Part III

Background material: Part II
AdS/CFT and Real Time Green’s Function

The main prescription of AdS/CFT correspondence [2] allows us to calculate quantum correlators in the boundary CFT by classical computations in the supergravity dual. It was originally formulated for Euclidean signature [3, 121]. In principle Minkowski formulation may be avoided by working in Euclidean space and then analytically continuing to find real-time propagators. But one needs to know Euclidean correlators for all Matsubara frequencies. In general it is difficult, and a direct computation in Minkowski space is preferable. Subtleties of Lorentzian signature AdS/CFT was discussed in [65, 66, 122]. A working recipe of Minkowski space correlators was given by Son-Starinets in [67] and later treated rigorously in [123]. An alternative formulation of the prescription was given by Iqbal-Liu in [124, 70]. We will review here the prescription of [67, 124, 70] and summarize their results.

7.1 Definitions of Correlators

We will consider here general properties of Minkowski Correlators at finite temperature. A retarded propagator for an arbitrary hermitian bosonic operator \( \mathcal{O} \) in thermal equilibrium (i.e. QFT at finite temperature) is defined as,

\[
G^R(k) = -i \int d^4x e^{-ik\cdot x} \theta(t) < [\mathcal{O}(x), \mathcal{O}(0)] >_\rho
\]  

(7.1)
Chapter 7. AdS/CFT and Real Time Green’s Function

where we have considered the \((- + + +)\) metric convention. \(\theta(t)\) is the Heaviside step function. The \(< A >_\rho = Tr(\rho A)\) denotes average over thermal ensemble, where \(\rho\) is the thermal density function. The advanced propagator is defined similarly,

\[
G^A(k) = i \int d^4xe^{-ikx}\theta(-t) < [\mathcal{O}(x), \mathcal{O}(0)] >_\rho
\]  

(7.2)

from these definitions we can show

\[
G^R(k)^* = G^R(-k) = G^A(k)
\]

(7.3)

If the system is parity invariant, then \(Re(G^{A,R})\) are even functions of \(\omega = k^0\) and \(Im(G^{A,R})\) are odd functions of \(\omega\). For the Euclidean formulation, one considers Matsubara correlators, given by,

\[
G^E(k) = \int d^4xe^{-ikxE}\mathcal{O}(x_E), \mathcal{O}(0)>_{S_E}
\]

(7.4)

where \(T_E\) denotes Euclidean time ordering, and \(< \cdots >_{S_E}\) denotes the vacuum expectation value w.r.t. Euclidean action \(S_E\), where the time direction is compact with radius \(\beta = \frac{1}{T}\). The Matsubara propagator is defined only at discrete values of the frequency \(\omega_E\). For bosonic \(\mathcal{O}\) these Matsubara frequencies are multiples of \(2\pi T\).

The Euclidean and Minkowski propagators are related to each other by analytic continuation. The retarded propagator \(G^R(k)\) can be analytically continued to whole upper half of the complex plane, and the relation is given by,

\[
G^R(2\pi iTn, k) = -G^E(2\pi iTn, k)
\]

(7.5)

The advanced propagator can be analytically continued to lower half plane and the relation to \(G^E\) is given as,

\[
G^A(-2\pi iTn, k) = -G^E(-2\pi iTn, k)
\]

(7.6)

In particular for \(n = 0\),

\[
G^R(0, k) = G^A(0, k) = -G^E(0, k)
\]

(7.7)
7.2 AdS/CFT prescription for Minkowski Correlators

Let us first recall the formulation of AdS/CFT Correspondence in Euclidean space [3]. We will here consider the famous example of the duality: Type IIB superstring theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ $SU(N)$ SYM theory on the 3+1-dimensional boundary of $AdS_5$. Let $g_{YM}$ be the Yang-Mills coupling. Then in the limit of large $N$ and large ‘t Hooft coupling $\lambda = g_{YM}^2 N$, the correspondence reduces to a duality between strongly coupled $\mathcal{N} = 4$ SYM theory and classical supergravity on $AdS_5 \times S^5$. The Euclidean version of the metric for $AdS_5 \times S^5$ is given by,

$$ds^2 = \frac{R^2}{z^2}(d\tau^2 + dx_1^2 + dz^2) + R^2 d\Omega_5^2$$  \hspace{1cm} (7.8)

which is a solution to Type II B supergravity with 5-form flux turned on. $R$ is the AdS radius and is related to the dual parameters by $R^4 = 4\pi g_s \alpha'^2 N$. This metric can be obtained as a field theory limit or near horizon limit of $N$ D3 branes solution described in the Section (3.0.1) for $p = 3$. In AdS/CFT correspondence, the $3+1$-dimensional QFT lives on the boundary of $AdS_5$ space given by $z = 0$. Suppose a bulk field $\phi$ is coupled to an operator $\mathcal{O}$ on the boundary in such a way that the interaction Lagrangian is $\phi \mathcal{O}$. Then the AdS/CFT correspondence can be formally write as,

$$< e^{\int_{b,m} \phi_0 \mathcal{O}} >_{CFT} = e^{-S_{AdS}[\phi]}$$  \hspace{1cm} (7.9)

where the left-hand side is the generating functional for correlators of $\mathcal{O}$ in the boundary field theory and right-hand side is the action of classical solution to the equation of motion for $\phi$ in the bulk metric (eqn. (7.8)) with the boundary condition $\phi|_{z=0} = \phi_0$. The metric (eqn. (7.8)) corresponds to zero temperature field theory at the boundary. To compute Matsubara correlator at finite temperature it has to be replaced with a non-extremal one,

$$ds^2 = \frac{R^2}{z^2}(f(z)d\tau^2 + dx_1^2 + \frac{dz^2}{f(z)}) + R^2 d\Omega_5^2$$  \hspace{1cm} (7.10)

where $f(z) = 1 - \frac{z}{z_H}$ and $z_H = (\pi T)^{-1}$, and $T$ is the Hawking temperature. The Euclidean time $\tau$ is periodic $\tau \sim \tau + T^{-1}$ and $z$ now runs from 0 to $z_H$. 

