5.1 Introduction

The concept of oriented coloring was introduced by Bruno Courcelle in [Cou94]. Since then, many researchers have worked on the problem, partly because of its applications in task assignment problems [CD06].

Sopena, in ([Sop97]), studied the notion of oriented chromatic number for oriented graphs. Recall that the oriented chromatic number of an oriented graph $\bar{G}$ is the minimum number of vertices of an oriented graph $\bar{H}$ such that there is a homomorphism from $\bar{G}$ to $\bar{H}$. The oriented chromatic number of $\bar{G}$ is denoted by $\chi_o(\bar{G})$ and the oriented chromatic number of an undirected graph $G$, denoted by $\chi_o(G)$, is the maximum oriented chromatic number of $\bar{G}$ taken over all orientations $\bar{G}$ of $G$. Upper bounds for the oriented chromatic number have been obtained in terms of the maximum degree and also for special families of graphs such as trees, planar graphs, partial $k$-trees [Sop97], for triangle-free planar graphs [Och04], for 2-outerplanar graphs [EO07], for arbitrary graphs in terms of maximum degree [KSZ97], maximum average degree [BKN+99] and in terms of treewidth [Sop97]. The following two results, in particular, are relevant to the main results of this chapter. They are:

(B1) The result of Sopena in [Sop97] that, for every $r \geq 1$, every partial $r$-tree has oriented chromatic number at most $(r + 1)2^r$.

(B2) The result of Raspaud and Sopena in [RS94] that if a graph has acyclic chromatic number at most $k$, then $\chi_o(G) \leq k2^{k-1}$. 
5.1.1 Our Results

We generalize the result (B2) by obtaining a relationship connecting the oriented chromatic number $\chi_o(G)$ of graphs and the $(j, \mathcal{F})$-subgraph chromatic numbers $\chi_{j, \mathcal{F}}(G)$ introduced and studied in Chapters 3 and 4. In particular, we relate the oriented chromatic number and the $(2, r)$-treewidth chromatic number and show that $\chi_o(G) \leq k((r + 1)2^r)^{k-1}$ for any graph $G$ having $(2, r)$-treewidth chromatic number at most $k$. We recall that the latter parameter is the least number of colors in any proper vertex coloring which is such that the subgraph induced by the union of any two color classes has treewidth at most $r$.

We also generalize a result of Alon, Mohar and Sanders [AMS96] on the acyclic chromatic number of graphs on surfaces to $(2, \mathcal{F})$-subgraph chromatic numbers. For certain families $\mathcal{F}$, we prove that $\chi_{2, \mathcal{F}}(G) = O(\gamma^{m/(2m-1)})$ for any graph $G$ of Euler characteristic $-\gamma$, where $\gamma \geq 0$. Here, $m = \min\{|E(H)| : H \in \mathcal{F}\}$. We also show that this bound is nearly tight. We then use this result to show that graphs of genus $g$ have oriented chromatic number at most $2^{O(g^{1/2+r})}$ for every fixed $\epsilon > 0$. This improves the currently best known bound of $2^{O(g^{1/7})}$ which follows from the result of [AMR91] (see subsection 5.1.4). We also refine the proof of a bound on $\chi_o(G)$ (in terms of maximum degree) obtained by Kostochka, Sopena and Zhu in [KSZ97] to obtain an improved bound on $\chi_o(G)$. In the following subsections of this section, we present the formal statements (without proofs) of the main results of this chapter.

5.1.2 Relating $\chi_{j, \mathcal{F}}(G)$ and $\chi_o(G)$

In this subsection, we state the following connection between $(j, \mathcal{F})$-subgraph colorings and oriented colorings. This result generalizes and was inspired by the connection between $a(G)$ and $\chi_o(G)$ established in [RS94]. Recall that for a family $\mathcal{F}$ of connected graphs, $Forb(\mathcal{F}) = \{G : G \text{ is } \mathcal{F} \text{ - free}\}$.

