillations in the entanglement. This phenomenon is a signature of the emergence of non-Markovian characteristics in the evolution and implies that the action of detuning changes the character of the dynamics itself, turning it non-Markovian.

### 10.2 Frequency modulation

Agarwal and coworkers [3, 5, 6] introduced an open-loop control strategy which involved modulation of the system-bath coupling, with the proviso that the frequency modulation (to be introduced below) should be carried out at a time scale which is faster than the correlation time scale of the heat bath. The technique of frequency modulation has been used earlier to demonstrate the existence of population trapping states in a two-level system [4]. Raghavan et al. [142] showed the connection between trapping in a two-level system under the action of frequency-modulated fields in quantum optics and dynamic localization of charges moving in a crystal under the action of a time-periodic electric field.

Consider the Hamiltonian given in Eq. (10.1). Frequency modulation essentially involves a modification of the coupling $g_k$ — the modulated coupling is $g_k \exp\{-im \sin \nu t\}$, where $m$ is the amplitude and $\nu$ is the frequency of the modulation. The decay of the excited state population can be significantly arrested by choosing $m$ such that $J_0(m) = 0$, where $J_0$ are the Bessel functions of order zero. The resulting master equation in the interaction picture, when applied to
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the evolution of a two-level system, is [3, 5, 6]:

\[
\frac{\partial \rho}{\partial t} = -\frac{2(\kappa - i\Delta)J_2^2(m)}{(\kappa - i\Delta)^2 + \nu^2} \left\{ C_0^- + (\sigma^+ \rho - \sigma^- \rho\sigma^+) + C_0^+ (\rho\sigma^- - \sigma^+ \rho\sigma^-) \right\} + h.c. \quad \Delta = (\omega_0 - \omega). \tag{10.5}
\]

Here \(\sigma^\pm\) are the Pauli matrices. We have used the Bessel function expansion

\[
e^{-im \sin(\nu t)} = \sum_{l=-\infty}^{\infty} J_l(m) e^{-il\nu t}
\]

where \(J_1(m)\) is the Bessel function of order one. Additionally, the modified bath correlation functions are assumed to have the forms \(C_{-+}^-(t) = C_{0-}^- e^{-\kappa t} e^{i\omega t}\) and \(C_{+-}^-(t) = C_{0+}^- e^{-\kappa t} e^{i\omega t}\), where \(\kappa\) is the bath correlation frequency. Now, we have

\[
\frac{\partial \rho}{\partial t} = \mathcal{L}_{fm}[\rho],
\]

\[
\Rightarrow \rho(t) = \exp(\mathcal{L}_{fm} t) \rho(0),
\]

\[
\Rightarrow \rho(t)_{ij} = \sum_{kl} \{ \exp(\mathcal{L}_{fm} t) \}_{ijkl} \rho(0)_{kl}, \tag{10.6}
\]

where \(V_{fm}(t) = \exp(L_{fm} t)\) and \(L_{fm}\) is the matrix representation of \(\mathcal{L}_{fm}\). We obtain the matrices \(L_{fm}\) and \(V_{fm}\) using Eq. (10.5).

\[
L_{fm} = \begin{pmatrix}
-2\text{Re}(\alpha)C_0^+ & 0 & 0 & 2\text{Re}(\alpha)C_0^- \\
0 & -\alpha(C_0^- + C_0^+) & 0 & 0 \\
0 & 0 & -\alpha^*(C_0^+ + C_0^-) & 0 \\
2\text{Re}(\alpha)C_0^+ & 0 & 0 & -2\text{Re}(\alpha)C_0^-
\end{pmatrix},
\]

\[
V_{fm} = \exp(L_{fm} t) \tag{10.7}
\]

\[
= \begin{pmatrix}
\frac{1}{T} \left( C_0^+ e^{-2\text{Re}(\alpha)Tt} + C_0^- \right) & 0 & 0 & \frac{1}{T} \left( C_0^- (1 - e^{-2\text{Re}(\alpha)Tt}) \right) \\
0 & e^{-\alpha Tt} & 0 & 0 \\
0 & 0 & e^{-\alpha^* Tt} & 0 \\
\frac{1}{T} \left( C_0^+ (1 - e^{-2\text{Re}(\alpha)Tt}) \right) & 0 & 0 & \frac{1}{T} \left( C_0^- e^{-2\text{Re}(\alpha)Tt} + C_0^+ \right)
\end{pmatrix}, \tag{10.9}
\]

where \(\alpha = \frac{2(\kappa - i\Delta)J_2^2(m)}{(\kappa - i\Delta)^2 + \nu^2}\) and \(T = C_0^- + C_0^+\). If \(M_{fm} = (I \otimes V_{fm})(|\phi^+\rangle\langle\phi^+|),\)
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then we have

\[ M_{fm} = \begin{pmatrix}
M_{11} & 0 & 0 & e^{-\alpha T t} \\
0 & M_{22} & 0 & 0 \\
0 & 0 & M_{33} & 0 \\
e^{-\alpha^* T t} & 0 & 0 & M_{44}
\end{pmatrix}, \quad (10.10) \]

where

\[ M_{11} = \frac{1}{T} \left(C_0^+ e^{-2\text{Re}(\alpha) T t} + C_0^+ \right), \]
\[ M_{22} = \frac{1}{T} \left(C_0^+ (1 - e^{-2\text{Re}(\alpha) T t}) \right), \]
\[ M_{33} = \frac{1}{T} \left(C_0^- (1 - e^{-2\text{Re}(\alpha) T t}) \right), \]
\[ M_{44} = \frac{1}{T} \left(C_0^+ e^{-2\text{Re}(\alpha) T t} + C_0^- \right). \]

If \( M_{fm} \) is separable at some time \( t \), the factorization law for entanglement decay [107] allows us to assert that all states will show ESD. The state \( M_{fm} \) is separable if only if it is positive under partial transposition, i.e,

\[ 1 + X^2 - 2X - \frac{T^2}{C_0^+ C_0^+} X \geq 0, \quad (10.11) \]

where \( X = \exp(-2\text{Re}(\alpha) T t) \). Therefore, \( M_{fm} \) is separable when LHS of Eq. (10.11) is zero. The roots of the above equation are

\[ X_{\pm} = \frac{1}{2} \left[ \left(2 + \frac{T^2}{C_0^+ C_0^+} \right) \pm \sqrt{\left(2 + \frac{T^2}{C_0^+ C_0^+} \right)^2 - 4} \right]. \quad (10.12) \]

The root \( X_- \) is less than unity, implying that there exists, always, a finite and positive time \( t_{ESD} \) at which the system loses all its entanglement. This is given by

\[ t_{ESD} = -\frac{1}{2\text{Re}(\alpha) T} \log(X_-). \quad (10.13) \]

The modulation factor \( \nu \) appears in the numerator of Eq. (10.13) and, therefore, it can be expected that a higher frequency of modulation should sustain entanglement longer. This is confirmed in the plot of \( t_{ESD} \) against \( \nu \) (FIG. (10.2)). This result is
not altogether surprising, for a higher degree of modulation is naturally expected to increase the coherence by filtering out the influence of the bath, which ultimately results in entanglement sustaining for a longer period of time.

