Part I

Introduction
Chapter 1

Introduction

1.1 Black Holes In General Relativity

Black holes are fascinating area of research both in theoretical physics and observational astrophysics. From experimental point of view it poses a challenging observational problem to find black holes in galaxies, however there are evidences of super-massive black holes at the center of most of the galaxies. From theoretical point of view it poses a challenging problem of unifying general relativity with quantum mechanics. Thus black holes are excellent theoretical laboratory for understanding some features of quantum gravity. The goal of the present thesis is to explore few aspects of the quantum theory of the black holes.

General theory of relativity, formulated by Einstein, is a classical theory of gravity according to which the gravity is a manifestation of the curvature of space time. According to Einstein, the gravitational field is not a new field but correspond to deviation of spacetime geometry from that of flat space time. The geometry of a spacetime is described in terms of metric or line element between two points.

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu \]  

Here \( \mu, \nu = 1, ..., 4 \). The deviation of the spacetime metric from flatness, which accounts for the physical effects usually ascribed to gravity, is determined in terms of Riemann curvature tensor \( R_{\mu\nu\rho\sigma} \). The Riemann tensor \( R_{\mu\nu\rho\sigma} \) is given in terms of Christoffel symbols \( \Gamma^\lambda_{\mu\nu} \) as

\[ R^\mu_{\nu\rho\lambda} = \partial_\rho \Gamma^\mu_{\nu\lambda} - \partial_\lambda \Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\rho} \Gamma^\mu_{\tau\lambda} - \Gamma^\tau_{\nu\lambda} \Gamma^\mu_{\tau\rho} \]  

(1.2)

The Christoffel symbols \( \Gamma^\lambda_{\mu\nu} \) are given as

\[ \Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho} \left( \partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right) \]  

(1.3)
Furthermore the curvature of the spacetime is related to the energy-momentum tensor of the matter in the spacetime via Einstein’s equation given as

\[ R_{\mu \nu} - \frac{1}{2} (R - 2\Lambda) g_{\mu \nu} = -8\pi T_{\mu \nu} \] (1.4)

Here \( \Lambda \) and \( T_{\mu \nu} \) are cosmological constant and energy-momentum tensor of the matter respectively\(^1\). The Ricci tensor \( R_{\mu \nu} \) and Ricci scalar \( R \) are given as

\[ R_{\mu \nu} = R^\rho_{\mu \rho \nu}, \quad R = g^{\mu \nu} R_{\mu \nu} \] (1.5)

The Einstein’s equation (1.4) is obtained as the Euler-Lagrange equation of motion from the action

\[ S = -\frac{1}{16\pi} \int d^4x \sqrt{-g} (R - 2\Lambda) + S_{\text{matter}} \] (1.6)

Here \( S_{\text{matter}} \) is the matter action.

Classically black holes are solutions of Einstein’s equations with special properties. They have a hypothetical surface - known as the event horizon - surrounding them such that no object inside the event horizon can escape the black hole. One of the simplest black hole solution in four dimension is the Schwarzschild solution. The Schwarzschild solution is a static, spherically symmetric solution of vacuum Einstein equation (1.4) (with \( T_{\mu \nu} = 0 \)) which describes the geometry outside the spherically symmetric energy-momentum distribution. The solution for mass \( M \) is given as

\[ ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \] (1.7)

There exist more general black hole solutions which in addition carry charge and angular momentum. In Einstein-Maxwell theory the most general stationary black hole solution is the Kerr-Newmann black hole, which is uniquely characterized by its mass, charge and angular momentum.

### 1.2 Laws of Black Hole Mechanics

One of the most remarkable results of black hole physics is that one can derive a set of laws, called the laws of black hole mechanics, which have the same structure as the laws of thermodynamics.

Before stating the laws of the black hole mechanics, we first give few definitions [1, 2, 3, 4, 5].

\(^1\)In this report we will put all fundamental constants, \( G_N, c \) and \( h \) to 1.
Null Hypersurface: Let $S(x)$ be a smooth function of the spacetime coordinates $x^\mu$ and consider a family of hypersurfaces $S = \text{constant}$. The vector fields normal to the hypersurface are

$$l = f(x)g^{\mu\nu}\partial_\nu S(x)\frac{\partial}{\partial x^\mu}$$  \hspace{1cm} (1.8)

where $f(x)$ is some arbitrary non-zero function. If $l^2 = 0$ for a particular hypersurface, $N$, in the family, then $N$ is said to be a null hypersurface.

A tangent vector field $t$ on $N$ is one for which $t \cdot l = 0$. Since for a null hypersurface $N$, $l \cdot l = 0$, $l$ is itself a tangent vector field i.e.

$$p^\mu = \frac{dx^\mu(\lambda)}{d\lambda}$$  \hspace{1cm} (1.9)

for some curve $x^\mu(\lambda)$ in $N$.

It is simple to prove that the curves $x^\mu(\lambda)$ generated by the vector field $l$ are geodesics. These are called null geodesics and are the generators of the null surface $N$.