72
Chapter 7. AdS/CFT and Real Time Green’s Function

The temperature $T$ then corresponds to the Euclidean boundary field theory with compact time direction with same period. The correlators can be found as,

$$< O(x_1)O(x_2) > = \frac{\delta S_{d} [\phi]}{\delta \phi_0(x_1) \delta \phi_0(x_2)}|_{\phi_0=0}$$  \hspace{1cm} (7.11)

One can formally write Minkowski version of AdS/CFT as,

$$< e^{i\phi O} >_{CFT} = e^{iS_{d}[\phi]}$$  \hspace{1cm} (7.12)

One of the problem with this formulation is the boundary condition of $\phi$ at horizon. For Minkowski space, the regularity of $\phi$ can not be demanded at $z = z_H$, as the solution wildly oscillates near the horizon and has two modes: ingoing and outgoing. For the in-coming wave boundary condition, where waves can only travel to the region inside the horizon, one may suspect to get Retarded Correlators for a finite temperature field theory and for outgoing wave Advanced correlator. But this simple correspondence fails to work.

Let us consider the Minkowski version of the AdS part of the metric (eqn. (7.10)),

$$ds^2 = R^2 \left(-f(z) d\tau^2 + dx^2 + \frac{dz^2}{f(z)} \right) = g_{zz}dz^2 + g_{\mu\nu}(z)dx^\mu dx^\nu$$  \hspace{1cm} (7.13)

with the following action for a bulk scalar field $\phi$ with mass $m$,

$$S = -\frac{1}{2} \int dx^4 \int_{z_B}^{z_H} \sqrt{-g} (g^{zz}(\partial_z \phi)^2 + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2)$$  \hspace{1cm} (7.14)

where for our case $z_B = 0$. The linearized field equation for $\phi$ is given by,

$$\frac{1}{\sqrt{-g}} \partial_z(\sqrt{-g}g^{zz} \partial_z \phi) + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi = 0$$  \hspace{1cm} (7.15)

The equation of motion has to be solved with a fixed value of $\phi$ at $z_B$, and can be write in the form,

$$\phi(z, x) = \int \frac{d^d k}{(2\pi)^d} e^{ik_{\mu}x^\mu} f_{k}(z) \phi_{0}(k)$$  \hspace{1cm} (7.16)

where $f_k(z_B) = 1$ and $\phi_0(k)$ is determined by the Fourier transform of source field $\phi(z_B, x) = \phi_0(x)$ at the boundary. the effective equation of motion for $f_k(z)$ is
Chapter 7. AdS/CFT and Real Time Green’s Function

given by,

\[ \frac{1}{\sqrt{-g}} \partial_z (\sqrt{-g} g^{zz} \partial_z f_k(z)) - (g^{\mu \nu} k_{\mu} k_{\nu} + m^2) \phi = 0 \]  \hspace{1cm} (7.17)

with the boundary condition \( f_k(z_B) = 1 \) and satisfying incoming-wave boundary condition at \( z = z_H \). The on-shell action reduces to

\[ S = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \phi_0(-k) \mathcal{F}(k, z) \phi_0(k) \bigg|_{z=z_H} \]  \hspace{1cm} (7.18)

where

\[ \mathcal{F}(k, z) = \sqrt{-g} g^{zz} f_{-k}(z) \partial_z f_k(z) \]  \hspace{1cm} (7.19)

If we assume the formal definition of Eqn.(7.12), the retarded Green’s function is given by taking two derivatives of the classical on-shell (eqn.(7.18)) and is given by,

\[ G^R(k) = \frac{1}{2} \mathcal{F}(k, z) \bigg|_{z_B}^{z_H} + \frac{1}{2} \mathcal{F}(-k, z) \bigg|_{z_B}^{z_H} \]  \hspace{1cm} (7.20)

It can be shown using \( f_k^*(z) = f_{-k}(z) \) (which also solves Eqn.(7.17) with in-going boundary condition), that the imaginary part of \( \mathcal{F} \) is conserved flux and is independent of \( z \). This means the imaginary part of the \( \mathcal{F} \) cancels for each term in the expression of \( G^R \), and the retarded Green’s function is real. One suggestion of avoiding the problem may be just dropping the contribution from horizon, but still the imaginary part cancels out as \( \mathcal{F}(-k, z) = \mathcal{F}^*(k, z) \). So by this prescription one can not get a retarded Green’s function which is in general complex.

### 7.2.1 Son-Starinets Prescription

Son and Starinets gave an ad hoc resolution to this problem of real time correlators from AdS/CFT in [67]. They gave a prescription and various checks on the validity of the formula. The prescription is given as follows,

- Solve the mode equation (7.17) with the boundary condition \( f_k(z_B) = 1 \) and the asymptotic solution is ingoing (outgoing) wave at the horizon for time-like momentum. For space-like momentum, the horizon boundary condition is similar to Euclidean case, and is taken to be the regular solution.
• The retarded (advanced) propagator is given by,

\[ G^R(k) = \mathcal{F}(k, z)|_{z_B} = \sqrt{-g} g^{zz} \partial_z f_k(z) \bigg|_{z=z_B} \quad (7.21) \]

where \( \mathcal{F} \) is given by (7.19). Notice the surface terms coming from IR or the horizon is dropped. This part of the metric influences the correlator through the boundary condition imposed on bulk field \( \phi \).