**Theorem 5.1** Let $\mathcal{F}$ be a family of connected graphs. Suppose there exists a natural number $t$ such that $\chi_o(F) \leq t$, for each $F \in Forb(\mathcal{F})$. Suppose $j \geq 2$. Then, for any graph $G \notin Forb(\mathcal{F})$ with $\chi_{j, \mathcal{F}}(G) \leq k$, its oriented chromatic number $\chi_o(G)$ is at most $kt^{\left\lfloor \frac{2k}{j-1} \right\rfloor}$ if $j$ is even and is at most $kt^{\left\lfloor \frac{2k}{j+1} \right\rfloor}$ if $j$ is odd.
In Section 5.2, we prove this theorem. By specializing to $j = 2$, we get the following theorem. This specialization is stated separately again since it plays an important role in other results of this chapter.

**Theorem 5.2** Let $\mathcal{F}$ be a family of connected bipartite graphs. Suppose there exists a $t$ such that $\chi_o(F) \leq t$, for each $F \in \text{Forb}(\mathcal{F})$. Then, for any graph $G \notin \text{Forb}(\mathcal{F})$ with $\chi_{2,\mathcal{F}}(G) \leq k$, its oriented chromatic number $\chi_o(G)$ is at most $kt^{k-1}$.

We now specialize Theorem 5.2 by choosing $\mathcal{F}$ to be the set of all connected bipartite graphs of treewidth $r + 1$ and apply the bound (B1) (mentioned before) on the oriented chromatic number of partial $r$-trees to obtain the following result as a consequence.

**Corollary 5.3** For $r \geq 1$, let $G$ be any graph with a $(2, r)$-treewidth chromatic number at most $k$. Then $G$ has oriented chromatic number at most $k((r + 1)2^r)^k$. 

**5.1.3 (2, $\mathcal{F}$)-subgraph colorings of graphs on surfaces**

It is known from the Map Color Theorem of Ringel and Youngs [RY68] that the chromatic number of an arbitrary surface of Euler characteristic $-\gamma$ is $\Theta(\gamma^{1/2})$. Using the upper bound of $O(\Delta^{4/3})$ bound on $a(\Delta)$, Alon, Mohar and Sanders proved in [AMS96] that the acyclic chromatic number of a (simple) graph embeddable on a surface of characteristic $-\gamma (\leq 0)$ is at most $100\gamma^{3/2} + 10^4$. It was also shown that this bound is nearly tight.

Generalizing these arguments and by using the bound of Theorem 5.1, we prove that this result can be extended to $(2, \mathcal{F})$-colorings as well provided that $\mathcal{F}$ does not contain connected graphs with pendant vertices. Our next main result in this chapter is this extension. Specifically, we prove (using essentially the arguments of [AMS96]) the following statement.

**Theorem 5.4** Let $\mathcal{F}$ be a family of connected bipartite graphs on at least 4 vertices each having minimum degree at least 2. Let $m$ be the smallest number of edges of any member of $\mathcal{F}$. If $G$ is a (simple) graph embeddable on a surface of Euler characteristic $-\gamma \leq 0$, then $\chi_{2,\mathcal{F}}(G) \leq A\gamma^{m-1} + B$ where $A$ and $B$ are constants depending only on $\mathcal{F}$. 

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When $\mathcal{F} = \{C_4, C_6, \ldots\}$ corresponding to the acyclic chromatic number, we have $m = 4$ and $m/(2m - 1) = 4/7$ and the result is consistent with the bound of [AMS96]. By choosing $\mathcal{F} = \mathcal{F}_r$ where $\mathcal{F}_r$ is the set of all minimal connected bipartite graphs of treewidth $r+1$, we get the following consequence of Theorem 5.4.

**Corollary 5.5** If $G$ is a simple graph embeddable on a surface of Euler characteristic $-\gamma \leq 0$, then, $\chi_{2,r}^\text{lw}(G) \leq A\gamma^{\frac{m}{2m-1}} + B$ for every $r \geq 1$. Here, $A$ and $B$ are suitable absolute positive constants and $m_r$ denotes the minimum number of edges in any member of $\mathcal{F}_r$.