10.3 Resonance fluorescence

In the previous section, we focused on the decrease in the time to ESD by increasing the degree of frequency modulation of the system-bath coupling. In this section, we study a system where a two-level atomic transition is driven by an external coherent single-mode field which is in resonance with the transition itself. We shall show that, in this situation, an increase in the Rabi frequency — which plays the role of the modulator — produces the opposite effect by speeding up ESD. The behavior of such driven systems has been well studied in the literature and has found many applications. In contrast to the situation here, Lam and Savage [110] have investigated a two-level atom driven by polychromatic light. The phenomenon of tunneling in a symmetric double-well potential perturbed by a monochromatic driving force was analyzed by Grossmann et al., [89], while photon-assisted tunneling in a strongly driven double-barrier tunneling diode has been studied by Wagner [173].

The analysis of the said driven system begins with its Hamiltonian which, when written in the interaction picture, is $H_{SR} = -E(t) \cdot D(t)$. Here $E(t) = \varepsilon e^{-i\omega_0 t} + \varepsilon^* e^{i\omega_0 t}$ is the electric field strength of the driving mode (treated classically), $\omega_0$ is the atomic transition frequency and $D(t)$ is the dipole moment operator in the interaction picture. The driven two-level system is coupled to a thermal reservoir of radiation modes. If $\gamma_0$ be the spontaneous rate due to coupling with the thermal reservoir and $N = N(\omega_0)$ be the Planck distribution at the atomic transition frequency $\omega_0$, the evolution of this composite system is given by the following master equation [33]:

$$\frac{d}{dt} \rho(t) = \frac{i\Omega}{2} [\sigma_+ + \sigma_-, \rho(t)] + \frac{\gamma_0(N + 1)}{2} [2\sigma_- \rho(t) \sigma_+ - \sigma_+ \sigma_- \rho(t) - \rho(t) \sigma_+ \sigma_-] + \frac{\gamma_0(N)}{2} [2\sigma_+ \rho(t) \sigma_- - \sigma_- \sigma_+ \rho(t) - \rho(t) \sigma_- \sigma_+], \quad (10.14)$$
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Figure 10.1: Evolution of entanglement in photonic band gap crystals at zero temperature.

(a) Entanglement (concurrence) of $M$ as a function of $\tau = \alpha^2 t$ for detuning parameter $\Delta = -0.1$. In this plot the behaviour of entanglement is different from the one in the case for $\Delta = 0$ (see inset). Here entanglement is seen to converge to a non-zero value at large $\tau$.

(b) Entanglement (concurrence) of $M$ as a function of $\tau = \alpha^2 t$ for detuning parameter $\Delta = -0.25$. This plot shows that higher the magnitude of detuning, larger will be the asymptotic value of entanglement.
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Figure 10.2: In this plot we have time of ESD, i.e, the time at which a maximally entangled initial state loses all its entanglement and become separable when exposed to a bath, against the frequency of modulation $\nu$. In this case we have kept the value of $m$ to be the first zero of the Bessel function $J_0$, i.e, $m = 2.4048$. Also, at $\nu = 0$ the value of $t_{ESD}$ is 1.4. The value of the other parameters are: $\kappa = 0.1$, $\Delta = 0.1$, $C_0^+ = C_0^- = 0.1$. 
where $\Omega = 2\varepsilon \cdot d^*$ is the Rabi frequency and $d$ is the transition matrix element of the dipole operator. The term $-(\Omega/2) [\sigma_+ + \sigma_-]$ characterizes the interaction between the atom and the external driving field in the rotating wave approximation. As usual, $\sigma_\pm$ are the atomic raising and lowering operators, respectively.

Let us consider two identical qubits and, as before, assume that one of them interacts locally with a thermal bath and is subject to monochromatic driving by an external coherent field. The master equation (Eq. 10.14) yields the corresponding matrices $V_{rf}$ and $M_{rf}$ (where the subscript $rf$ stands for resonance fluorescence):

$$V_{rf} = \begin{pmatrix} a_1 & a_2 & a_2^* & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ b_1^* & b_3^* & b_2^* & b_4^* \\ d_1 & -a_2 & -a_2^* & d_4 \end{pmatrix},$$  

(10.15)

$$M_{rf} = \frac{1}{2} \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_2^* & a_4 & b_3 & b_4 \\ b_1^* & b_3^* & d_1 & -a_2 \\ b_2^* & b_4^* & -a_2^* & d_4 \end{pmatrix},$$  

(10.16)

where

$$a_1 + a_4 = 1 + \left( 1 - X^3 \left( \cos(\mu t) - \frac{\gamma}{4\mu} \sin(\mu t) \right) \right) S_3 + \frac{i\Omega}{\mu} X^3 \sin(\mu t) (S_- + S_+),$$

$$a_1 - a_4 = X^3 \left[ \cos(\mu t) - \frac{\gamma}{4\mu} \sin(\mu t) \right],$$

$$a_2 = \frac{i\Omega}{\mu} X^3 \sin(\mu t),$$

$$b_1 + b_4 = -X^2 (S_+ + S_-) - \frac{i\Omega}{\mu} X^3 \sin(\mu t) S_3 + X^3 \left( \cos(\mu t) + \frac{\gamma}{4\mu} \sin(\mu t) \right) (S_- - S_+),$$

$$b_1 - b_4 = \frac{i\Omega}{\mu} X^3 \sin(\mu t),$$

$$b_{2,3} = \frac{1}{2} X^2 \pm X^3 \left( \cos(\mu t) + \frac{\gamma}{4\mu} \sin(\mu t) \right),$$

$$d_1 + d_4 = 2 - (a_1 + a_4),$$
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\[ d_1 - d_4 = -(a_1 - a_4), \]

\[ X = e^{-\frac{\gamma t}{4}}, \]

\[ S_+ = -\frac{i\Omega \gamma_0}{\gamma^2 + 2\Omega^2}, \]

\[ S_- = S_+^*, \]

\[ S_3 = -\frac{\gamma_0 \gamma}{\gamma^2 + 2\Omega^2}, \]

\[ \gamma = \gamma_0 (2N + 1), \]

\[ \mu = \sqrt{\Omega^2 - (\gamma/4)^2}. \]

