Chapter 6

$S_3$ Flavor Symmetry and Lepton Mass and Mixing

6.1 Introduction

In the previous chapter we have seen the implication of the discrete symmetry group $A_4$ in explaining the tribimaximal mixing matrix. Another discrete group which has been extensively discussed in the literature as a family symmetry group is the $S_3$ permutation group [1–7]. In this chapter we show how the $S_3$ symmetry group can be used to generate appropriate neutrino mass and mixing, and as well as the charged lepton mass hierarchy. To explain neutrino mass and mixing, we have extended our particle content by SU(2)$_L$ triplet Higgs fields and the standard model singlets.

The $S_3$ group has the $S_2$ permutation group as its subgroup. If one identifies this subgroup with the $\mu - \tau$ exchange symmetry, then it is straightforward to get vanishing $\theta_{13}$ and maximal $\theta_{23}$ for the neutrinos. However, since the same group acts on the charged leptons as well, this would lead to $\mu$ and $\tau$ masses of the same order. In addition this would lead to a highly non-diagonal mass matrix for the charged leptons, which is undesirable in this case. Therefore, the $S_3$ group should be broken in such a way that $\mu - \tau$ permutation symmetry remains intact for the neutrinos but gets badly broken for the charged leptons.

In the following, we propose a model with $S_3 \times Z_4 \times Z_3$ family symmetry. The additional $Z_3$ symmetry is required for obtaining the correct form of the charged lepton mass matrix. We preserve the $\mu - \tau$ symmetry in the neutrino sector while breaking it maximally for the charged leptons. In particular, we introduce two SU(2)$_L$ triplet Higgs in our model for generating the neutrino masses. The charged lepton masses are generated by the standard Higgs doublet. The particle content of our model has been tabulated in Tab. (6.1). We postulate two additional $S_3$ doublets of Higgs which are SU(2)$_L \times U(1)_Y$ singlets, to generate the desired lepton mass matrices. The $S_3$ group is broken spontaneously when the singlet Higgs acquire VEVs. The VEVs are aligned in such
a way that the residual $\mu - \tau$ symmetry is intact for the neutrinos but broken maximally for the charged leptons. We explicitly minimize our scalar potential and discuss the VEV alignment. We show that under the most general case, the minimization condition of our scalar potential predicts a very mild breaking of the $\mu - \tau$ symmetry for the neutrinos.

Since the Higgs triplet $\Delta$ interacts with the gauge bosons via their kinetic terms, they can be produced at the LHC and then can be traced via their subsequent decays. The most crucial feature of the Higgs triplet is the presence of the doubly charged Higgs. The doubly charged Higgs can decay to different states such as dileptons, gauge bosons, singly charged Higgs $H^+$. We discuss the doubly charged decay modes, specially the decay to the dileptonic mode and relate the doubly charged decay with the neutrino phenomenology. In our model the mixing between the two doubly charged Higgs is very closely related with the extent of the $\mu - \tau$ symmetry in the neutrino sector. In the exact $\mu - \tau$ limit the mixing angle $\theta$ between the two doubly charged Higgs is $\theta = \frac{\pi}{4}$, whereas mild breaking of the $\mu - \tau$ symmetry results in a mild deviation from $\theta \sim \frac{\pi}{4}$. This close connection between the $\mu - \tau$ symmetry and the doubly charged mixing angle significantly effects the doubly charged Higgs-dileptonic vertices. In the exact $\mu - \tau$ limit, the vertex factors $H_2^{++} - \mu - \tau$ and $H_2^{++} - e - e$ are zero, hence the doubly charged Higgs $H_2^{++}$ never decays to $\mu^+ + \tau^+$ or to $2e^+$ states. Other than this, the non observation of the $e\mu$ and $e\tau$ states in the dileptonic decay of the doubly charged Higgs would possibly disfavor the inverted mass hierarchy of the standard model neutrino. We study the phenomenological viability and testability of our model both in the exact as well as approximate $\mu - \tau$ symmetric cases. We have also very briefly commented about lepton flavor violation in our model. We give predictions for the mass squared differences, mixing angles, absolute neutrino mass scale, beta decay and neutrino-less double beta decay.

The chapter is organized as follows. In section 6.2 we introduce the particle content of our model and write down the mass matrices for the neutrinos and charged leptons. In section 6.3 we present the phenomenological implications of our model in the exact and approximate $\mu - \tau$ symmetric case. We discuss in detail the possible collider phenomenology and lepton flavor violating channels which could be used to provide smoking gun evidence for our model. Section 6.4 is devoted to justifying the alignment needed for the Higgs VEVs. We end in section 6.5 with our conclusions.