We also establish that the upper bound of Theorem 5.4 is tight up to a $\text{polylog}(\gamma)$ multiplicative factor. This generalizes a similar tightness result presented in [AMS96] for acyclic chromatic numbers.

**Theorem 5.6** Let $\mathcal{F}$ and $m$ be as described in Theorem 5.4. For every sufficiently large $\gamma \geq 0$, there is a graph $G$ embeddable on a surface (orientable or non-orientable) with Euler characteristic $-\gamma$ such that $\chi_{2,r}(G) \geq cr\gamma^{\frac{m}{2m-1}}/(\log \gamma)^{1/(2m-1)}$ for some positive constant $c$ which depends only on $\mathcal{F}$.

### 5.1.4 Oriented chromatic numbers of graphs on surfaces

For graphs of Euler characteristic $-\gamma \leq 0$, by combining the upper bound of $O(\gamma^{4/7})$ on oriented chromatic number (obtained in [AMS96]) with the bound (B2) of [RS94] (mentioned before), we get an upper bound of $O(\gamma^{4/7}2^{O(\gamma^{4/7})}) = 2^{O(\gamma^{4/7})}$ for the oriented chromatic number $\chi_o(G)$. The next main result of this chapter is an improvement of this bound and is obtained by combining Corollary 5.3 and Corollary 5.5. Recall that Corollary 5.3 is a generalization of bound (B2) and Corollary 5.5 is a generalization of the bound obtained in [AMS96].

**Theorem 5.7** Let $r \geq 0$ be any fixed integer. There exists a positive constant $c_r$ and a positive integer $m_r$, both depending only on $r$, such that the following holds: For any simple graph $G$ embeddable on a surface of Euler characteristic $-\gamma \leq 0$,

$$\chi_o(G) \leq c_r(\gamma^{\frac{m_r}{2m_r-1}})((r + 1)2^\gamma)^{O(\gamma^{\frac{m_r}{2m_r-1}})} \leq 2^{O(\gamma^{m_r/(2m_r-1)})}$$

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Here, \( m_r = \min \{ E(H) : tw(H) > r \} \). It can be seen that \( m_r \geq r + 1 \), so that \( m_r \to \infty \). Thus for every \( \epsilon > 0 \), there exists \( c_\epsilon \) such that \( \chi_o(G) \leq 2^{c_r \gamma^{(1/2)+\epsilon}} \).

**Proof** Follows as a consequence of combining Corollary 5.3 and Corollary 5.5 with the bound (B1) (mentioned earlier).

Note that this significantly improves the bound \( 2^{O(\gamma^{4/7})} \) mentioned before.

### 5.1.5 An improved bound on the oriented chromatic number

In [KSZ97], Kostochka, Sopena and Zhu showed that the oriented chromatic number of any graph \( G \) of maximum degree \( \Delta \) is at most \( 2\Delta^2 2^{\Delta} \). They prove this result by showing (with the help of probabilistic arguments) the existence of a tournament on \( t = 2\Delta^2 2^{\Delta} \) vertices possessing a nice property which enables one to obtain an oriented coloring of any orientation of \( G \) with \( t \) colors.

We show that this proof can in fact be refined so that we obtain the following improvement of this result.

**Theorem 5.8** If \( G \) is any graph of maximum degree \( \Delta \) and degeneracy \( d \), then its oriented chromatic number \( \chi_o(G) \) is at most \( 16\Delta d 2^d \).

This replaces a factor \( \Delta^2 2^\Delta \) by \( d^2 2^d \) and will result in a better bound for those \( G \) having \( d \ll \Delta \).

### 5.1.6 Outline of this chapter

We prove Theorem 5.1 in Section 5.2. Theorems 5.4 and 5.6 are proved in Section 5.3. In Section 5.4, we prove Theorem 5.8. Finally, in Section 5.5, we conclude with some remarks and open problems.