Since imaginary part of \( \mathcal{F} \) is independent of the radial coordinate \( z \), \( \text{Im}(G^{R, A}) \) can be computed by evaluating \( \text{Im}(\mathcal{F}(k, z)) \) at any convenient value of \( z \). This ad hoc prescription was later confirmed rigorously in [123].

### 7.2.2 Iqbal-Liu Prescription

An alternative equivalent prescription was given in [124, 70]. According to the prescription the retarded Green’s function for a boundary operator \( \mathcal{O} \) corresponding to a bulk field \( \phi \) is given by,

\[ G^R(k_\mu) = \left( \lim_{z \rightarrow z_B} \frac{\Pi(z; k_\mu)|_{\phi_R}}{\phi_R(z; k_\mu)} \right) \bigg|_{\phi_0=0} \quad (7.22) \]

where \( \Pi \) is the canonical momentum conjugate to \( \phi \) with respect to radial \( (z) \) foliation. \( \phi_R(z, k_\mu) \) is the solution to the equations of motion with in-falling boundary condition at horizon and \( \lim_{z \rightarrow z_B} \phi_R(z, k_\mu) \rightarrow \phi_0(k_\mu) \). The notation \( \Pi(z; k_\mu)|_{\phi_R} \) indicates that \( \Pi \) has to be evaluated on the classical solution \( \phi_R \). It was shown in [124, 70], that this definition is equivalent to the Son-Starinets prescription at level of linear approximation. Recall, the standard result from linear response theory, that if one considers a system in equilibrium at \( t \rightarrow -\infty \), and then perturbs its action with the term \( \int d^d x \phi_0(x) \mathcal{O}(x) \), the one point function in presence of the source is given at linearized level by,

\[ \langle \mathcal{O}(k_\mu) \rangle_{\phi_0} = G^R(k_\mu) \phi_0(k_\mu) \quad (7.23) \]
Then the equivalent statement of the prescription (7.22) is,

\[ < \mathcal{O}(k_\mu) >_{\phi_0} = \lim_{z \to z_B} \Pi(z; k_\mu) \bigg|_{\phi_R} \]  

(7.24)

Above is true in general for Euclidean version of AdS/CFT given by (7.9), where,

\[ < \mathcal{O}(x) >_{\phi_0} = -\frac{\delta S_{cl}}{\delta \phi_0(x)} = -\lim_{z \to z_B} \Pi_E(z, x) \bigg|_{\phi_E} \]  

(7.25)

where \( \Pi_E \) is the canonical momentum conjugate to \( \phi \) w.r.t radial \( (z) \) foliation in Euclidean signature evaluated at the classical solution \( \phi_E \). The last equality comes from the well known fact in classical mechanics that derivative of an on-shell action with respect to the boundary value of a field is equal to the canonical momentum conjugate to the field, evaluated at the boundary. Eqn.(7.23) is Lorentzian version of the statement (7.25), but in absence of a proper statement equivalent to (7.9), status of the statement (7.22) is at the level of conjecture, and it was applied for known results in [124, 70], and so far has passed all the tests. Also in [70], it was shown by analytic continuation that (7.22) gives the correct Euclidean action. We will show the equivalence of the statement (7.22) with that of Son-Starinets (7.21) for bulk scalar field and move on to applications.

Consider the action for bulk scalar field given by (7.14), the canonical momentum conjugate to \( \phi \) is given by,

\[ \Pi(z; x) = \sqrt{-g(z)g^{zz}(z)} \partial_z \phi(z; x) \]  

(7.26)

and its Fourier transform given by (7.16),

\[ \Pi(z; k) = \sqrt{-g(z)g^{zz}(z)}(\partial_z f_k(z))\phi_0(k) \]  

(7.27)

where \( \phi_0(k) \) is the Fourier transform of \( \phi_0(z; x) \) which is the boundary value of a classical solution when \( f_k \) satisfies (7.17) with ingoing boundary condition and proper normalization \( f_k(z_B) = 1 \). Then using the prescription (7.22), the retarded Green’s function is same as that obtained by Son-Starinets prescription (eqn.(7.20)).
Chapter 7. AdS/CFT and Real Time Green’s Function

Although we mainly talked about $AdS_5/CFT_4$ correspondence, the conjecture of AdS/CFT has expanded its horizon to general duality between systems in background of classical gravity solutions and QFT living on the boundary of this gravity solution. In particular, this duality is assumed to be true for asymptotically AdS space times. Consider bulk fields in a background metric,

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + g_{ii}(dx^i)^2$$  \hfill (7.28)

The boundary is taken to be at $r = \infty$ (to match with previous discussion $r = \frac{1}{\lambda}$), where various components of the metric have asymptotic AdS behavior,

$$g_{tt}, g^{rr}, g_{ii} \simeq r^2, \quad r \to \infty$$  \hfill (7.29)

Also assume the theory is translationally invariant in $x^\mu$ directions, i.e., the metric components are function of $r$ only and work in momentum space along these directions,

$$\phi(r, x^\mu) = \phi(r, k_\mu)e^{-i\omega t + ik \cdot x}, \quad k_\mu = (-\omega, k)$$  \hfill (7.30)

where $\phi$ represents a general bulk field.