## 6.2 The Model

We present in Table 6.1 the particle content of our model and their transformation properties under the discrete groups $S_3$, $Z_4$ and $Z_3$. The Higgs $H$ is the usual SU(2)$_L$ doublet,

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}, \quad (6.1)$$
which transforms as singlet under $S_3$. The Higgs $\Delta_1$ and $\Delta_2$ are $SU(2)_L$ triplets with hypercharge $Y = +2$,

$$
\Delta_i = \begin{pmatrix}
\Delta_i^+/\sqrt{2} \\
\Delta_0^i \\
-\Delta_i^+/\sqrt{2}
\end{pmatrix},
$$

(6.2)

which transform as a doublet

$$
\Delta = \begin{pmatrix}
\Delta_1 \\
\Delta_2
\end{pmatrix},
$$

(6.3)

under $S_3$. Other than these fields, we also have two additional $S_3$ scalar doublets $\phi_e$ and $\xi$,

$$
\phi_e = \begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix},
\xi = \begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix},
$$

(6.4)

which are singlets under $SU(2)_L \times U_Y(1)$. The $SU(2)_L \times U_Y(1)$ lepton doublets are distributed in the $S_3$ multiplets as follows:

$$
D_l = \begin{pmatrix}
L_2 \\
L_3
\end{pmatrix},
$$

(6.5)

transforms as a doublet under $S_3$, where $L_2 = L_\mu = (\nu_{\mu L}, \mu_L)^T$ and $L_3 = L_\tau = (\nu_{\tau L}, \tau_L)^T$, while

$$
L_1 = L_e = \begin{pmatrix}
\nu_{e L} \\
\ell_{e L}
\end{pmatrix},
$$

(6.6)

transforms as a singlet. The right-handed fields $e_R$, $\mu_R$ and $\tau_R$ transform as 1 under $S_3$. The corresponding charges of the particles under $Z_4$ and $Z_3$ has been summarized in Table 6.1.

### 6.2.1 Neutrino Masses and Mixing

Given the field content of our model and their charge assignments presented in Table 6.1, the most general $S_3 \times Z_4 \times Z_3$ invariant Yukawa part of the Lagrangian (leading order) giving the neutrino mass can be written as

$$
-\mathcal{L}_\nu = \frac{y_2}{\Lambda} (D_l D_l) L_{\nu} (L_{\nu} L_{\nu})^T \Delta + \frac{y_1}{\Lambda} (D_l D_l) L_{\nu} \Delta + 2 y_3 l_1 D_l \Delta + \frac{y_4}{\Lambda} l_1 d_1 \phi_e + h.c. + ...$

(6.7)
where $\Lambda$ is the cut-off scale of the theory and the underline sign in the superscript represents the particular $S_3$ representation from the tensor product of the two $S_3$ doublets. Since $(D_l D_l)$ and $\xi \Delta$ are $2 \times 2$ products which could give either 1 or 2, and since we can obtain 1 either by $1 \times 1$ or $2 \times 2$, we have two terms coming from $(D_l D_l \xi \Delta)$. The $(D_l D_l)(\xi \Delta)$ as $1' \times 1'$ term does not contribute to the neutrino mass matrix. In this model the presence of the $Z_4$ symmetry prevents the appearance of the usual 5 dimensional $D_l D_l H H$ and $l_l l_l H H$ Majorana mass term for the neutrinos. In fact, the neutrino mass matrix is completely independent of $H$ due to the $Z_4$ symmetry. In addition, there are no Yukawa couplings involving the neutrinos and the flavon $\phi_e$ due to $Z_4$ or/and $Z_3$ symmetry. The $S_3$ symmetry is broken spontaneously when the flavon $\xi$ acquires a vacuum VEV:

$$\langle \xi \rangle = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$  

(6.8)

The $SU(2)_L \times U_Y(1)$ breaks at the electroweak scale by the VEV of the $SU(2)$ doublet Higgs $H$. The VEV of the Higgs triplet is

$$\langle \Delta \rangle = \begin{pmatrix} \langle \Delta_1 \rangle \\ \langle \Delta_2 \rangle \end{pmatrix}, \quad \text{where} \quad \langle \Delta_i \rangle = \begin{pmatrix} 0 \\ v_i \\ 0 \end{pmatrix}.$$  

(6.9)

The neutrino get masses due to the VEV of the Higgs triplet field $\Delta$ and as well as the standard model gauge singlet field $\xi$. The mass matrix of the neutrino is given as

$$m_\nu = \begin{pmatrix} 2 y_4 \frac{w}{\Lambda} & 2 y_3 v_2 & 2 y_3 v_1 \\ 2 y_2 \frac{w}{\Lambda} & 2 y_2 \frac{w}{\Lambda} & 2 y_1 \frac{w}{\Lambda} \\ 2 y_3 v_1 & 2 y_2 \frac{w}{\Lambda} & 2 y_1 \frac{w}{\Lambda} \end{pmatrix},$$  

(6.10)

where $w = u_1 v_2 + u_2 v_1$. For the VEV alignments

$$v_1 = v_2, \quad \text{and} \quad u_1 = u_2,$$  

(6.11)

the neutrino mass matrix reduces to the form

$$m_\nu = \begin{pmatrix} 2 y_4 \frac{w_{u_1}}{\Lambda} & 2 y_3 v_1 & 2 y_3 v_1 \\ 2 y_2 \frac{w_{u_1}}{\Lambda} & 2 y_2 \frac{w_{u_1}}{\Lambda} & 2 y_1 \frac{w_{u_1}}{\Lambda} \\ 2 y_3 v_1 & 2 y_2 \frac{w_{u_1}}{\Lambda} & 2 y_1 \frac{w_{u_1}}{\Lambda} \end{pmatrix}. $$  