Using these, we plot, in FIG. (10.3), concurrence vs the time to ESD for different values of the Rabi frequency \( \Omega \) and observe that \( t_{ESD} \) decreases for an increase in \( \Omega \). This is contrary to the result derived in the previous section, where an increase in the modulation frequency \( \nu \) delayed the loss of entanglement. The decrease in \( t_{ESD} \) does not however continue indefinitely, but rather saturates to a certain value for large values of the Rabi frequency. Figure (10.4) depicts an increase in the single-qubit coherence with an increase in the Rabi frequency \( \Omega \), bringing out the fact that here coherence and entanglement behave in a different fashion. This puts into perspective the fact that coherence, a local property, need not be monotonic with entanglement, a non-local property of quantum correlations.

Let us now consider the situation where the system, consisting of the excited two-level atom, is at zero temperature. Let us also consider the evolution of entanglement for two cases demarcated by the relation between the Rabi frequency and the spontaneous rate of coupling with the thermal reservoir. For the underdamped case when \( \Omega > \gamma_0/4 \), the quantity \( \mu \) is real (since \( N = 0 \) at \( T = 0 \)) and hence both the upper level occupation and coherence exhibit exponentially damped oscillations. Conversely, in the overdamped case, \( \Omega < \frac{\gamma_0}{4} \Rightarrow \mu \) is purely imaginary and both these quantities decay monotonically to their stationary values. The evolution of entanglement, however, works in an opposite way. Entanglement decays faster for the underdamped case than for overdamping, where the \( t_{ESD} \) is higher. One possible reason for this could be the relative positions of the three Lorentzian peaks of the inelastic part of the resonance fluorescence spectrum. The central peak is at \( \omega = \omega_0 \) and the rest are at \( \omega = \omega_0 \pm \mu \) [33] for the underdamped case,
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Figure 10.3: Entanglement (concurrence) of $M_{ij}$ as a function of time for different values of Rabi frequency $\Omega$, varying from 0 to 0.5: 0 (the last curve on the right hand side) corresponding to pure damping and 0.5 (first curve on the left hand side) corresponding to the underdamped case, i.e., it covers both the overdamped as well as the underdamped cases. In the inset one can see that as we increase the $\Omega$ the $t_{ESD}$ seem to converge at $t = 17.0$. Here $\gamma = 0.1$. 

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whereas all three peaks are at $\omega = \omega_0$ for the overdamped case. This indicates that the decay of entanglement in the underdamped should be closely dependent on the quantity $\mu$. This in turn depends on both the dissipation parameter $\gamma$ and the Rabi frequency, the latter in itself a function of the driving strength of the external field and the dipole transition matrix elements. Thus, in the underdamped case, there exists greater avenues for the decay of quantum coherences as well as entanglement than the overdamped case. Phenomenologically, for the underdamped case ($\Omega > \gamma_0/4$), the two-level atom interacts with the external monochromatic field multiple times before spontaneously radiating a photon (see, for example, chapter 10 of ref. [125]). Such numerous interactions allows quantum correlations to develop between the two atomic levels and the quantized levels of the field. The phenomenon of monogamy of entanglement [131] thus ensures that the amount of quantum correlation between the two qubits will decrease. Additionally, it can be seen that at a higher Rabi frequency, $\Omega$ dominates the dissipation and thus causes a saturation of the time to ESD, as shown in FIG. (10.3).
10.4 Dynamic decoupling and the effect on ESD

As discussed earlier, open-loop control strategies involve the application of suitably tailored control fields on the system of interest, with the aim of achieving dynamic decoupling of the system from the environment [170, 171, 169, 3, 5, 6]. Bang-Bang control is a particular form of such decoupling where the decoupling interactions are switched on and off at a rate faster than the rate of interaction set by the environment. The application of suitable radio frequency (RF) pulses, applied fast enough, averages out unwanted effects of the environment and suppresses decoherence. In this section, we compare the effect of Bang-Bang decoupling on the evolution of entanglement, using channel-state duality and factorization law of entanglement decay, in systems connected to two different types of baths. One bath type is composed of infinitely many harmonic oscillators at a finite non-zero temperature $T$ and couples locally to a two-level atom acting as the qubit, while the other adds random telegraph noise to a Josephson-junction charge qubit. It should be kept in mind that our result in [86] that every two-qubit state shows ESD at all non-zero temperature, was based on a particular type of Markovian master equation. So for the case of telegraphic noise, it may not be wise to extend our result bluntly.

10.4.1 Bang-Bang decoupling when the bath consists of harmonic oscillators

Quantum Non-Demolition Interaction

Let us consider the interaction of a qubit with a bath of harmonic oscillators where the system Hamiltonian commutes with the interaction Hamiltonian so that there is no exchange of energy between the system and the bath — this is quantum non-demolition dynamics [11, 14]. The only effect of the bath will be on the coherence elements of the qubit density operator, which will decay in time at the rate $\gamma$. The
total Hamiltonian for the system-bath combine is:

\begin{align}
H_0 &= H_q + H_B + H_I; \quad (10.17) \\
H_q &= \omega_0 \sigma_z, \\
H_B &= \sum_k \omega_k b_k^\dagger b_k, \\
H_I &= \sum_k \sigma_z (g_k b_k^\dagger + g_k^* b_k). 
\end{align}