### 5.2 Relating \( \chi_j, \mathcal{F}(G) \) and \( \chi_o(G) \)

We now prove Theorem 5.1 which relates oriented chromatic number and the forbidden subgraph colorings.
Proof of Theorem 5.1 Let $G = (V, E)$ be an undirected graph such that $G \not\in \text{Forb}(\mathcal{F})$ and let $\vec{G} = (V, A)$ be an arbitrary orientation of $E(G)$. Since $G \not\in \text{Forb}(\mathcal{F})$, we have $k \geq \chi_{j^{-}}(G) \geq j + 1$. Let $V_1, \ldots, V_k$ be the color classes of $V$ with respect to a $(j, \mathcal{F})$-subgraph coloring $c$ of $V(G)$ using $k$ colors. Let $T$ be the collection of subsets obtained by partitioning $[1, k]$ into at most $\left\lceil \frac{k}{j/2} \right\rceil$ subsets of size at most $\lfloor j/2 \rfloor$ each. Note that $|T|$ is at most $\left\lceil \frac{2k}{j} \right\rceil$ if $j$ is even and is at most $\left\lceil \frac{2k}{j-1} \right\rceil$ if $j$ is odd. Let $S$ be the collection defined by

$$S = \{T \cup T' : T, T' \in T, T \neq T'\}.$$  

It follows that

(i) Each $S \in S$ is a set of size at most $j$.

(ii) for every $l, m \in [1, k]$, there exists a $S \in S$ with $l, m \in S$.

(iii) for each $i \in [k]$, $i$ is a member of at most $\left\lceil \frac{k}{j/2} \right\rceil - 1$ sets in $S$. Let $S_i$ be defined by $S_i = \{S \in S : i \in S\}$.

For each $S \in S$, let $\vec{G}_S$ denote the induced subgraph $\vec{G}[\bigcup_{i \in S} V_i]$. Clearly $G_S \in \text{Forb}(\mathcal{F})$, since $(V_1, \ldots, V_k)$ is a $(j, \mathcal{F})$-subgraph coloring.

Let $c_S$ be an oriented coloring of $\vec{G}_S$ using at most $t$ colors.

Assume an ordering $\{S_1, S_2, \ldots\}$ on the members of $S$. We now define a new coloring $\phi$ of $V(G)$: Fix any $i$ and let $S_i = \{S_{i_1}, \ldots, S_{i_l}\}$ be the members of $S_i$ where we have $l \leq \left\lfloor \frac{k}{j/2} \right\rfloor - 1$. For each $v \in V_i$,

$$\phi(v) = \{c(v), (c_{S_{i_1}}(v), S_i), \ldots, (c_{S_{i_l}}(v), S_i)\}.$$ 

Clearly, $\phi$ is a proper coloring of $V(\vec{G})$ because of the component $c$. We now prove that it is an oriented coloring. If it is not an oriented coloring, then there are four vertices $x, y, z, t$ of $\vec{G}$ such that $(x, y) \in A$ and $(z, t) \in A$ with $\phi(x) = \phi(t)$ and $\phi(y) = \phi(z)$. By the definition of $\phi$, $x$ and $t$ (respectively $y$ and $z$) belong to the same $V_i$ (respectively $V_j$) where $i = c(x) = c(t)$ and $j = c(y) = c(z)$. Let $S$ be any set in $S$ containing $i$ and $j$ where $S \in S_i \cap S_j$ and $x, y, z, t \in V(\vec{G}_s)$. For each $u \in \{x, y, z, t\}$, the pair $(c_S(u), S) \in \phi(u)$. By the definition of $\phi$, we
have \( c_S(x) = c_S(t) \) and \( c_S(y) = c_S(z) \). But this contradicts the fact that \( c_S \) is an oriented coloring of \( \tilde{G}_S \).