7.2.2.1 Example: Massive Scalar field

Let us consider a massive scalar field action (7.14) in a background metric (7.28) in $(d + 1)$-dimensional space-time. Then $\phi_R$ has asymptotic behavior,

$$\phi_R(r, k_\mu) \sim A(k_\mu)r^{\Delta - d} + B(k_\mu)r^{-\Delta}, \quad r \to \infty$$  \hfill (7.31)

where $\Delta = \frac{d}{2} + \sqrt{m^2 + \frac{d^2}{4}}$ is to be interpreted as the scaling dimension of the boundary operator. For massless case ($m = 0$), $r \to \infty$, $\phi = A(k_\mu)$, which corresponds to the boundary value or the dual field theory source. But in general it differs by a power of $r$, and the $r \to \infty$ has to be taken carefully to compute $G^R$. This implies,

$$<\mathcal{O}(k_\mu)>_A = \lim_{r \to \infty} r^{\Delta - d}\Pi(z; k_\mu) \bigg|_{\phi_R}$$  \hfill (7.32)
where as before $\Pi = -\sqrt{-g}g^{\nu r}\partial_r \phi$ is the momentum conjugate to $\phi$. So asymptotic expansion of $\Pi$ becomes,

$$\Pi(r, k_\mu) \simeq -(\Delta - d) A(k_\mu) r^\Delta + \Delta B(k_\mu) r^{d-\Delta} \quad (7.33)$$

Then by prescription (7.22),

$$G_R(k_\mu) = \lim_{r \to \infty} r^{2(\Delta-d)} \left. \frac{\Pi(r, k_\mu)}{\phi_R(r, k_\mu)} \right|_{\phi_R} = (2\Delta - d) \frac{B(k_\mu)}{A(k_\mu)} \quad (7.34)$$

Note there is change in (7.22), a power of $r$ multiplied to extract the finite piece in $r \to \infty$ limit. The general folklore is to take the limit in such a way that we extract finite piece and which is in general also non-analytic in $k_\mu$. As finite terms which are analytic in momentum corresponds to local terms in correlators (delta-function terms in the position space) and is of not much interest. Also note in the asymptotic solution of $\phi_R$, has two independent components given by coefficients $A$ and $B$, which corresponds to normalizable and non-normalizable modes in Lorentzian signature discussed in [66]. According to the identification in [66], coefficient of the solution $A$ corresponds to the source in dual theory and $B$, the normalizable mode corresponds to operator expectation value. So the prescription (7.22) correctly reproduces the Green’s function defined in [66].

### 7.2.2.2 Example: Vector Field

Consider a bulk vector field $A_M$ with Maxwell action and gauge coupling $g_{\text{eff}}$. This is dual to a conserved current $J^\mu$, and the prescription (7.23) becomes,

$$< J^\mu > = - \lim_{r \to \infty} \frac{1}{g_{\text{eff}}} \sqrt{-g} F^{\nu \mu} \quad (7.35)$$

where $F^{\mu \nu}$ is the field strength corresponding to $A_\mu$. The right hand side of the equation is momentum conjugate to $A_\mu$. 

78
7.2.2.3 Example: Fermion field

As seen in previous subsections the Iqbal-Liu prescription is directly applicable in case of scalars and vector operators, applicability to fermionic case is little subtle as discussed in [70], we will briefly summarize their results here. A very nice review and detailed analysis for fermions can also be found in [125]. Fermionic real time propagators was constructed in [70], by first applying the prescription (7.22) in Euclidean space, and then analytically continuing to real time. Consider the Minkowski action for fermions in the asymptotically AdS metric (eqn.(7.28)),

\[ S = i \int d^{d+1}x \sqrt{-g} i(\bar{\Psi} \Gamma^M \partial_M \Psi - m \bar{\Psi} \Psi) + S_{\partial M} \]  \hspace{1cm} (7.36)

where

\[ \bar{\Psi} = \Psi^\dagger \Gamma^t, \quad \partial_M = \partial_\mu + \frac{1}{4} \omega_{abM} \Gamma^{ab} \]  \hspace{1cm} (7.37)

and \( \omega_{abM} \) is the spin connection. We will denote \( M \) and \( a, b \) to denote bulk space-time and tangent space indices respectively, and \( \mu, \nu \cdots \) to denote indices along the boundary directions, i.e. \( M = (t, \mu) \). The \( \Gamma^a \) obey \( \{ \Gamma^a, \Gamma^b \} = 2i \epsilon^{ab} \), where \( \eta = \text{Diagonal}(-1,1,\cdots,1) \) and \( \Gamma^M = e^a_M \Gamma^a \), where \( e^a_M \) is the vielbein. Also \( (t, r, x_\mu) \) denote space time indices and \( (t, \rho, x_\mu) \) the tangent space indices.

\( S_{\partial M} \) is required to make the action stationary under variations of the spinor field. It was shown in [63], that \( S_{\partial M} \) can be fixed uniquely (demanding stationary action, locality, absence of derivatives and invariance under asymptotic AdS symmetry) to,

\[ S_{\partial M} = -i \int_{\partial M} d^dx \sqrt{-g} g^{tt} \bar{\Psi} \Psi_+ - \Psi_- \]  \hspace{1cm} (7.38)

where \( \Psi_{\pm} = \frac{1}{2} (1 \pm \Gamma^t) \Psi \). The addition of the boundary term makes the on-shell action independent of \( \Psi_- \). So with fixing \( \Psi_{\pm} \) in-going, we have the freedom to fix only half of the components at the boundary for the fermionic fields. As discussed in [70, 125], this related to the fact that equation of motion of fermions are first order.