Here the system Hamiltonian \( H_q \) commutes with the interaction Hamiltonian \( H_I \) and the evolution of such a system is called \textit{pure dephasing}. For simplicity we will work in the interaction picture where the density matrix of the system-bath combine and the interaction Hamiltonian transform as:

\begin{align}
\tilde{\rho}(t) &= e^{i(H_q + H_B) t} \rho(t) e^{-i(H_q + H_B) t}, \quad (10.18) \\
\tilde{H}(t) &= \sigma_z \sum_k (g_k b_k^\dagger e^{i\omega_k t} + g_k^* b_k e^{-i\omega_k t}). \quad (10.19)
\end{align}

From here we can write the total time evolution operator for the system plus bath as

\begin{align}
\tilde{U}(t_0, t) &= T \exp \left\{ -i \int_{t_0}^t ds \tilde{H}(s) \right\} \\
&= \exp \left\{ \frac{\sigma_z}{2} \sum_k \left[ b_k^\dagger e^{i\omega_k t_0} \xi_k(t - t_0) - b_k e^{-i\omega_k t_0} \xi_k^*(t - t_0) \right] \right\}, \quad (10.20)
\end{align}

where \( \xi_k(t) = \frac{2g_k}{\omega_k} (1 - \exp(i\omega_k t)) \). We are interested in calculating

\begin{align}
\tilde{\rho}_{01}(t) = \langle 0 | \text{Tr}_B \left\{ \tilde{U}(t_0, t) \tilde{\rho}(t_0) \tilde{U}^\dagger(t_0, t) \right\} | 1 \rangle. \quad (10.21)
\end{align}

Assuming that the bath and the qubit were uncorrelated in the beginning and that the bath is in a thermal state, we have [170]:

\begin{align}
\tilde{\rho}_{01}(t) = \tilde{\rho}_{01}(t_0) e^{-\gamma(t_0, t)}, \quad (10.22)
\end{align}


Figure 10.5: Sketch of the pulse sequence used in bang-bang decoupling procedure.

where

\[
\gamma(t_0, t) = \sum_k \frac{|\xi_k(t - t_0)|^2}{2} \coth \left( \frac{\omega_k}{2T} \right).
\]  

(10.23)

The matrix representation of the evolution operator \( V_{QND} \) can be written from here as:

\[
V_{QND} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{-\gamma(t_0, t)} & 0 & 0 \\
0 & 0 & e^{-\gamma(t_0, t)} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]  

(10.24)

The evolution of the maximally entangled state \( |\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \) provided sufficient information considering the evolution of entanglement. The evolution of one subsystem in state \( |\phi^+\rangle \) gives rise to the density matrix:

\[
M_{QND} = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & e^{-\gamma(t_0, t)} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
e^{-\gamma(t_0, t)} & 0 & 0 & 1
\end{pmatrix}.
\]  

(10.25)

The concurrence in the state \( M \) is directly proportional to \( e^{-\gamma(t_0, t)} \).

**Dephasing under Bang-Bang dynamics**

The function of Bang-Bang decoupling is to hit the system of interest with a sequence of fast radio-frequency pulses with the aim of slowing down decoherence (see FIG. 10.5). Adding the radio frequency term to the system-plus-bath Hamiltonian
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$H_0$ (Eq. (10.17)), we get

$$H(t) = H_0 + H_{RF}(\omega, t),$$

(10.26)

$$H_{RF}(t) = \sum_{n=1}^{n_p} U^{(n)}(t)\{\cos[\omega(t - t_p^{(n)})]\sigma_x + \sin[\omega(t - t_p^{(n)})]\sigma_y\},$$

(10.27)

where $t_p^{(n)} = t_0 + n\Delta t$, $n = 1, 2, \cdots, n_p$, and

$$U^{(n)}(t) = \begin{cases} U & t_p^{(n)} \leq t \leq t_p^{(n)} + \tau_p \\ 0 & \text{elsewhere.} \end{cases}$$

(10.28)

The term $H_{RF}$ acts only on the system of interest which here is the qubit. It represents a sequence of $n_p$ identical pulses, each of duration $\tau_p$, applied at instants $t = t_p^{(n)}$. The separation between the pulses is $\tau = \Delta t$. The decay rate for this pulsed sequence evolution is [170]:

$$\gamma_p(N, \Delta t) = \sum_k |\eta_k(N, \Delta t)|^2 \coth \left( \frac{\omega_k}{2T} \right),$$

(10.29)

where

$$|\eta_k(N, \omega_k \Delta t)|^2 = 4(1 - \cos(\omega_k \Delta t))^2 \times \left( N + \sum_{n=0}^{N-1} 2n \cos[2(N - n)\omega_k \Delta t] \right).$$

(10.30)

In [170] it has also been shown that $|\eta_k|^2 \leq |\xi_k|^2$ which implies that decoherence is suppressed. Also, it is evident that a lower value of $\eta$ implies a lower value of $\gamma$. Consequently, we conclude that Bang-Bang decoupling slows down entanglement decay.

10.4.2 Josephson Junction qubit

Although solid state nanodevices satisfy the requirements of large scale integrability and flexibility in design, they are subject to various kinds of low-energy excitations in the environment and suffer from decoherence problems. There have been a number of proposals in this context about the implementation of quantum
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Figure 10.6: Schematic diagram for Josephson-junction charge qubit

computers using superconducting nanocircuits [118, 68]. Experiments highlighting the quantum properties of such devices have already been performed [127, 74]. Here the concept of a Josephson-junction qubit comes into prominence. A charge-Josephson qubit is a superconducting island connected to a circuit via a Josephson junction and a capacitor. The computational states are associated with charge $Q$ in the island and are mixed by Josephson tunneling. For temperatures much lower than the Josephson energy, $k_B T \ll E_j$ [153, 119, 133, 132], we have the Hamiltonian

$$H_Q = \frac{\epsilon}{2} \sigma_z - \frac{E_j}{2} \sigma_x,$$  \hspace{1cm} (10.31)

with the charging energy $E_C$ dominating the Josephson energy. Here, $\epsilon \equiv \epsilon(V) = 4E_C(1 - C_2V/e)$, $C_2$ is the capacitance of the capacitor connected to the island and $V$ is the external gate voltage (see Fig. (10.6)).