The number of possible values of \( \phi(v) \) is at most \( kl^{(4\gamma-1)/v} \). This number is \( kl^{(4\gamma-1)/v} \) if \( j \) is even and is \( kl^{(4\gamma+1)/v} \) if \( j \) is odd. This proves Theorem 5.1.

5.3 \((2, \mathcal{F})\)-subgraph colorings of graphs on surfaces

By applying the bound of Theorem 4.7 which holds for general graphs, we obtain a bound on \( \chi_{2,\mathcal{F}}(G) \) for graphs embeddable on surfaces, provided the members of \( \mathcal{F} \) have minimum degree at least 2. This bound was stated in Theorem 5.4 and is proved in this section.

The proof is essentially the proof of [AMS96] extended to a more general setting. Hence, we do not provide the complete proof but only provide the sketch to give an idea of the proof.

5.3.1 Proof of Theorem 5.4

We follow the proof of [AMS96]. Assume the theorem is false for a surface \( S \) with Euler characteristic \( -\gamma \leq 0 \), and let \( G \) be a graph embeddable on it, with a minimum number of vertices, which is a minimal counterexample to the theorem. Let \( H \) be \( G \) with (possibly multiple) edges added to triangulate \( S \). Clearly \( \text{deg}_{G}(v) \leq \text{deg}_{H}(v) \) for all vertices \( v \) of \( G \). Suppose \( V(G) = V(H) = \{ v_1, \ldots, v_n \} \), where \( \text{deg}_{H}(v_1) \leq \text{deg}_{H}(v_2) \leq \ldots \leq \text{deg}_{H}(v_n) \). If \( \gamma = 0 \), define \( h_1 = 0 \) and \( h_2 = 0 \). Otherwise, define \( h_1 := \lceil c\gamma^{m/(m-1)} \rceil \) and \( h_2 := \lceil 6\gamma h_1 \rceil (\leq 6\gamma^{m/(m-1)} / c) \), where \( c \) is an absolute constant, to be chosen later. Let \( d := \text{deg}(v_{n-h_1}) \). The proof will split on the size of \( d \).

**Case 1:** \( d \leq (4/3)h_2 + 9 \). In this case, the induced subgraph of \( G \) on \( \{ v_1, \ldots, v_n \} \) has maximum degree at most \( d \), and thus has a \((2, \mathcal{F})\)-subgraph coloring using at most \( \lfloor C d^{m/(m-1)} \rfloor \) colors, by Theorem 5.2. Coloring the remaining vertices of \( G \) with \( h_1 \) new colors that have not been used before gives a \((2, \mathcal{F})\)-subgraph coloring of \( G \) with at most

\[
\lfloor C ((4/3)h_2 + 9)^{m/(m-1)} \rfloor + h_1 \leq C (8\gamma^{m/(2m-1)}/c + 9)^{m/(m-1)} + 1 + c\gamma^{m/(2m-1)} + 1
\]
colors. An appropriate choice of constant values (independent of \( \gamma \)) for \( A, B \) and \( c \) shows that this is smaller than \( A\gamma^{m/(2m-1)} + B \), implying that in this case \( G \) cannot be a counterexample.

**Case II:** \( d \geq (4/3)h_2 + (28/3) \). We charge each vertex as follows. Define \( \text{charge}'(v_i) = 6 - \deg_H(v_i) \) for \( 1 \leq i \leq n - h_1 \), and \( \text{charge}'(v_i) = -\deg_H(v_i)/4 \) for \( n - h_1 + 1 \leq i \leq n \).