Killing Horizon: A null hypersurface $N$ is a Killing horizon of a Killing vector field $\xi$ if, on $N$, $\xi$ is normal to $N$. Then on $N$

$$\xi = f(x)l$$  \hspace{1cm} (1.10)

for some function $f(x)$ and $l$ is normal to $N$.

For killing horizon one defines the surface gravity $k_s^2$ as

$$\xi^\mu D_\mu \xi^\nu = k_s \xi^\nu, \quad \text{on} \quad N$$  \hspace{1cm} (1.11)

were $D_\mu$ is covariant derivative and $k_s = \xi \cdot \partial \ln |f(x)|$ on $N$.

Then by simple calculation one can show that on the killing horizon $N$ the surface gravity $k_s$ is constant on the orbits of the killing vector field $\xi$ generating $N$.

Event horizons of black holes are null hypersurfaces. In Einstein’s theory of gravity all event horizons of stationary black holes are killing horizon. From here onwards we will assume that the event horizon of a stationary asymptotic flat black hole in arbitrary higher derivative theory of gravity is a killing horizon. This is trivially true for static black hole since in this case there exist a time like hypersurface orthonormal killing vector field.

Let us consider the case of non-degenerate killing horizons ($k_s \neq 0$). We choose a coordinate on $N$ such that

$$\xi = \frac{\partial}{\partial \alpha}$$  \hspace{1cm} (1.12)

with group parameter $\alpha$ as one coordinate. Consider the orbit $\alpha = \alpha(\lambda)$ with affine parameter $\lambda$ generated by $\xi$. Then

$$\xi|_{\text{orbit}} = \frac{\partial \lambda}{\partial \alpha} \frac{\partial}{\partial \lambda} = fl$$  \hspace{1cm} (1.13)

\footnote{In defining surface gravity we choose standard normalisation for $\xi$ at infinity, e.g. for time translation killing field $\xi$, in asymptotic flat spacetime, we choose normalisation such that $\xi^2 \to -1$ as $r \to \infty$.}
Here \( f = \frac{\partial \lambda}{\partial \alpha} \) and \( l = \frac{\partial}{\partial \lambda} \).

Now we have

\[
\frac{\partial \ln |f|}{\partial \alpha} = k_s
\]

(1.14)

Solving this we get

\[
f = \frac{\partial \lambda}{\partial \alpha} = \pm f_0 e^{k_s \alpha}
\]

(1.15)

Since there is a freedom to shift \( \alpha \) by a constant, we can choose \( f_0 = k_s \). Hence we get

\[
\lambda = \pm e^{k_s \alpha} + \text{constant}
\]

(1.16)

We can choose the constant to be zero. Hence when the group parameter \( \alpha \) ranges from \( -\infty \) to \( +\infty \), \( \lambda \) is either \( >0 \) or \( <0 \). Thus \( \xi \) generates two null surface \( \mathcal{N}_A \) and \( \mathcal{N}_B \) intersecting at \( \lambda = 0 \) hypersurface \( \Sigma \) corresponding to fixed point of \( \xi \). The killing horizon \( \mathcal{N} \) which is union of two null hypersurface \( \mathcal{N}_A \) and \( \mathcal{N}_B \) is called bifurcate killing horizon and \( \Sigma \) \((\xi|_{\Sigma} = 0)\) is called bifurcation surface.

As it is evident, in degenerate case \((k_s = 0)\) there are no bifurcate killing horizon and bifurcation surface.

The fact that the entropy \( S \) of a stationary black hole with bifurcate killing horizon is \( 2\pi \) times the Noether charge associated with the horizon killing field, normalised so as to have unit surface gravity, was first proved in [6, 7]. The expression for the entropy was derived for a general, classical theory of gravity in \( d \)-dimension arising from diffeomorphism invariant Lagrangian with arbitrary matter fields. It is given as

\[
S = 2\pi \int_{\Sigma} \tilde{Q}
\]

(1.17)

where \( \Sigma \) is the bifurcation surface where killing field \( \xi = 0 \) and \( \tilde{Q} \) is Noether charge associated with killing field \( \xi \). In 4-dimension, for a Lagrangian which does not depend on derivatives of the Riemann tensor, the entropy is given by

\[
S = 2\pi \int_{\Sigma} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \varepsilon_{\mu\nu} \varepsilon_{\rho\sigma} \sqrt{h} dA
\]

(1.18)

where \( \varepsilon^{\mu\nu} \) is binormal\(^3\) of \( \Sigma \). As an example, consider a 4-dimensional theory described by Lagrangian \( \mathcal{L} = -\frac{1}{16\pi} R \).