Fluctuating background charges (BCs) (charge impurities) are an important source of decoherence in the operation of Josephson charge qubits. These are believed to originate in random traps for single electrons in dielectric materials surrounding the superconducting island. These fluctuations cause at low frequencies, the $1/f$ noise which is also known as random telegraph noise, and is directly observed in single electron tunneling devices [190, 128]. This has also been studied in the context of fractional statistics in the Quantum Hall Effect [105]. This noise,
arising out of decoherence, is modeled \[133, 132\] by considering each of the BCs as a localized impurity level connected to a fermionic band, i.e., the quantum impurity is described by the Fano Anderson model. This is the quantum analogue of the classical model of \(N\) independent, randomly activated bistable processes. For a single impurity, the total Hamiltonian is:

\[
H = H_Q - \frac{v}{2} b^\dagger b \sigma_z + H^I, \tag{10.32}
\]

where

\[
H^I = \epsilon_c b^\dagger b + \sum_k \left[ T_k c_k^\dagger b + \text{h.c.} \right] + \sum_k \epsilon_k c_k^\dagger c_k. \tag{10.33}
\]

Here \(H^I\) describes the BC Hamiltonian, \(b\) represents the impurity charge in the localized level \(\epsilon_c\), \(c_k\) the electron in the band with energy \(\epsilon_k\), and \(H_Q\) is as in Eq. (10.31). The impurity electron may tunnel to the band with amplitude \(T_k\). The BC produces an extra bias \(v\) for the qubit via the coupling term \((v/2)b^\dagger b \sigma_z\). An important scale is the switching rate \(\gamma = 2\pi \rho(\epsilon_c) |T|^2\), where \(\rho(\epsilon_c)\) is the density of states of the band. It is assumed that we are working in the the relaxation regime of the BC where the tunneling rate to all fermionic bands are approximately same. The fraction \(v/\gamma\) determines whether the operational regime of the qubit is weak \((v/\gamma \ll 1)\) or strong \((v/\gamma > 1)\). Studying the single BC case is important, since it has been shown \[133\] that the effect of multiple BCs can be trivially extended from that of a single BC. For multiple strongly coupled BCs producing \(1/f\) noise, the effect of a large number of slow fluctuators is minimal and pronounced features of discrete dynamics such as saturation and transient behavior are seen. There are two special operational points for the qubit related to Eq. (10.31): (a) \(\epsilon = 0\), corresponding to charge degeneracy and (b) \(E_j = 0\), for the case of pure dephasing \[26, 2\], where tunneling can be neglected. We will consider this case later in detail and make a comparison of ESD, for the case of pure dephasing, between the harmonic oscillator and \(1/f\) baths.

The general procedure for studying the effect of the BC on the dynamics of the qubit is to calculate the unitary evolution of the entire system-bath combine and then trace out the bath degree of freedom. Thus, i.e, \(\rho_Q(t) = \text{tr}_B\{ W(t) \}\), \(W(t)\) being the the full density matrix. In the weak coupling limit a master equation for
\( \rho_Q(t) \) can be written [53]. The results in the standard weak coupling approach are obtained at lowest order in the coupling \( v \), but it has been pointed out that higher orders are important for a \( 1/f \) noise [153, 119, 133].

The failure of the standard weak coupling approach is due to the fact that the \( 1/f \) environment includes fluctuators which are very slow on the time scale of the reduced dynamics. To circumvent this problem one considers another approach in which a part of the bath is treated on the same footing as the system [132]. We study the evolution of this new system and later trace out the extra part which belongs to the bath, i.e., \( \rho(t) = \text{Tr}_b\{W(t)\} \). We then obtain \( \rho_Q(t) \) from \( \rho(t) \) as \( \rho_Q(t) = \text{Tr}_b\{\rho(t)\} \), where the subscript \( fb \) stands for fermionic band. In that context we split the Hamiltonian (10.32) into a system Hamiltonian \( H_0 = H_Q - \frac{v}{2} b^\dagger b \sigma_z + \epsilon_c b^\dagger b \) and environment Hamiltonian \( H_E = \sum_k \epsilon_k c_k^\dagger c_k \) coupled by \( V = \sum_k [T_k c_k^\dagger b + \text{h.c.}] \). The eigenstates of \( H_0 \) are product states of the form \( |\theta\rangle|n\rangle \), e.g.,

\[
|a\rangle = |\theta_+\rangle|0\rangle, \\
|b\rangle = |\theta_-\rangle|0\rangle, \\
|c\rangle = |\theta'_+\rangle|1\rangle, \\
|d\rangle = |\theta'_-\rangle|1\rangle,
\]

with corresponding energies

\[
-\frac{\Omega}{2}, \frac{\Omega}{2}, \frac{\Omega'}{2} + \epsilon_c, \frac{\Omega'}{2} + \epsilon_c.
\]

Here \( |\theta_\pm\rangle \) are the two eigenstates of \( \sigma_n \), the direction being specified by the polar angle \( \theta \) and \( \phi = 0 \). The two level splittings are \( \Omega = \sqrt{\epsilon^2 + E_j^2} \) and \( \Omega' = \sqrt{(\epsilon + v)^2 + E_j^2} \), and \( \cos(\theta) = \epsilon/\Omega, \sin(\theta) = E_j/\Omega, \cos(\theta') = (\epsilon + v)/\Omega', \sin(\theta') = E_j/\Omega' \).

The master equation for the reduced density matrix \( \rho(t) \), in the Schrödinger representation and in the basis of the eigenstates of \( H_0 \) reads:

\[
\frac{d\rho_{ij}(t)}{dt} = -i\omega_i \rho_{ij}(t) + \sum_{mn} R_{ij,mn} \rho_{mn}(t),
\]

(10.34)
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where \( \omega_{ij} \) is the difference of the energies and \( R_{ij,mn} \) are the elements of the Redfield tensor [53]. These are given by:

\[
R_{ij,mn} = \int_0^\infty d\tau \left\{ c_{njmi}^>(\tau) e^{i\omega_{mi}\tau} + c_{njmi}^<(\tau) e^{i\omega_{jn}\tau} - \delta_{nj} \sum_k c_{ikmk}^>(\tau) e^{i\omega_{mk}\tau} - \delta_{im} \sum_k c_{nkjk}^<(\tau) e^{i\omega_{kn}\tau} \right\},
\]

(10.35)

where

\[
c_{ijkl}^>(t) = \left[ \langle i| b| j \rangle \langle l| b^\dagger| k \rangle + \langle i| b^\dagger| j \rangle \langle l| b| k \rangle \right] i G^>(t).
\]

(10.36)

Here \( G^>(\omega) = \gamma/(1 - e^{-\beta\omega}) \) is the Fourier transform of \( G^>(t) \) and \( G^<(\omega) = G^>(-\omega) \), therefore, \( G^<(t) = G^>(-t) \). This problem has a very interesting symmetry: the diagonal and off diagonal elements do not mix if the initial state of the charge particle is a diagonal density matrix in the BC. Therefore, we can divide the Redfield tensor elements in two parts, one corresponding to population (diagonal elements) and other corresponding to coherence (off diagonal elements).