We will be working in momentum space with the Fourier transform of \( \Psi \) denoted
Chapter 7. AdS/CFT and Real Time Green’s Function

by \( \Psi(r, k_\mu) \). Consider the momentum conjugate to \( \Psi_+ \) with respect to \( r \)-foliation,

\[
\Pi_+(r, k_\mu) = i\sqrt{-g}g^{rr}(r)\bar{\Psi}_-(r, k_\mu)
\]  

(7.39)

The scaling dimension of an boundary fermionic operator (\( \Delta \)) is given in terms of the mass of the bulk fermion field (in asymptotically \( AdS_{d+1} \) space-time),

\[
\Delta = \frac{d}{2} + m
\]  

(7.40)

Then the boundary value for \( \Psi_- \) or the source has a scaling dimension \( \Delta - d \) (as the boundary action \(-i \int d^d x (\bar{\chi}_0 \mathcal{O} + \bar{\mathcal{O}} \chi_0)\) should be scale invariant). Then,

\[
\chi_0(k_\mu) = \lim_{r \to \infty} r^{d-\Delta} \Psi_+(r, k_\mu)
\]  

(7.41)

where \( \chi_0(k_\mu) \) is the boundary source. It will be related to the boundary value of \( \Psi_- \) (which will correspond to operator expectation value) by equation of motion, as with the action (7.36) and in-going boundary condition for both \( \Psi_\pm \), only \( \Psi_+ \) is independent. In general we expect a relation of the form,

\[
\psi_0(k_\mu) = \mathcal{S}(k_\mu) \chi_0(k_\mu)
\]  

(7.42)

where \( \mathcal{S} \) is a matrix and

\[
\psi_0(k_\mu) = \lim_{r \to \infty} r^\Delta \Psi_-(r, k_\mu)
\]  

(7.43)

Now we should take care in defining Gamma matrices in the boundary theory which is one dimension less compared to the bulk. A detailed analysis of relation between Gamma matrices of various dimension can be found in Appendix B of [126]. We will here just state the result (capital Gamma denotes bulk matrices and small Gamma denotes boundary matrices),

- For \( d \) even, \( \Gamma^\mu = \gamma^\mu \) and \( \Gamma^z = \gamma^{d+1} \), where \( \gamma^{d+1} \) is the chirality operator in \( d \)-dimension. Thus from boundary point of view \( \Psi_\pm \) transform as \( d \)-dimensional Weyl spinor of opposite chirality. Then \( \chi_0 \) and \( \mathcal{O} \) are \( d \)-dimensional boundary spinor of definite chirality, i.e. a bulk Dirac spinor is mapped to a chiral spinor operator at boundary.
Chapter 7. AdS/CFT and Real Time Green’s Function

- For $d$ odd, $\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix}$ and $\Gamma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ In this basis the two components $\Psi_\pm$ each transform as a $d$-dimensional Dirac spinor; thus $\chi_0$ and $\mathcal{O}$ are both Dirac spinors.

Notice that in all dimensions the number of components of $\mathcal{O}$ is always half of that of $\Psi$.

For $d$ odd, let us the spinor field as,

\[
\Psi_+ = \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix}, \quad \Psi_- = \begin{pmatrix} 0 \\ \psi_- \end{pmatrix}
\]  

(7.44)

And the corresponding boundary values of $\psi_\pm$ be $\bar{\chi}_0$ and $\bar{\psi}_0$ respectively as defined in (7.41) and (7.43). Then the relation corresponding to (7.42), takes the form,

\[
\bar{\psi}_0(k_\mu) = \bar{\mathcal{S}}(k_\mu)\bar{\chi}_0(k_\mu)
\]  

(7.45)

Also write the conjugate momentum as,

\[
\pi_+(r, k_\mu) = i\sqrt{-gg^{rr}(r)}\bar{\psi}_-(r, k_\mu)
\]  

(7.46)

So the Iqbal-Liu prescription takes the form,

\[
< \bar{\mathcal{O}} >_{\chi_0} = \lim_{r \to \infty} r^{d-d} \pi_+ = i\bar{\psi}_0(k) = i\bar{\chi}_0(k)^\dagger \bar{\mathcal{S}}(k)^\dagger \gamma^L
\]  

(7.47)

which implies,

\[
< \mathcal{O} >_{\bar{\chi}_0} = -i\bar{\mathcal{S}}(k)\bar{\chi}_0(k)
\]  

(7.48)

By definition of retarded correlator $G^R \sim < \mathcal{O} \mathcal{O}^\dagger >$ (from action $-i \int \! d^d x (\bar{\chi}_0 \mathcal{O} + \bar{\mathcal{O}} \chi_0)$),

\[
< \mathcal{O}(k) >_{\bar{\chi}_0} = -iG^R(k) \gamma^L \bar{\chi}_0(k)
\]  

(7.49)
So comparing above two equations, we have,

\[ G^R(k) = -i\tilde{S}(k)\gamma^L \]  

(7.50)

where \( \tilde{S} \) is given by (7.45). There is an overall sign ambiguity, the sign is chosen demanding unitarity or demanding imaginary part of the Green’s function is positive for all \( \omega \) [70].

For \( d \) even, we do not need such an decomposition, and similarly, the Green’s function is given as,

\[ G^R(k) = -iS(k)\gamma^L \]  

(7.51)

where \( S \) is given by (7.42).