We consider a static and spherically symmetric metric of the form

\[
ds^2 = -e^{2g(r)} dt^2 + e^{2f(r)} (dr^2 + r^2 d\Omega^2)
\]

(1.19)

\(^3\)In general expression for the Noether charge one of the binormal arises through the relation \( D_\mu \xi_\nu = k_s \varepsilon_{\mu\nu} \) on the bifurcation surface \( \Sigma \).
For this metric the binormal takes the form \( \varepsilon_{tr} = -\varepsilon_{rt} = \varepsilon^{g(r) + f(r)} \). We also have

\[
8\pi \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \varepsilon_{\mu\nu} \varepsilon_{\rho\sigma} = -\frac{1}{2} g^{\mu\nu} g^{\rho\sigma} \varepsilon_{\mu\nu} \varepsilon_{\rho\sigma} = -(\varepsilon_{tr})^{2} g^{tt} g^{rr} = 1 \tag{1.20}
\]

And the entropy is

\[
S = \frac{2\pi}{8\pi} \int_{\Sigma} \sqrt{h} d\Omega = \frac{A}{4} \tag{1.21}
\]

where \( A \) is the area of the event horizon at \( r = 0 \).

Now consider a stationary rotating black hole with bifurcate killing horizon. Let the killing field which vanishes on the bifurcation surface be

\[
\xi = \xi_{t} + \Omega \xi_{\phi} \tag{1.22}
\]

where \( \xi_{t} \) and \( \xi_{\phi} \) are timelike and axial killing field respectively. Then the surface charges associated with these killing field at asymptotic infinity are the mass \( (M) \) and the angular momentum \( (J) \) of the black hole and their expression in terms of killing field are similar to Komar’s formula of mass and angular momentum.

We shall now state the laws of black hole mechanics [8, 9, 10, 3].

**Zeroth Law:** The zeroth law states that the surface gravity \( k_s \) of a stationary black hole is constant over the event horizon,

\[
k_s = \text{constant} \tag{1.23}
\]

**First Law:** The first law, in general theory of gravity, states\(^4\) that if one considers two infinitesimally close stationary rotating black hole solutions then the change \( \delta M \) of the mass is related to change of entropy, \( \delta S \) and of the angular momentum, \( \delta J \) via:

\[
\delta M = k_s \frac{2\pi}{\Omega} \delta S + \Omega \delta J \tag{1.24}
\]

Here \( \Omega \), defined via (1.22), is the angular velocity at the horizon. Second Law: If \( T_{\mu\nu} \) satisfies the weak energy condition and assuming that the cosmic censorship conjecture is true then the total area of the event horizon of black holes in a non-stationary process (e.g. collisions and fusion of black holes) in an asymptotically flat spacetime is a non-decreasing function of time.

The weak energy condition means that \( T_{\mu\nu} u^\mu u^\nu \geq 0 \) at any point in spacetime for any time like vector field \( u^\mu \). The cosmic censorship conjecture states that naked singularities can not form from a gravitational collapse in an asymptotically flat spacetime which was non singular on some initial spacelike hypersurface.

The zeroth law of the black hole mechanics resembles the zeroth law of thermodynamics,

\(^4\)In Einstein’s theory of gravity it takes the form \( \delta M = k_s \frac{2\pi}{A + \Omega} \delta A \), where \( A \) is the area of the event horizon.
which says that the temperature is constant in a thermodynamic equilibrium. The first law of black hole mechanics resembles the energy conservation and the second law resembles the thermodynamic law that the entropy in a given non-equilibrium process always increases. These resemblance led one to think that black hole has temperature and thermodynamic entropy which is proportional to surface gravity and area of the event horizon. However the black hole laws are a priori not linked to thermodynamics in any obvious way because they are derived using geometrical properties of event horizons and general covariance. The most obvious reason for not believing in a thermodynamic content is that a classical black hole is just black. It cannot radiate and therefore one should assign temperature zero to it, so that the interpretation of the surface gravity as temperature has no physical content.

This changes dramatically when taking into account quantum effects [11]. One can analyse black holes in the context of quantum field theory in curved backgrounds, where matter is described by quantum field theory while gravity enters as a classical background. In this framework it was discovered that black holes can emit radiation, called Hawking radiation, and the spectrum is Planckian with a temperature, the so-called Hawking temperature, which is indeed proportional to the surface gravity,

\[ T_H = \frac{k_s}{2\pi} \]  

(1.25)

This motivates one to take the laws of the black hole mechanics and the relation between area of the event horizon and the thermodynamic entropy more seriously. In fact one now interprets the laws of black hole mechanics as the laws of black hole thermodynamics. The universal expression for the thermodynamic entropy of the stationary black hole in Einstein’s theory of gravity, which we also got in (1.21), is called Bekenstein-Hawking entropy.

\[ S_{BH} = \frac{A}{4} \]  

(1.26)

However in general theory of gravity with arbitrary higher derivative terms, the expression for the entropy deviates from the Bekenstein-Hawking area law. The entropy also deviate from the area law when we take into account the quantum gravity effects.

### 1.3 Black Holes in String Theory

String theory is a quantum theory of relativistic strings. The low energy limit of string theory gives rise to (super-)gravity coupled to other fields. As a result these theories typically have black hole solutions\(^5\). At present the string theory is the most prominent candidate for a quantum theory of gravity. Thus it is natural to expect that the string theory is a framework for studying classical and quantum properties of black holes.