The \( R_{ii,nn} \) elements which affect the population are:

\[
R_{ii,nn} = \int_0^\infty \left\{ \chi_{in} i G^>(\tau) e^{i\omega_{ni}\tau} + \chi_{in} i G^<(\tau) e^{-i\omega_{ni}\tau} \right\} = \chi_{in} \left[ i G^>(\omega_{ni}) \right].
\]

(10.37)

Here \( n \neq i \) and \( \chi_{in} = (|\langle n| b| i \rangle|^2 + |\langle n| b^\dagger| i \rangle|^2) \), and

\[
R_{ii,ii} = -\sum_k \chi_{ik} \left[ i G^>(\omega_{ik}) \right].
\]

(10.38)

Now we calculate the elements which are responsible for the coherence part. In the adiabatic regime we have \( \gamma \sim \Omega - \Omega' \ll \Omega \sim \Omega' \), i.e., where the BCs are not static and the mixing of \( \rho_{ab} \) and \( \rho_{cd} \) (10.34), as well as their conjugates, cannot be neglected. Hence the non-zero elements of \( R \) tensor – which affect the coherence
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– are the following:

\[ R_{ab,ab} = -\frac{\gamma}{2} \left[ 1 - c^2 \delta - s^2 \delta' + i(c^2 w + s^2 w') \right], \]

\[ R_{cd,cd} = -\frac{\gamma}{2} \left[ 1 + c^2 \delta + s^2 \delta' + i(c^2 w - s^2 w') \right], \]

\[ R_{ab,cd} = \frac{c^2 \gamma}{2} [1 + \delta + iw], \]

\[ R_{cd,ab} = \frac{c^2 \gamma}{2} [1 - \delta - iw]. \]

Here

\[ c = \cos[(\theta - \theta')/2], \]

\[ s = \sin[(\theta - \theta')/2], \]

\[ \delta = t_{ca} + t_{db}, \]

\[ \delta' = t_{da} + t_{cb}, \]

\[ w = w_{ca} - w_{cb}, \]

\[ w' = w_{da} - w_{cb}, \]

\[ t_{ij} = \frac{1}{2} \tanh \left( \frac{\beta \omega_{ij}}{2} \right), \]

\[ w_{ij} = -\frac{1}{\pi} \Re \left\{ \psi \left( \frac{\pi + i\beta \omega_{ij}}{2\pi} \right) \right\}, \]

and \( \psi(z) \) is the digamma function.
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Now we can construct the explicit form of the matrix $R$:

\[
R = \begin{pmatrix}
R_{1,1} & 0 & 0 & 0 & 0 & R_{1,2} & 0 & 0 & 0 & R_{1,3} & 0 & 0 & 0 & 0 & R_{1,4} \\
0 & z_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_+ & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z_+^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_+^* & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & R_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
R_{3,1} & 0 & 0 & 0 & 0 & R_{3,2} & 0 & 0 & 0 & 0 & R_{3,3} & 0 & 0 & 0 & R_{3,4} \\
0 & y_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_+ & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & y_+^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_+^* & 0 \\
R_{4,1} & 0 & 0 & 0 & 0 & R_{4,2} & 0 & 0 & 0 & 0 & R_{4,3} & 0 & 0 & 0 & R_{4,4}
\end{pmatrix}
\]

(10.39)

where

\[
z_- = -\frac{\gamma}{2} \left[1 - c^2 \delta - s^2 \delta' + i(c^2 w + s^2 w') \right],
\]

\[
z_+ = -\frac{\gamma}{2} \left[1 + c^2 \delta + s^2 \delta' + i(c^2 w - s^2 w') \right],
\]

\[
y_+ = \frac{c^2 \gamma}{2} \left[1 + \delta - i w \right],
\]

\[
y_- = \frac{c^2 \gamma}{2} \left[1 - \delta - i w \right],
\]

\[
R_{i,j} = R_{ii,jj}.
\]

Channel-state duality implies that the exponential of the matrix $R$ is the matrix representation of the evolution channel. Therefore, we have $V = \exp(Rt)$ which gives us the evolution for the qubit plus the charge impurity. This map is in the basis \{|$\theta_\pm$|$i$\} $\otimes$ \{|$\theta_\pm$|$j$\}. To make it computationally easier we need to write it
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in the basis $\{|\theta_\pm\rangle|\theta_\pm\rangle\} \otimes \{|i\rangle|j\rangle\}$, since the matrix representation of the map acting on the qubit is in the basis $\{|\theta_\pm\rangle|\theta_\pm\rangle\}$. To change the basis we need the assistance of a unitary matrix (in this case permutation matrix) $P$ which is defined as:

$$P(|a\rangle|b\rangle|c\rangle|d\rangle) = |a\rangle|c\rangle|b\rangle|d\rangle;$$  \hspace{1cm} (10.40)

$$|a\rangle|b\rangle|c\rangle|d\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \otimes \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$  \hspace{1cm} (10.41)

$$|a\rangle|c\rangle|b\rangle|d\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \otimes \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.$$  \hspace{1cm} (10.42)

$$\Rightarrow P = \mathcal{I} \otimes p \otimes \mathcal{I},$$  \hspace{1cm} (10.43)

where

$$p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

After conjugating the matrix $V$ by $P$ we get:

$$\tilde{V} = PV P^T.$$  \hspace{1cm} (10.44)

We can write this $16 \times 16$ matrix $\tilde{V}$ as a $4 \times 4$ matrix, where each of the element itself is a $4 \times 4$ matrix $Z_{ij}$ where $i, j \in \{1, 2, 3, 4\}$. Then the map $V_s$ acting on qubit is simply $V_{sij} = \text{Tr}(Z_{ij})$. From here we can get the corresponding $M$ matrix. It is not easy to solve it analytically in the present case. Therefore, we use numerical methods to calculate the evolution operator and entanglement evolution for a system of qubits.