As shown in [AMS96],

\[
\sum_{1 \leq i \leq n} \text{charge}'(v_i) = \left( \sum_{i \leq n-h_1} 6 - \deg_H(v_i) + \sum_{i > n-h_1} -\deg_H(v_i)/4 \right) > 0.
\]

Following [AMS96], we define new charges \( \text{charge}''(v) \) for each vertex by the following discharging rules. (i) Send a charge of \( 1/2 \) from each vertex of degree 4 to each of its neighbors of degree at least 8. (ii) Send a charge of \( 1/4 \) from each vertex of degree 5 to each of its neighbors of degree at least 7. The degrees are with respect to \( H \). By conservation of total charges, we have \( \sum_{i \leq n} \text{charge}''(v_i) > 0 \). Hence for some \( j \), we have \( \text{charge}''(v_j) > 0 \).

Using the definition of \( \text{charge}''(v_j) \), we see that \( \deg_H(v_j) \neq 6 \). Now consider the following cases:

**Case 1:** \( \deg_H(v_j) \leq 3 \). Then, \( \deg_G(v_j) \leq 3 \) and we delete \( v_j \) from \( G \) and join every pair of its neighbors by an edge (if it is not there) in the embedding of \( G - v_j \). Since \( G \) is a counterexample on minimum number of vertices, \( G - v_j \) is \( (2, \mathcal{F}) \)-colorable using the allowed number of colors where neighbors of \( v_j \) get different colors. Now we can extend this coloring by coloring \( v_j \) with any permissible color and it will continue to be a \( (2, \mathcal{F}) \)-coloring of \( G \) contradicting our assumption.

**Case 2:** \( \deg_H(v_j) = 4 \). In this case, \( v_j \) should have a neighbor \( v_k \) with \( \deg_H(v_j) \leq 7 \). Let \( K \) be the graph obtained by removing \( v_j \) and making every pair of neighbors other than \( v_k \) adjacent. From a \( (2, \mathcal{F}) \)-coloring of \( K \), we can obtain a \( (2, \mathcal{F}) \)-coloring of \( G \) by assigning \( v_j \) with any color not used on its neighbors or the neighbors of \( v_k \). This contradicts our assumption.
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Case 3: $\deg_H(v_j) = 5$. Now $\text{charge}(v_j) = 1$, thus $v_j$ must have two neighbors, say $v_k$ and $v_m$ of degree at most 6. Let $K$ be $G$ with $v_j$ deleted, and edges added so that the neighbors of $v_j$ in $G$ (except possibly $v_k$, $v_m$ are pairwise adjacent. Give $K$ a $(2, \mathcal{F})$-coloring by induction, this can be extended to $G$ by coloring $v_j$ with a color different form each of its neighbors as well as the neighbors of $v_k$ and $v_m$.

As shown in [AMS96], the other cases reduce to the three previous ones. This completes the proof.

Remark: For any graph $G$, $\chi_{2tw}(G) = \chi(G)$ when $r = tw(G)$. When $r$ becomes large, the bound of Corollary 5.5 approaches the Heawood bound of $O(g^{1/2})$ for the chromatic number of genus $g$ (fixed $g$) graphs. Hence, the upper bound of Corollary 5.5 approximates the Heawood bound more closely in the case of graphs of large treewidth.

5.3.2 Proof of Theorem 5.6

The proof is based on an approach similar to the one used in [AMS96]. It uses the following lemma whose proof follows from the proof of Theorem 4.1 presented in Chapter 4 of this thesis. The proof is based on analyzing a random graph $G(n, p)$ for a suitably chosen $p$.

Lemma 5.9 Let $\mathcal{F}$ and $m$ be as described in Theorem 5.6. Let $G = G(n, p)$ be the random graph on $\{1, \ldots, n\}$ where each potential edge is chosen independently with probability $p = c \left(\frac{\log n}{n}\right)^{1/m}$ for a suitable positive constant $c$ which depends only on $\mathcal{F}$. Then, almost surely, $G$ is connected and has at most $cn^{(2m-1)/m} (\log n)^{1/m}$ edges and satisfies $\chi_{2,\mathcal{F}}(G) = \Omega(n)$.