\(^5\)For nice review on the black holes in string theory look at [12, 13].
In most physical systems, the thermodynamic entropy has a statistical description in terms of microstates that correspond to the same macrostate. Thus one can ask: in string theory can we understand the thermodynamic entropy (1.26) from a statistical viewpoint, i.e., as the logarithm of the number of quantum states associated with the black hole? Although one does not yet have a complete answer to this question, for a special class of black holes in string theory, known as extremal black holes, this question has been answered in the affirmative. These black holes have zero temperature and hence do not radiate and are usually stable. Technically, an extremal black hole is one which has near horizon geometry of the form \( AdS_2 \times K \), for some compact space \( K \). Often, but not always, extremal black holes are also invariant under certain numbers of supersymmetry transformations. In that case they are called BPS black holes. In string theory, the BPS black holes can be realised in terms of various configurations of solitonic objects like D-branes, fundamental strings, etc. The charges carried by the black hole are realised in terms of quantum numbers, like number of D-brane, Kaluza-Klein momenta, winding number, etc., carried by the configuration of solitonic objects. This in turn allows us to calculate the degeneracy of such states at weak coupling where gravitational backreaction of the system can be ignored. Supersymmetry allows us to continue the result to strong coupling where gravitational backreaction becomes important and the system can be described as a black hole. In string theory, one finds that for a wide class of extremal BPS black holes, we have, in the limit of large charges,

\[
S_{BH}(Q) = S_{micro}(Q)
\]

where \( S_{micro} \) is defined as

\[
S_{micro}(Q) = \ln d_{micro}(Q)
\]

where \( d_{micro}(Q) \) is the degeneracy of BPS states in the theory carrying the same set of charges \( Q \).

The initial comparison between \( S_{BH} \) and \( S_{micro} \) was carried out in the limit of large charges. Typically in this limit the horizon size is large so that the curvature and other field strengths at the horizon are small and hence we can calculate the entropy via (1.26) without worrying about the higher derivative corrections to the effective action of string theory. In this limit, \( S_{micro} \) also get simplified and often it is given in terms of the degeneracy of a state in \((1+1)\) dimensional CFT with the spatial coordinate compactified on a circle \([14, 15]\). In this case the BPS black hole with large charges corresponds to a state in the CFT with large \( L_0 \) (or \( \bar{L}_0 \)) eigenvalue \( h_L \) (or \( h_R \)) and zero \( \bar{L}_0 \) (or \( L_0 \)) eigenvalue. The degeneracy of such states (e.g., for \( h_R = 0 \) case) are given in terms of Cardy formula

\[
d_{micro} \simeq \exp[2\pi \sqrt{\frac{c_L h_L}{6}}]
\]

where \( c_L \) is the left moving central charge.

One of the success of the string theory is that these two completely different computations, one for \( S_{BH}(Q) \) from gravity side and the other for \( S_{micro}(Q) \) from CFT side, give
the same answer. Given this success, it is natural to carry out this comparison to finer
details. When we move away from the large charge limit, the curvature and other field
strengths at the horizon are no longer negligible. In string theory one finds that the low
energy effective action contains higher curvature terms. In fact at tree level it contains an
Einstein-Hilbert term together with an infinite series of higher curvature terms that are
suppressed by powers of $\alpha'$, so that they are subleading at low energy. Thus, string theory
deviates from Einstein gravity already at the classical level. On top of these terms the
effective action also gets contribution from string loop corrections which involves powers
of string coupling $g_s$. For a large but finite size black hole we expect the effect of these cor-
trections at the horizon will be small but non-zero, giving rise to small modifications of the
black hole entropy. On the other hand for finite but large charges the statistical entropy
computed from the Cardy formula will also receive corrections which are suppressed by
inverse powers of charges. Thus it would be natural to ask: Does the agreement between
$S_{BH}(Q)$ and $S_{micro}(Q)$ continue to hold even after taking into account the effects of higher
derivative corrections on the black hole side, and deviation from the Cardy formula on
the statistical side?

In order to address this problem we need to open two fronts. First of all we need to
compute the degeneracy of states of black holes to greater accuracy so that we can com-
pute corrections to $S_{micro}$ given in (1.28). Conceptually this is a straightforward problem
since $d_{micro}$ is a well defined number, especially in the case of BPS extremal black holes,
since the BPS property gives a clean separation between the spectrum of BPS and non-
BPS states. Technically, counting of $d_{micro}$ is a challenging problem, although this has
now been achieved for a class of black holes in $N = 4$ supersymmetric string theories
[16, 17, 18]. Significant progress has also been made for half BPS black holes in a class
of $N = 2$ supersymmetric string theories [19]. The other front involves understanding
how higher derivative corrections / string loop corrections affect the black hole entropy.
Wald’s formalism [6, 7] gives a clear prescription for calculating the effect of tree level
higher derivative corrections on the black hole entropy, and for extremal black holes this
leads to the entropy function formalism. Thus here there is no conceptual problem, but in
order to implement it we need to know the higher derivative terms in the action. Inclusion
of quantum corrections into the computation of black hole entropy is more challenging
both conceptually and technically.