The two parameters $\epsilon$ and $E_j$ in the Hamiltonian for the charge Josephson qubit $H_Q$ play a crucial role in the decoherence properties of the system. For example, if $E_j = 0$, the system Hamiltonian $H_Q$ commutes with the interaction Hamiltonian. This situation, as mentioned earlier, is called non-demolition evolution or pure dephasing. In this case there is no energy exchange between system and the bath.
Figure 10.7: Contour of the entanglement after time $t = 5$ for all values of $E_j$ and $\epsilon$ (Josephson junction Hamiltonian parameters) The temperature in this case is equal to 0, $\gamma = 1$, $\kappa = v/\gamma = 0.45$. 

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On the other hand when we have $\epsilon = 0$, the system Hamiltonian does not commute with the interaction Hamiltonian. Therefore, the two situations are qualitatively different. We present, in Fig. (10.7), a plot of entanglement in the phase space of $\epsilon$ and $E_j$ by evolving a maximally entangled state of two qubits with the bath (of charge impurities) acting only on one qubit. The qubit is evolved for a fixed time $t$ and the entanglement is calculated for different values of $\epsilon$ and $E_j$. Fig. (10.7) shows that the entanglement in the system increases with an increase in $E_j$ when $\epsilon$ is held fixed, but decreases with an increase in $\epsilon$ when $E_j$ is held fixed. This is counterintuitive because dissipation increases with the increase in the value of $E_j$. Fig. (10.8) compares the time-evolution of entanglement for the harmonic oscillator bath with the charge-impurities bath, both under pure dephasing. While entanglement decay is exponential for the case of $1/f$ noise, it is slower for a bath of harmonic oscillators. We compare, in FIG. (10.9), the time-evolution of entanglement for various values of Josephson energy ($E_j$) starting with the pure dephasing case given by $E_j = 0$. We see that the entanglement remaining in the system increases with an increase in the value of Josephson energy. This is consistent with FIG. (10.7).

Decoherence produced by background charges depends qualitatively on the ratio $\kappa = v/\gamma$, where $\kappa \ll 1$ denotes the weak-coupling regime and $\kappa > 1$ is the strong coupling regime. The latter gives rise to qualitatively new properties. We find that (see FIG. (10.10(b))), for $\kappa > 1$, the time-evolution of entanglement does not depend on $\kappa$. This is in contrast to the weak coupling regime, where the time-evolution of entanglement does depend on $\kappa$, as seen in FIG. (10.10(a)), where an increase in $\kappa$ leads to a decrease in entanglement. Naturally, decoherence due to the bath forces entanglement to decay with time for both cases.

**Evolution operator with Bang-Bang interaction**

The Josephson charge qubit in contact with an $1/f$ bath is now subject to fast pulses, under the Bang-Bang dynamical decoupling scheme. The Hamiltonian for this radio frequency pulse is the same as in Eq. (10.27). If the time for which a pulse is active is $\pi$, then the evolution operator for the pulse may be written as
Figure 10.8: Evolution of entanglement for the pure dephasing case, i.e, $E_j = 0$. We can see that whereas entanglement decay is exponential for the $1/f$ noise, it slows down for a bath of harmonic oscillator (inset) at zero temperature.
Figure 10.9: Evolution of entanglement for $1/f$ (telegraph) noise at zero temperature. Here different curves represent the evolution of entanglement for different values of $E_j$ (from 0 to 1), with the curve at the bottom corresponding to $E_j = 0$ and that at the top to $E_j = 1$, while $\epsilon$ is fixed and equal to 1.
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(a) Plot of the entanglement (concurrence) as a function of time (t) for different values of coupling strength $\kappa = v/\gamma$, in the weak coupling regime, i.e, $\kappa \ll 1$. Here the range of $\kappa$ is from 0.05 to 0.5, with 0.05 corresponding to the uppermost curve, and 0.5 to the lowest (bottom) one. We can see that as we increase $\kappa$, entanglement decreases.

(b) Plot of the entanglement (concurrence) as a function of time (t) for different values of coupling strength $\kappa = v/\gamma$, in the strong coupling regime, i.e, $\kappa > 1$. Here the range of $\kappa$ is from 5.05 to 5.5. We can see that in the strong coupling region all the curves converge.

Figure 10.10: Evolution of entanglement with respect to time, for different coupling strengths and temperature $T = 0$, $E_j = 1 = \epsilon$. 


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\[ V_p = I \otimes i\sigma_x \] where \[ 2U\tau_p = \pm \pi. \] The total evolution can therefore be written as

\[ V_{total} = (V_p V_S(\tau))^{2N} \tag{10.45} \]

where \[ 2N\tau = t. \] Since the RF pulses act on the system for very short amounts of time, the evolution of the system can safely be assumed to be governed only by the dynamical map \( V_p \) for the time period during which the pulse is operating. As can be seen from Figs. (10.11) to (10.13) and FIG. (10.16), the system exhibits the phenomenon of ESD on the application of bang-bang pulses.

Let us consider the case where \( E_j = \epsilon = 1 \). Let us also fix the pulse strength to be \( U = 50\pi \) and ensure that the pulses act for very short times. As defined earlier, the ratio of the BC bias \( v \) and the switching rate \( \gamma \) defines the weak and strong coupling regimes, the former designated by \( \frac{v}{\gamma} \ll 1 \) and the latter by \( \frac{v}{\gamma} > 1 \).

In FIG. (10.11), we plot the time-evolution of entanglement, with the coupling strength as parameter. For weak coupling, we find that \( t_{ESD} \) initially increases with coupling strength. This continues till a turning point is reached at \( \frac{v}{\gamma} = 0.38 \) when \( t_{ESD} \simeq 880 \). After this, with increase in coupling strength, \( t_{ESD} \) starts to decrease. As a result, a kink appears in the corresponding entanglement vs time plot, FIGS. (10.11), (10.12). The receding of \( t_{ESD} \) with increase in coupling strength continues well into the strong coupling regime, i.e. for \( 5.05 < \frac{v}{\gamma} < 5.5 \).