**7.2.2.3.1 Spinor in pure \( AdS_3 \):** Consider the bulk geometry to be pure \( AdS_3 \) given by,

\[ ds^2 = r^2(-dt^2 + dx^2) + \frac{dr^2}{r^2} \]  

(7.52)

We will use the prescription described in previous subsection to calculate the boundary \( (r \to \infty) \) retarded Green’s function. Since we have considered pure \( AdS_3 \), the boundary field theory will be a 1 + 1-dimensional CFT at zero temperature. The equation of motion in this background geometry is given by,

\[
\begin{align*}
\Psi_+ & = -i\frac{\gamma^L k^L}{k^2} \hat{A}(-m)\Psi_- \\
\Psi_- & = i\frac{\gamma^L k^L}{k^2} \hat{A}(m)\Psi_+ 
\end{align*}
\]  

(7.53)

where \( \gamma^L k = -\gamma^0 \omega + \gamma^L \kappa \), \( k^2 = -\omega^2 + \kappa^2 \), \( k^L = (\omega, \kappa) \) and

\[ \hat{A}(m) = r(r\partial_r + 1 - m) \]  

(7.54)

and,

\[
\begin{pmatrix}
\gamma^0 \\
\gamma^1 
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-1 & 0 
\end{pmatrix} ; \quad \begin{pmatrix}
\gamma^1 \\
\gamma^0 
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
1 & 0 
\end{pmatrix}
\]  

(7.55)

The equation of motion for \( \psi_{\pm} \), which in 2 + 1-dimensions are just two complex functions is given by,

\[ \hat{A}(\pm m)\psi_{\pm} = i(\omega \mp k^1)\psi_{\mp} \]  

(7.56)
and
\[ \hat{A}(-m)\hat{A}(m)\psi_+ = k^2\psi_+ \]  
(7.57)

Let us consider the case of \( m = 0 \). The scaling dimension of the boundary operator is \( \Delta = 1 \) (as \( m = 0 \) and \( d = 2 \)) and \( \hat{A}(0) = r(r\partial_r + 1) \). We have,
\[ r^2\partial^2_r\psi_+ + 4r\partial_r\psi_+ + (2 - \frac{k^2}{r^2})\psi_+ = 0 \]  
(7.58)

which has a solution of the form,
\[ \psi_+ = \frac{A(k)}{r}e^{\frac{\kappa}{r}} + \frac{B(k)}{r}e^{-\frac{\kappa}{r}} \]  
(7.59)

we will assume time-like momentum i.e., \( k^2 < 0 \). If we now demand in-going boundary condition at \( r = 0 \) (Poincare horizon, as described in next section), we have,
\[ \psi_+ = \frac{A(k)}{r}e^{i\text{sgn}(\omega)\frac{\sqrt{-k^2}}{r}} \]  
(7.60)

Now putting the solution of \( \psi_+ \) back in (7.56), we get a relation between \( \psi_{\pm} \),
\[ \psi_- = \sqrt{\frac{\omega + \kappa}{\kappa - \omega}}\psi_+, \quad \omega > |\kappa| \]
\[ \psi_- = \sqrt{\frac{\omega + \kappa}{\kappa - \omega}}\psi_+, \quad \omega < -|\kappa| \]  
(7.61)

Let us consider the case \( \omega > |\kappa| \). Also from definitions of previous section,
\[ \psi_0 = \lim_{r \to \infty} r\Psi_-; \quad \chi_0 = \lim_{r \to \infty} r\Psi_+ \]  
(7.62)

So we get using (7.53),
\[ S = -\frac{\gamma k}{\sqrt{-k^2}} \]  
(7.63)

So the Green’s function is given by (Eqn.(7.50), but with an overall negative sign),
\[ G^R(k) = iS\gamma^L = i\gamma \frac{k}{\sqrt{-k^2}}\gamma^L = \begin{pmatrix} i\sqrt{\frac{\omega - \kappa}{\omega + \kappa}} & 0 \\ 0 & i\sqrt{\frac{\omega + \kappa}{\omega - \kappa}} \end{pmatrix} \]  
(7.64)
Chapter 7. AdS/CFT and Real Time Green’s Function

Notice \( G_{11}^R(\omega, \kappa) = \frac{1}{G_{22}(\omega, \kappa)} = G_{22}^R(\omega, -\kappa) \). It is sufficient to study one of the components. In particular for \( 2 + 1 \) dimensional case we can formally define the retarded Green’s function as,

\[
G^R(k_\mu) = i \lim_{r \to \infty} \frac{\psi_-}{\psi_+} 
\]  

(7.65)

with overall sign to be fixed demanding \( Im(G^R) > 0 \) for all \( \omega > 0 \). Above definition is true for any asymptotically \( AdS_3 \) space-time, and we will use this definition to calculate Green’s function in later chapters.

### 7.3 In-going boundary condition

Consider the metric,

\[
ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + a(r)^2d\vec{x}^2
\]

(7.66)

with \( f(r), a(r) \to r^2 \) as \( r \to \infty \), Horizon at \( f(r_0) = 0 \), Boundary at \( r \to \infty \). The in-falling boundary condition is given by,

- **Non-degenerate Horizon**: Simple pole at \( r = r_0 \) i.e. \( f(r) \sim \frac{4\pi}{f'}(r - r_0) \).

  Solution near horizon: \( e^{-i\omega t}(r - r_0)^{-i\frac{\omega}{4\pi}} \).

- **Degenerate horizon**: Double pole at \( r = r_0 \) i.e. \( f(r) \sim c(r - r_0)^2 \). Solution near horizon: \( e^{-i\omega t}e^{i\frac{\omega}{c(r - r_0)}} \).