1.4 AdS/CFT Correspondence

According to [20, 21, 22] string theory on $AdS_{d+1}$ times a compact space is dual to a $d$
dimensional conformal field theory ($CFT$) living on the boundary of $AdS_{d+1}$. According
to this correspondence, in the approximation where the bulk fields are treated classically,
the boundary fields parameterizing the boundary conditions of the bulk fields are identified
with the source for the dual operators and the classical supergravity partition function
acts as a generating function for the correlation function of the dual operator in the boundary CFT.

In the present thesis we will concentrate on $d = 1$ and $2$ case. The $d = 1$ case will be dealt in detail in the next section. For $d = 2$ the AdS/CFT correspondence implies an equivalence between string theory on $\text{AdS}_3$ and $\text{CFT}_2$ living at the boundary of $\text{AdS}_3$. $\text{AdS}_3$ is a homogeneous space of constant negative curvature and its isometry group is $SL(2, \mathbb{C}) \cong SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. In fact a long time ago Brown and Henneaux [23] made a remarkable observation that the asymptotic symmetry group of $\text{AdS}_3$ is generated by two copies of Virasoro algebra (whose global part is $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$) and therefore the states in the Hilbert space of quantum theory of gravity on $\text{AdS}_3$ must be in the representation of these Virasoro algebra and hence it must be a conformal field theory. They also computed the value of the central charge as $c_L = c_R = \frac{3}{2\sqrt{\Lambda}}$ where $\Lambda$ is the cosmological constant. The vacuum of this CFT with $L_0 = \bar{L}_0 = 0$ correspond to empty $\text{AdS}_3$ and is invariant under only global part of the Virasoro algebra.

In three dimension the Weyl tensor vanishes and hence Riemann tensor can be expressed in terms of metric, Ricci tensor and Ricci scalar. As a result in case of pure gravity there are no gravitational waves (however it becomes nontrivial in presence of matter fields and higher derivative terms) but there are a black hole solutions, discovered by Bañados, Teitelboim, and Zanelli [24]. BTZ solution describe a rotating black hole solution and often appears as a factor in the near horizon geometry of higher dimensional black holes in string theory. For some extremal black holes in string theory the $\text{AdS}_2$ component of the near horizon geometry together with the an internal circle describes a locally $\text{AdS}_3$ space. Specifically the near horizon geometry of these extremal black hole correspond to that of extremal BTZ black hole. BTZ black hole corresponds to states in the $\text{CFT}$ and has non zero $L_0$ and $\bar{L}_0$ which is determined in terms it’s mass and angular momentum. The degeneracy of such states with large $L_0$ and $\bar{L}_0$ is given in terms of cardy formula and hence one can compute the entropy of such black hole.

1.5 Quantum Entropy of Extremal Black Hole

Wald’s formula [6, 7] computes the entropy of a non-extremal black hole in a classical theory of gravity with arbitrary higher derivative terms coupled to arbitrary set of matter fields. It gives entropy in terms of field configurations near the horizon of the black hole. For extremal black holes, defined as the limit of a non-extremal black holes, Wald’s formula reduces to a simple algorithm known as the entropy function formalism. We shall review this formalism below. Since the main ingredient of this analysis is the appearance of an $\text{AdS}_2$ factor in the near horizon geometry of extremal black holes, we shall begin by describing the origin of $\text{AdS}_2$ factor.
As an example consider a \((3 + 1)\)-dim. Reissner-Nordstrom black hole. It is given by
\[
\begin{align*}
\text{ds}^2 &= -(1 - a/\rho)(1 - b/\rho)\text{d}\tau^2 + \frac{d\rho^2}{(1 - a/\rho)(1 - b/\rho)} + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) \\
\end{align*}
\]
(1.30)

Here \((\tau, \rho, \theta, \phi)\) are the coordinates of space-time and \(a\) and \(b\) \((a > b)\) are two parameters labelling the position of the outer and inner horizon of the black hole respectively. The extremal limit corresponds to \(b \to a\). We take this limit keeping the coordinates \(\theta, \phi\) and \(r = 2(\rho - a + b^2)/(a - b)\), \(t = (a - b)\tau/2a^2\) \((1.31)\) fixed.

In this limit the metric takes the form
\[
\begin{align*}
\text{ds}^2 &= a^2 \left( -(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1} \right) + a^2(d\theta^2 + \sin^2\theta d\phi^2) \\
\end{align*}
\]
(1.32)

This is the metric of \(AdS_2 \times S^2\), with \(AdS_2\) labelled by \((r, t)\) and \(S^2\) labelled by \((\theta, \phi)\). As we mentioned, all known extremal black hole solutions have an \(AdS_2\) factor in their near horizon geometry; furthermore the other near horizon field configurations remain invariant under the \(SO(2,1)\) isometry of \(AdS_2\). We shall take this as the definition of extremal black hole even in presence of higher derivative terms.