It, however, does not go to zero, but rather chooses to saturate at the threshold value of \( t_{ESD} \simeq 10 \), see FIG. (10.13). The “turning” and the “saturation” features are well captured in FIG. (10.16), where we plot \( t_{ESD} \) against \( \frac{v}{\gamma} \) and keep the pulse strength and durations fixed. We observe a crossover phenomenon around \( \frac{v}{\gamma} \simeq 0.38 \), where the value of \( t_{ESD} \) rises sharply, only to fall back again even quicker.

The evolution of coherence with respect to time, for the Josephson charge qubit subjected to \( 1/f \) noise, is shown in Figs. (10.14), (10.15), for the weak and strong coupling regimes, respectively. Both show an improvement in the coherence with the application of the bang-bang decoupling pulses, in contrast to the corresponding behavior of entanglement, thereby reiterating that coherence is not synonymous with entanglement.

In FIG. (10.17), we plot the behavior of \( t_{ESD} \) with \( E_j \) and find that, as we
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Figure 10.11: Effect of bang-bang decoupling on entanglement, in the weak coupling regime. If we compare this plot with FIG. (10.10(a)), with the curves corresponding to the same values of $\kappa$, we see that the bang-bang decoupling causes entanglement to disappear faster in time for a fixed value of the coupling strength. Here the parameters are same as in FIG. (10.10(a)) and the pulse strength is $U = 50\pi$ with time for which the pulse was activated is $\tau_p = 0.01$. In the inset we have the evolution of entanglement for very small range (0.01 to 0.1) of coupling $\kappa$. The thickest curve is the one corresponding to $\kappa = 0.38$. This curve is important in the sense that it has the largest $t_{ESD}$.

10.5 Summing up

As stated earlier, the aim of most control procedures is to suppress decoherence. For the case of photonic crystals, the design allows the system to conserve coherence when it is within the photonic band gap. Modulating the frequency of the system-bath coupling aims to suppress decoherence by shifting the system out of the spectral influence of the bath. In both these cases it is found that the suppression increase $E_j$ and thus move away from the pure dephasing situation, the time to ESD keeps increasing. As discussed earlier, this is a counterintuitive result because dissipation increases with $E_j$. 

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Figure 10.12: Entanglement evolution for the coupling parameter range $0.3 < \kappa < 0.4$, with the uppermost curve corresponding to $\kappa = 0.3$ and the lowest (bottom) curve corresponding to 0.4. One can see from this plot the formation and disappearance of the kink.
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Figure 10.13: Effect of bang-bang decoupling on entanglement in the strong coupling region. Here again we can see the effect of bang-bang decoupling on the entanglement if we compare this plot with FIG.(10.10(b)). Here the parameters are same as in FIG. (10.10(b)) and the pulse strength is $U = 50\pi$ with the pulse duration $\tau = 0.01$.

Figure 10.14: Plot for the evolution of coherence in the case of Telegraph noise in weak coupling region, i.e, $\kappa < 1$. Here all the parameters has the value same as in Fig. (10.12) and (10.11) and $\kappa = 0.38$. 

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Figure 10.15: Plot for the evolution of coherence in the case of Telegraph noise in strong coupling region, i.e, $\kappa \geq 1$. Here the value of all the parameters are same as in Fig. (10.13) and $\kappa = 5.38$.

Figure 10.16: $t_{ESD}$ is plotted as a function of coupling strength $\kappa$. Here we can see that there is a clear distinction between the strong and the weak coupling region. As we increase $\kappa$ the $t_{ESD}$ tends to freeze and asymptotic value of $t_{ESD}$ is around 10. The parameters used are as in the previous plots.
Figure 10.17: $t_{ESD}$ is plotted as a function of $E_j$. This shows that as we go away from pure dephasing the $t_{ESD}$ increases. The parameters used are same as in the previous plots.
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of decoherence is accompanied by a corresponding increase in $t_{ESD}$.

However, it will be erroneous to naïvely suppose that this is the norm. Exactly the opposite phenomenon is observed for the case of resonance fluorescence, where the coupling between the bath and a two-level atomic system forced by an external resonant field, is modulated. It is seen that an increase in the external field frequency $\Omega$, the Rabi frequency, results in a faster decay of entanglement (FIG. (10.3)). A further non-trivial effect observed is the saturation in the time to ESD: $t_{ESD}$ does not go below a threshold value no matter what the Rabi frequency. In what could point towards a possible explanation of this phenomenon, we observe that the sudden death time stops being dependent on the Rabi frequency at $\Omega = \frac{\gamma_0}{4}$; strikingly, this happens to be the boundary between the overdamped $\Omega < \frac{\gamma_0}{4}$ and underdamped $\Omega > \frac{\gamma_0}{4}$ regimes.

In dynamic decoupling schemes RF pulses, applied at short time-intervals, smooth out unwanted effects due to environmental interactions. We discuss two qualitatively different system-bath models: the first being the usual qubit and harmonic oscillator bath pair with pure dephasing or QND interaction; and the second being a bath of charge impurities, simulating $1/f$ (telegraph) noise, acting on a Josephson-junction charge qubit. Entanglement decays to zero asymptotically in both these models. The application of fast RF pulses to the former manages to speedup the rate of the still-asymptotic loss of entanglement, whereas the same RF pulse applied to the latter kills off entanglement in finite time and thus shows ESD. A very interesting phenomenon, observed in the strong coupling regime, is the decrease in the time to ESD with increasing pulse strengths. This is extremely counterintuitive, and brings into perspective the fact that in the non-Markovian strong coupling regime, the dynamics of entanglement can be different than that of decoherence. This feature gets further highlighted by the behavior of coherence with time, both for the case of resonance fluorescence and Josephson-junction charge qubit subjected to $1/f$ noise. Here coherence—which is a local property—is seen to vary in a non-monotonic fashion with entanglement which happens to be a non-local property of the system.
Table 10.1: Summary of the results.

<table>
<thead>
<tr>
<th>Control Procedure</th>
<th>Decoherence suppression</th>
<th>Entanglement decay suppression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Photonic Crystals</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Frequency Modulation</td>
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<td>✓</td>
</tr>
<tr>
<td>Resonance Fluorescence</td>
<td>✓</td>
<td>×</td>
</tr>
<tr>
<td>DD in EMF bath</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>DD in Telegraph noise</td>
<td>✓</td>
<td>×</td>
</tr>
</tbody>
</table>