Let $G$ be a connected graph on at most $O(n^{(2m-1)/m} (\log n)^{1/m})$ edges and satisfying $\chi_{2,\mathcal{F}}(G) = \Omega(n)$ as guaranteed by Lemma 5.9. Let $G$ be embedded on a surface of characteristic $-\gamma$ for the smallest $\gamma \geq 0$ possible. Let $e = |E(G)|$. By an application of Euler's formula, one can show (as shown in [AMS96]) that $\gamma > n^{(2m-1)/m}$, and hence $\log \gamma > (2m - 1)(\log n)/m$ and also that $\gamma = O \left( n^{(2m-1)/m} (\log \gamma)^{1/m} \right)$. Hence, $\chi_{2,\mathcal{F}}(G) = \Omega(n) = \Omega \left( \gamma^{m/(2m-1)} / (\log \gamma)^{1/(2m-1)} \right)$. 

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5.4 Proof of Theorem 5.8

As in [KSZ97], we prove (using probabilistic arguments) the following lemma. Before that, we recall the following notation from [KSZ97]. For an oriented graph $G = (V, A)$ and a subset $I = \{x_1, \ldots, x_i\}$ of $V$ and a vertex $v \in V \setminus I$ such that $v$ is adjacent to each $x_j$, we use $F(I, v, G)$ to denote the vector $a = (a_1, \ldots, a_i)$ where, for each $j \leq i$, $a_j = 1$ if $(x_j, v) \in A$ and $a_j = -1$ if $(v, x_j) \in A$.

**Lemma 5.10** Let $d, k$ be positive integers with $d \leq k$ and $k \geq 5$. There exists a tournament $T = (V, A)$ on $t = 16kd2^d$ vertices with the following property:

For each $i, 0 \leq i \leq d$, for each $I \subseteq V$, $|I| = i$, and for each $a \in \{1, -1\}^i$, there exist at least $kd + 1$ vertices $v \in V \setminus I$ with $F(I, v, T) = a$.

**Proof of Lemma 5.10**: Consider a random tournament $T = (V, A)$ on $t$ vertices obtained by randomly and independently orienting each edge of $K_t$ (complete undirected graph on $t$ vertices) in one of the two directions with equal probability.

Fix an $i \leq d$ and fix any $I \subseteq V$ of size $i$. Also, fix a vector $a \in \{1, -1\}^i$. Define the random variable

$$X_{I,a} = |\{u \in V \setminus I : F(I, u, T) = a\}|.$$

It is easy to verify that $X_{I,a}$ is the sum of $t - i$ independent and identically distributed indicator random variables each having the common expectation $2^{-i}$. Hence it follows that

$$\mu_{I,a} = E(X_{I,a}) = (t - i)2^{-i} \geq (t - d)2^{d-d}.$$

Also, by the well-known Chernoff-Hoeffding bounds (see Chapter 4 of [MR95]), it also follows, using $k \geq 5$ and $d \geq 2$, that

$$\Pr(X_{I,a} \leq kd) = \Pr(X_{I,a} - \mu_{I,a} \leq kd - \mu_{I,a}) \leq e^{-\mu_{I,a}(1-kd/\mu_{I,a})^2/3} \leq e^{-\mu_{I,a}/4} \leq e^{-(3.75)kd}.$$
Hence, for the event $\mathcal{E}$ defined by $\mathcal{E} = \exists I, a : |I| \leq d, X_{I,a} \leq kd$, we have

$$\Pr(\mathcal{E}) \leq d \cdot \binom{t}{d} \cdot 2^d \cdot e^{-(3.75)kd} \leq e^{-d((3.75)k-\ln(2e) - \frac{\ln d}{d})} \leq e^{-d((3.75)k-\ln(2e) - \ln 16 - d(\ln 2) - \ln k)} < 1$$

where the last strict inequality uses the definition of $t$ and the assumption $k \geq 5$, $d \geq 2$. This shows that, with positive probability, there exists a tournament with desired properties, completing the proof of the lemma.

We now give the proof of Theorem 5.8 where we shall make use of Lemma 5.10.