One can give a uniform treatment of all such extremal black holes by regarding the angular directions as part of compact coordinates. Thus we have an effective two dimensional theory of gravity coupled to (infinite number of) other fields obtained by compactifying the fundamental theory on the compact directions. Among them of particular importance are the set of massless fields like abelian gauge fields \(A^{(i)}_\mu\) and a set of neutral scalar fields \(\phi_s\). Let \(L_0\) be the classical Lagrangian density and \(A\) be the classical action describing the dynamics of these massless fields
\[
A[g_{\mu\nu}, \{A^{(i)}_\mu\}, \{\phi_s\}] = \int d^2x \sqrt{-g}L_0 
\]
(1.33)

The most general near horizon geometry consistent with the \(SO(2,1)\) isometries of \(AdS_2\) is given by
\[
\begin{align*}
\text{ds}^2 &= v \left( -(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1} \right), \quad F^{(i)}_{rt} = e^i, \quad \phi_s = u_s \\
\end{align*}
\]
(1.34)

where \(F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu\) are the gauge field strengths, \(v, \{e^i\}\) and \(\{u_s\}\) are constants. Note that there are no parameter explicitly labelling the magnetic charges; they are encoded in the component of the gauge field strengths along the compact directions and appear as the parameters labelling the two dimensional theory. Let us denote by \(f(\bar{u}, v, \bar{v})\) the Lagrangian density \(\sqrt{-g}L_0\) evaluated for the near horizon geometry \((1.34)\)
\[
f(\bar{u}, v, \bar{v}) = \sqrt{-g}L_0 = vL_0 
\]
(1.35)
1.5. QUANTUM ENTROPY OF EXTREMAL BLACK HOLE

Then the classical black hole entropy is given by

$$S_{BH} = 2\pi(e^i q_i - f(\vec{u}, v, \vec{e}))$$  \hspace{1cm} (1.36)

evaluated at

$$\frac{\partial f}{\partial u_s} = 0, \quad \frac{\partial f}{\partial v} = 0, \quad \frac{\partial f}{\partial e^i} = q_i$$  \hspace{1cm} (1.37)

The function $2\pi(e^i q_i - f(\vec{u}, v, \vec{e}))$ is called the classical entropy function.

We now make an analytic continuation of the solution (1.34) by defining a new coordinate

$$t = -i\theta, \quad r = \cosh \eta$$  \hspace{1cm} (1.38)

In these coordinates the solution (1.34) takes the form

$$ds^2 = v(d\eta^2 + \sinh^2 \eta d\theta^2), \quad \phi_s = u_s, \quad F_{\theta_0}^i = ie^i \sinh \eta$$  \hspace{1cm} (1.39)

Also the action (1.33) becomes

$$A = -i \int d\eta d\theta \sqrt{g} F_0$$  \hspace{1cm} (1.40)

In this coordinate the $AdS_2$ is a disk with boundary at $\eta = \infty$. We can regulate the volume of the $AdS_2$ by putting an upper cut-off $\eta_{\text{max}}$ on $\eta$. The metric in non-singular at the origin $\eta = 0$ if we choose $\theta$ to have period $2\pi$.

Integrating the field strength we can get the form of the gauge field (in $A_\eta^{(i)} = 0$ gauge)

$$A_\theta^{(i)} = -ie^i (\cosh \eta - 1)$$  \hspace{1cm} (1.41)

The $(-1)$-factor inside the parenthesis is required in order to make the gauge field non-singular at the origin $\eta = 0$.

We can formally define the partition function of string theory in this background (1.39) as the path integral over all the string fields. In order to properly define this path integral we need to fix the boundary condition on various fields at $\eta = \eta_{\text{max}}$. Special care is needed to fix the boundary condition on the gauge fields. In $AdS_{d+1}$ for general $d$ the classical Maxwell equations for a gauge field near the boundary has two independent solutions. One of these represent the constant mode of the asymptotic gauge field and the other one measures the asymptotic electric field or equivalently the charge carried by the solution. Requiring the absence of singularity in the interior of $AdS_{d+1}$ gives a relation between the two coefficients. Thus in defining the path integral over $AdS_{d+1}$ we fix one of the coefficients and allow the other one to fluctuate. For $d \geq 3$ the constant mode of the gauge field is dominant near the boundary; hence it is natural to fix this and allow the mode measuring the charge to be determined dynamically in the classical limit and to fluctuate in the full quantum theory. However for $d = 1$ the mode that
measures charge is the dominant one near the boundary; thus it is more natural to think of this as a parameter of the boundary CFT and let the constant mode of the gauge field be determined dynamically. This can be seen for example in (1.41) where the term proportional to \cosh \eta measures the charge and the constant term in the expression for the gauge field is determined in terms of the electric field by requiring the gauge fields to be non-singular at the origin. Thus a more natural definition of the partition function on AdS$_2$ will be to fix the coefficient of the linear term in \cosh \eta and allow the constant term to fluctuate. In general this is achieved by requiring that

$$\lim_{r \to \infty} \frac{\delta A_{\text{bulk}}}{\delta F_{r\theta}^{(i)}} = iq_i,$$