- **Poincare Horizon**: (near \( z \to \infty \)) Consider the metric (with \( r = \frac{1}{z} \)),

\[
ds^2 = \frac{1}{z^2}[-dt^2 + d\vec{x}^2 + dz^2]
\]

(7.67)

Solution near horizon: \( e^{-i\omega t}e^{-kz} \) where

\[
k = \begin{cases} 
-i\sqrt{\omega^2 - k^2} & \omega > |\vec{k}| \\
+i\sqrt{\omega^2 - k^2} & \omega < |\vec{k}|
\end{cases}
\]

(7.68)

Assuming time-like momentum \( |\omega| > |\vec{k}| \).
8

Lessons from Condensed Matter

Recently a lot of work has been initiated in studying strongly coupled Condensed matter systems by using AdS/CFT conjecture discovered in String theory. As described in the earlier chapters, the term AdS/CFT conjecture refer to a general class of symmetries observed in systems including gravity and those without gravity. In particular in these conjecture classical gravity theories in $d + 1$-dimension space-time are dual to strongly coupled Quantum Field Theories living on boundary of this space-time i.e. in $d$-dimension. The extra dimension in gravity system or the bulk acts as a energy scale in the boundary theory, and renormalization group flows in the boundary theory can be studied by studying variation of the classical geometry in this extra dimension. As classical calculations are more tractable, this “AdS/CFT tool” becomes extremely useful in extracting quantum strongly coupled theories, where usual perturbation technique fails. In this thesis we will study one such system using this tool-kit. A review of application of AdS/CFT to Condensed matter systems can be found in [60, 127]. In this chapter we will introduce various Condensed matter terms which we will use in this thesis.

8.1 Fermi Gas

It is a non interacting translationally invariant system of non interacting electrons, with the single-particle eigenstates are plane waves with energy $\epsilon_k = \frac{k^2}{2m}$ ($k$ is the momentum vector and $m$ is the mass of the particles). The ground state of an $N$ non-interacting particle system is the well-known Fermi sea: all states up to the Fermi wave vector $k_F$ are filled, all the other states are empty. The energy of the
last occupied state is the Fermi energy, $E_F = \frac{k_F^2}{2m} = \mu(T = 0)$ ($\mu$ is the chemical potential). The elementary excitations are creation of particles ($|k| > k_F$) and destruction of particles at $|k| < k_F$ called holes. We can also construct particle-hole excitations which keep the total particle number fixed, *i.e.* one takes one particle from some state $k$, with $|k| < k_F$, and puts it into a state $k'$, with $|k'| > k_F$. These particle-hole excitations are parametrized by the two quantum numbers $k, k'$ and thus form a continuum.

### 8.2 Fermi and Non-Fermi Liquids

Landau Fermi liquid is essentially a Fermi gas when interactions of the electrons with each other are included. Landau’s theory of Fermi liquid is based on the idea of a continuous and one-to-one correspondence between eigen states of non-interacting and interacting system. Fermi Liquid Theory is expected to break down in many situations involving "strongly correlated electrons". A nice review on this subject can be found in [128, 129]. Let us start with a filled fermi sphere and imagine adding one more fermion in a momentum eigenstate. One can imagine doing this with a Fermi gas first, and then quickly increasing ("adiabatic") the strength of the interaction parameter. The particle becomes a *quasi-particle* with the same momentum. But it is not an energy eigenstate and can decay. Due to phase space limitations the life time increases as $\frac{1}{(k-k_F)^2}$ as $k \to k_F$. So near $k_F$ ($k_F$ remains same as the Fermi gas) these are legitimate excitations. It is not possible in frame work of Landau’s to derive microscopic parameters, but a general form of one-particle Green’s function can be predicated based on the existence of quasi-particle *i.e.* quasiparticle poles in the correlators. Consider one-particle Euclidean Green’s function ¹,

$$G(k, \omega) = \frac{1}{i\omega - \epsilon_k^{00} - \Sigma(k, \omega)} \quad (8.1)$$

where $\epsilon_k^{00}$ is the bare particle energy. Excitation energies of the system are given by the poles of the correlator. Landau’s assumption of existence of quasi-particles close to Fermi surface, amounts to demanding regular behavior of the self-energy correction $\Sigma(k, \omega)$. A expansion of the Green’s function up to second order in

¹The retarded Green’s function is given by $G^R(k, \omega) = G(k, i\omega \to \omega + i\epsilon)$, where $\epsilon \to 0$. 

86
Chapter 8. Lessons from Condensed Matter

$\langle |k| - k_F \rangle$ gives,

$$G(k, \omega) = \frac{z_k}{i \omega - \epsilon_k^0 + i \text{sgn}(\omega) \tau(\omega)^{-1}} \quad (8.2)$$

where $z_k$, quasi-particle weight gives the jump in momentum distribution function at $k_F$ compared to unity for non-interacting case. $\tau \sim \frac{1}{(k - k_F)^2}$ gives the quasi-particle lifetime and $\epsilon_k^0$ is the effective energy of the quasi-particle including effective chemical potential (renormalized by $\Sigma(k_F, 0)$). As the energy of the quasiparticle on the Fermi surface is zero, and we are concerned with the excitations near the Fermi surface, dispersion relation is taken to be linear of form $\epsilon_k = \omega = \frac{k_F}{m^*}(\langle |k| - k_F \rangle)$. $m^*$ is called the effective mass of this quasiparticles. The difference between the bare mass $m$ and $m^*$ is due to interaction effects which in principle can be calculated from a microscopic theory of the system in question. Given that the non-interacting particles obey Fermi-Dirac statistics, the quasiparticles also obey the same. In practice, this means that Landau’s theory is useful for phenomena at energy scales much smaller than the Fermi energy, but inapplicable otherwise. Also ground state energy $E_F$ receives contribution from states well below the Fermi energy, so excitation above the ground state are the fundamental objects of Landau’s theory. When the Green’s function does not have the properties of Fermi liquid it is called a non-Fermi liquid.