**Proof of Theorem 5.8** Let $G = (V, E)$ be any graph of maximum degree $\Delta$ and degeneracy $d$. If $d \leq 1$, then $G$ is a forest and hence its $\chi_o(G) \leq 3$ as shown in [Sop97]. For $d \geq 2$ and $\Delta \leq 4$, the result follows from a bound of $(2\Delta - 1)2^{2\Delta - 2}$ derived in [Sop97]. Hence, we assume that $\Delta \geq 5$ and $d \geq 2$. Consider a linear ordering $(v_n, \ldots, v_1)$ of $V$ such that for each $i \leq n$, $v_i$ has at most $d$ neighbors in the subgraph $G_i$ induced by $V_i = \{v_1, \ldots, v_i\}$. Let $T$ be the tournament on $t = 16kd2^d$ vertices specified in Lemma 5.10, with $k = \Delta$. Let $G'$ be any orientation of $G$. We inductively color vertices of $G'$ in the order $(1, \ldots, n)$ in such a way that after the coloration of the first $m$ vertices:

1. the partial coloring $f(v_1), \ldots, f(v_m)$ is a valid oriented coloring of $G'_m$ using vertices of $T$;

2. for each $v_j$ with $j > m$, all neighbors of $v_j$ in $V_m$ are colored with distinct colors.

Now, we need to color $v_{m+1}$ so that (1) and (2) hold for $f(v_{m+1})$ as well. For this, let $\{y_1, \ldots, y_i\} \subseteq V_m$ be the neighbors of $v_{m+1}$ in $V_m$ each colored with distinct colors (because of (2)) from $I = \{f(y_1), \ldots, f(y_i)\}$. Note that $i \leq d$. Let $a = F(\{y_1, \ldots, y_i\}, v_{m+1}, G'_{m+1})$. Let $K = \{w \in V(T) \setminus I : F(I, w, T) = a\}$. By Lemma 5.10, we know that $|K| \geq kd + 1$. Now, there can be at most $kd$ paths of the form $(v_{m+1}, u, v_j)$ such that $u \in V \setminus V_{m+1}$ is a neighbor of $v_{m+1}$ in $G$ and $v_j, j \leq m$ is a neighbor of $u$ in $V_m$. Let $B \subseteq V_m$ be the set of all such $v_j$'s and let
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\( f(B) \) be the set of their colors with \( |f(B)| \leq kd \). Now, color \( v_{m+1} \) with any color from \( K \setminus f(B) \) and one can easily check that \( f(v_{m+1}) \) satisfies both (1) and (2), thus extending the coloring inductively. This proves Theorem 5.8.

5.5 Conclusions and Open Problems

We obtained a relation between forbidden subgraph colorings and oriented colorings. In particular, we obtained an upper bound for the oriented chromatic number in terms of low treewidth chromatic numbers and found an upper bound of \( O(2^{g^{1/2+a(1)}}) \) for the oriented chromatic number of graphs of genus \( g \). However, we believe that this bound is not tight. In fact, we believe in the following conjecture:

**Conjecture**: There exist absolute positive constants \( c_1, c_2 \) such that: if \( G \) is a graph of genus at most \( g \), then \( \chi_o(G) \leq c_1 2^{c_2 \sqrt{g}} \).

Further, it would be interesting to obtain bounds for the \((j,k)\)-treewidth chromatic number (for graphs of bounded genus), when \( j > 2 \). We also pose the following interesting and challenging open problem.

**Open Problem**: Determine if there is a \( k \) such that \( \chi_{tw}^{2,k}(G) \leq 4 \) for all planar graphs \( G \) and find the smallest such \( k \) if it exists.

Note that if we replace 4 by 5 in the above inequality, then the answer is yes for \( k = 1 \) since it has been shown by Borodin [Bor79] that \( a(G) \leq 5 \) for any planar graph \( G \). Also, this bound is tight as Grünbaum [Grü73] obtained an infinite family of planar graphs having no acyclic 4-coloring.