Here $A_{\text{bulk}}$ is the bulk part of the action $iA$ and $r = \cosh \eta$. In this case under infinitesimal variation of the gauge field component $A_\theta^{(i)}$ we have

$$\delta A_{\text{bulk}} = (\text{e.o.m}) + iq_i \oint d\theta \delta A_\theta^{(i)}$$

In order to cancel this boundary term, we need to add a boundary term $-iq_i \oint d\theta A_\theta^{(i)}$ into the action $iA$. Similarly boundary conditions on the other fields are fixed in the standard manner, e.g. in the $g_{\eta\eta} = v$, $g_{\theta\eta} = 0$ gauge we freeze the mode of $g_{\theta\theta}$, proportional to $e^{2\eta}$ near the boundary, to $\frac{v}{4}$ and allow the mode independent of $\eta$ to fluctuate. Appropriate boundary terms must be added to the action so that the variation of the action under arbitrary variation of the various fields, subject to the boundary conditions, vanishes when equations of motion are satisfied. With these suitable boundary conditions we introduce the quantity

$$Z_{\text{AdS}_2} = \left\langle \exp \left[ -iq_i \oint d\theta A_\theta^{(i)} \right] \right\rangle_{\text{AdS}_2}$$

where $<>_{\text{AdS}_2}$ denotes the unnormalised path integral over various fields on AdS$_2$ with weight factor $e^{iA}$. Let us evaluate the partition function (1.44) in the classical limit.

On-shell the bulk part of the action is given by

$$A_{\text{bulk}} = -i \int d\eta d\theta \sqrt{g} \mathcal{L}_0 = -2\pi iv \int_{0}^{\eta_{\text{max}}} d\eta \sinh \eta \mathcal{L}_0 = -2\pi iv (\cosh \eta_{\text{max}} - 1) \mathcal{L}_0 = i(\cosh \eta_{\text{max}} - 1)(S_{\text{BH}} - 2\pi e^\eta g_i)$$

Here we have used (1.36). This is however not the complete contribution to $A$; we can get additional contribution from the boundary terms at $\eta = \eta_{\text{max}}$. To determine the form of the boundary contribution, we make a change of coordinates

$$\tilde{\eta} = \eta_{\text{max}} - \eta, \quad \tilde{\theta} = \frac{1}{2} e^{\eta_{\text{max}}} \theta$$

(1.46)
In this coordinate the boundary has period
\[ \beta = \pi e^{\eta_{\text{max}}} \]  
(1.47)

Furthermore the metric and the gauge field strengths near the boundary take the form
\[ ds^2 = v[d\eta^2 + e^{-2\eta}d\theta^2] + \mathcal{O}(\beta^{-2}) \]
\[ F_{\tilde{\eta}\tilde{\theta}} = ie^i e^{-\eta} + \mathcal{O}(\beta^{-2}) \]  
(1.48)

And the Wilson loop at the boundary becomes
\[ -iq_e \oint d\theta A^{(i)} = 2\pi q_i - \beta q_i + \mathcal{O}(\beta^{-1}) \]  
(1.49)

Now the boundary term in the action is given by some local expression constructed from
the metric, the gauge field strength and their derivatives integrated along the boundary.
Due to translation symmetry along \( \tilde{\theta} \), the integration along the boundary gives a factor of \( \beta \) multiplying the integrand. On the other hand the form of the solution given in (1.48)
shows that the integrand is given by a \( \beta \)-independent term plus a contribution of order \( \beta^{-2} \). Thus up to correction terms of order \( \beta^{-1} \), the boundary contribution must be
proportional to the length \( \beta \) of the boundary circle. Together with (1.45), the complete
classical partition function (1.44) is
\[ Z_{\text{AdS}_2} = \exp\left[ S_{\text{BH}} + \beta K(q) + \mathcal{O}(\beta^{-1}) \right] \]  
(1.50)

Since the term \( K(q) \) contains the boundary terms and hence ambiguous but unambiguous
and the finite part of the partition function is given by
\[ Z_{\text{AdS}_2}^{\text{finite}} = e^{S_{\text{BH}}} \]  
(1.51)

Thus in the classical limit the finite part of the partition function (1.44) reduces to expo-
nential of Wald entropy.

In full quantum theory we hope to represent the effect of this path integral by a modifi-
cation of the original Lagrangian density to an appropriate effective Lagrangian density.
In flat spacetime the one particle irreducible action is non-local and hence causes an ob-
struction to express the action as an integral over a local Lagrangian density. However
the situation in \( \text{AdS}_2 \) background is better since the non vanishing curvature of \( \text{AdS}_2 \)
puts a natural infrared cut-off. Thus the quantum \( Z_{\text{AdS}_2} \) is expected to be given by an
expression similar to (1.50) but with quantum corrected entropy (determined in terms of
effective Lagrangian density).