8.3 Luttinger Liquid

The best understood example of an Non-Fermi Liquid is $1+1$ dimensional interacting electrons, also known as Luttinger Liquid. Extensive review of this subject can be found in [130, 128, 129]. Luttinger Liquid is an exactly solvable model, in particular it can be solved by “Bosonization formalism”. In one dimension, the Fermi surface is replaced by two Fermi points at $k_F$ and $-k_F$. The assumption in Luttinger liquid model is that the spectrum is linear about Fermi points. So the particle-hole excitations with a given momentum has same kinetic energy, compared to higher dimensional electron gas. In Luttinger liquid, the elementary excitations are not quasi-particles but collective oscillations of the charge and spin density, propagating coherently, but in general with different velocities. The correlation function in contrary to Fermi liquid shows non-universal power laws with interaction-dependent exponents i.e. the quasiparticle pole is replaced by power-law
edge singularity. Also even at zero temperature the momentum distribution function does not show a jump like Fermi liquid, but the derivative becomes infinite at Fermi points. Another characteristic of the Luttinger liquid is particle-hole symmetry, i.e. identical behavior of the particle and hole part of the spectral function. The spinless Luttinger liquid has a spectral function (given by imaginary part of retarded Green’s function) of the form,

\[ A(k, \omega) \sim \frac{1}{|\omega - v_c k|^{1 - \gamma}} \]  

(8.3)

near the singularity where \( \gamma \) measures the interaction strength and \( \tilde{k} = (k - k_F) \). \( \gamma = 0 \) corresponds to the non-interacting case. In the case with both spin and charge, there are two such singularities, with different velocities for spin \( (v_s) \) and charge \( (v_c) \) excitations (Spin-charge separation). A schematic comparison of Spectral functions of non-interacting electrons, fermi Liquid and Luttinger liquid is given in Fig.(8.1).

![Diagram](image)

Figure 8.1: Spectral function or imaginary part of Green’s function for non-interacting electrons \( (\omega_N = \frac{k_F}{m} (k - k_F)) \), Fermi Liquid \( (\omega_f = \frac{k_F}{m^*} (k - k_F)) \), and a Luttinger liquid. \( v_s \) and \( v_c \) are spin and charge velocity respectively.

### 8.3.1 Bosonization

In 1 + 1 dimensions fermions can be bosonized, i.e. there exists an duality between fermionic degrees of freedom and corresponding bosonic degrees. This was first applied to show equivalence of sine-Gordon and Thirring Model in [131], where
a map was given between bosonic degrees of freedom of an interacting scalar field action (sine-Gordon) in 1 + 1 dimension with fermionic degrees of freedom of a theory (Thirring Model) of single Dirac field action in 1 + 1-dimension. This technique can be used to solve various 1 + 1 models, like Luttinger liquid. A review of bosonization technique can be found in [132]. The Bosonization map is given by:

$$
\psi_L \approx e^{i\phi_L}, \quad \psi_R \approx e^{-i\phi_R}
$$

$$
\bar{\psi} \gamma^\mu \partial_\mu \psi \approx (\partial_\mu \phi)^2 = (\partial_t \phi)^2 - v^2 (\partial_x \phi)^2
$$

$$
\psi_L^\dagger \psi_R \approx e^{-i(\phi_L + \phi_R)}
$$

$$
\bar{\psi} \gamma^\mu \psi \approx e^{i\mu} \partial_\mu \phi
$$

(8.4)

where $v$ is velocity of the excitation. $\psi$ is a fermionic field and $\phi$ is bosonic field. $L/R$ represents left-moving and right-moving fields. This has a Lorentz invariant form with $v$ as velocity of light. So an interaction of type $(\bar{\psi} \gamma^\mu \psi)^2$ renormalises the kinetic term and hence also renormalises the dimension of the operator $e^{i\phi_L}$. The correlation becomes $(\omega - v(k - k_F))^\alpha$ where $\alpha$ depends on this interaction strength. Non Lorentz invariant $\rho^2 (\rho \sim \bar{\psi} \gamma^0 \psi, "\text{charge-charge}"	ext{ interaction})$ term adds $(\partial_x \phi)^2$ in the dual bosonic theory. This renormalizes the velocity of the excitation.

### 8.4 Fermi-Luttinger Liquid

Deviation from Luttinger liquid behavior can be seen in modified Luttinger models e.g. Fermi-Luttinger liquid [82, 133]. The formulation and exact solvability of Luttinger liquid relies on the assumption of linear dispersion. In [82] the spectral function was modified by an non-linear term for spinless Luttinger liquid.

$$
\omega_\pm = \pm v \tilde{k} + \frac{\tilde{k}^2}{2m} + \cdots
$$

(8.5)

where $\omega_\pm$ represents left-moving and right-moving particles. The presence of finite mass $m$ breaks the particle-hole symmetry, and modifies the spectral function near particle and hole differently. As a consequence the edge singularity for particle is replaced by a Lorentzian peak like Fermi-liquid with finite decay rate, but corre-
sponding behavior for hole remains same ("Particle-hole asymmetry"). The scaling exponent now also becomes function of momentum $k$. Very close to the singularity the behavior of the particle spectral function become similar to a Fermi-liquid, but far away it behaves like an Luttinger liquid.