We will now give an interpretation of the entropy \( S_{\text{BH}} \) appearing in (1.50) in terms of an
appropriate conformal quantum mechanics living at the boundary of \( \text{AdS}_2 \). Since \( Z_{\text{AdS}_2} \) is
the partition function of the theory on \( \text{AdS}_2 \), according to \( \text{AdS/CFT} \) correspondence\[20, 21, 22\] one expects this to be the partition function of the dual quantum mechanics living
at the boundary $\eta = \eta_{\text{max}}$. Thus if $H$ denotes the Hamiltonian generating $\tilde{\theta}$ translation in the dual quantum mechanics living at $\eta = \eta_{\text{max}}$ then, according to $\text{AdS/CFT}$ correspondence,

$$Z_{\text{AdS}_2} = Tr \left( e^{-\beta H} \right) \quad (1.52)$$

In the $\eta_{\text{max}} \to \infty$ which corresponds to $\beta \to \infty$ limit, the right hand side of (1.52) will get dominant contribution from the ground state. If $d(q)$ is the degeneracy of the ground states, then in this limit we get

$$Z_{\text{AdS}_2} = e^{-\beta E} d(q) \quad (1.53)$$

Here $E$ is the ground state energy.

Comparing with (1.50) and identifying $E = -K(q)$.

$$d(q) = e^{S_{BH}(q)} = \left\langle \exp \left[ -i q_i \oint d\theta A^{(i)}_\theta \right] \right\rangle_{\text{finite } \text{AdS}_2}, \quad (1.54)$$

The right hand side of (1.54) is called quantum entropy function.

In case of supersymmetric extremal black hole one expects to make a precise comparison between macroscopic and microscopic entropy. Let $d_{\text{micro}}(\vec{q})$ denotes the degeneracy of the BPS microstates carrying total charge $\vec{q}$ in string theory. Then on general grounds one expects the following relation between $d_{\text{micro}}(\vec{q})$ and the macroscopic quantities associated with the black hole:

$$d_{\text{micro}}(\vec{q}) = d_{\text{macro}}(\vec{q}) \quad (1.55)$$

where

$$d_{\text{macro}}(\vec{q}) = \sum_n \sum_{\sum_{i=1}^n \vec{q}_i = \vec{q} \text{ and } \vec{q}_{\text{hair}}} \left\{ \prod_{i=1}^n d_{\text{hor}}(\vec{q}_i) \right\} d_{\text{hair}}(\vec{q}_{\text{hair}}; \{\vec{q}_i\}), \quad (1.56)$$

The $n$-th term on the right hand side of (1.56) represents the contribution to the degeneracy from $n$-centered black hole configuration and $d_{\text{hor}}(\vec{q}_{\text{hair}}; \{\vec{q}_i\})$ is the degeneracy of the hair degrees of freedom [26]. $d_{\text{hor}}(\vec{q}_i)$ is the degeneracy associated with the horizon degrees of freedom given by (1.54).

One issue which will be not dealt here in detail but need to mention is related to infrared divergence and it’s regularisation. In proving that the quantum entropy function (1.54) in classical limit reduces to exponential of Wald entropy we have used, in order to regularise the infrared divergence arises because of infinite volume of $\text{AdS}_2$, a specific infrared cutoff in which we put $\eta = \eta_{\text{max}}$. However in [25] it was shown that the right hand side of (1.54) is insensitive to any choice of infrared cutoff in full quantum theory.
1.6 Plan of the report

This thesis is divided into two parts. In the first part we describe the field redefinition and consistent truncation in general three dimensional theory of (super-)gravity and compute the classical correction to BTZ black hole entropy. In the second part we describe the quantum correction to entropy of extremal black hole. The plan of the chapters are as follows:

As we mentioned for some extremal black hole the near horizon geometry correspond to that of extremal BTZ black hole. The entropy of the BTZ black hole are given in terms of central charges. In chapter 2 we describe field redefinition and consistent truncation in three dimensional general higher derivative theory of (super-)gravity coupled to arbitrary set of matter fields. After field redefinition and consistent truncation the action reduces to standard (super-)gravity action which is sum of three terms, Einstein-Hilbert term, a cosmological constant term and the Chern-Simons term. We describe the procedure of computing the parameters of the final truncated theory. We apply these procedure in case of 3-dimensional theory obtained by dimensional reduction of 5-dimensional four derivative theory of $\mathcal{N}=2$ supergravity. We then compute cosmological constant and the central charges of the boundary CFT which will give higher derivative correction to entropy of the BTZ black hole.

In chapter 3 we describe a check of quantum entropy function proposal. We consider extremal supersymmetric charged BTZ black hole in 3-dimension for which there exist independent definition of entropy based on $AdS_3/CFT_2$ correspondence. We compare this definition of entropy with that of quantum entropy function proposal. We will show that these two different definition of entropy agrees.

In chapter 4 we tried to find the field configuration which could give non vanishing contribution to quantum entropy function. Using supersymmetry and localization techniques we show that the path integral could receive non-vanishing contribution only from a special class of field configurations which preserve a particular subgroup (we call it $H_1$) of $SU(1,1|2)$. We identify this subgroup and showed that path integral around such field configuration reduces to integration over $H_1$ invariant slice passing through the field configuration. This analysis is useful to find the saddle points, i.e. classical string field configuration which could give non-perturbative corrections to the quantum entropy. We also find out some examples of such field configurations.