(6.12)

We discuss about the VEV alignments in section 6.4. Denoting $\frac{w}{\Lambda}$ as $u_1'$ the mass matrix becomes

$$m_\nu = 2 v_1 \begin{pmatrix} 2 y_4 u_1' & y_3 & y_3 \\ y_3 & y_1 u_1' & 2 y_2 u_1' \\ y_3 & 2 y_2 u_1' & y_1 u_1' \end{pmatrix},$$  

(6.13)

The term $(l_i D_l \Delta)$ denotes $(l_i^T C \tau_2 D_l \Delta)$, where $C$ is the charge conjugation operator.
where \( u_1' = \frac{u_1}{\Lambda} \) and it is less than 1. Redefining \( 2y_4u_1' \) as \( y_4 \), \( y_1u_1' \) as \( y_1 \) and \( 2y_2u_1' \) as \( y_2 \), the final form of the mass matrix is

\[
\begin{pmatrix}
y_4 & y_3 & y_3 \\
y_3 & y_1 & y_2 \\
y_3 & y_2 & y_1
\end{pmatrix}.
\]

(6.14)

In the above matrix, for the VEV \( u_1 = 0 \), we would obtain the matrix

\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

(6.15)

This is a very well known form of the neutrino mass matrix which returns inverted neutrino mass spectrum with eigenvalues \( \{-2\sqrt{2}v_1y_3, 2\sqrt{2}v_1y_3, 0\} \), and bimaximal mixing with \( \theta_{23} = \theta_{12} = \pi/4 \) and \( \theta_{13} = 0 \). The family symmetry considered in the literature for obtaining the form of the mass matrix given by Eq. (6.15) is \( L_e - L_\mu - L_\tau \) [8]. However, exact bimaximal mixing is ruled out by the solar neutrino and KamLAND data [9]. Besides, as one can see from the eigenvalues of this neutrino mass matrix, that \( \Delta m^2_{21} = 0 \). This is untenable in the light of the experimental data. In order to generate the correct \( \Delta m^2_{21} \) and deviation of \( \theta_{12} \) from maximal that is consistent with the data, one has to suitably perturb \( m_\nu \) (for instance, in [10] the authors have build a model in the framework of the Zee-Wolfenstein ansatz). In the \( S_3 \) model that we consider here, this is very easily obtained if we allow non-zero VEV for \( \xi \). The strength of the additional terms is linearly proportional to \( u_1/\Lambda \) and could be relatively small.

In what follows, we will consider all values of \( u_1/\Lambda \) from very small to \( \sim 1 \). The eigenvalues of the most general matrix given by Eq. (6.14) are\(^2\)

\[
m_i = v_1 \left(y_1 + y_2 + y_4 - \sqrt{y_1^2 + y_2^2 + y_4^2 + 8y_3^2 + 2y_1y_2 - 2y_1y_4 - 2y_2y_4} \right), \\
m_j = v_1 \left(y_1 + y_2 + y_4 + \sqrt{y_1^2 + y_2^2 + y_4^2 + 8y_3^2 + 2y_1y_2 - 2y_1y_4 - 2y_2y_4} \right), \\
m_3 = 2v_1(y_1 - y_2).
\]

(6.16) (6.17) (6.18)

In the above, \( i, j = 1, 2 \) and the only difference between \( m_i \) and \( m_j \) comes in the sign of the quantity within square root. We know that the solar neutrino data provides evidence for \( \Delta m^2_{21} > 0 \) at more than 6\( \sigma \) [9]. Therefore, the choice of \( m_1 \) and \( m_2 \) in Eqs. (6.16) and (6.17) is determined by the condition \( m_2 > m_1 \text{ viz., the larger eigenvalue corresponds to} \)

\(^2\)For all analytical results given in this section we have assumed the model parameters to be real for simplicity. We check the phenomenological viability and testability of our model in the next section for complex Yukawa couplings.
The eigenvectors are given as

\[ U_i = \begin{pmatrix} -y_1 + y_2 - y_4 + \sqrt{a} \\ \frac{2y_1 b}{\sqrt{b}} \end{pmatrix}, \quad U_j = \begin{pmatrix} -y_1 + y_2 - y_4 - \sqrt{a} \\ \frac{2y_1 c}{\sqrt{c}} \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \] (6.19)

where \( U_i \) corresponds to the eigenvalue given in Eq. (6.16) and \( U_j \) to that in Eq. (6.17). Whether \( U_1 \equiv U_i \) or \( U_j \) depends on whether \( m_i \) is smaller or larger than \( m_j \). The quantities \( b \) and \( c \) are the normalization constants given by

\[ b^2 = 2 + \frac{(y_1 + y_2 - y_4 + \sqrt{a})^2}{(2y_3)^2}, \] (6.20)

and

\[ c^2 = 2 + \frac{(y_1 + y_2 - y_4 - \sqrt{a})^2}{(2y_3)^2}, \] (6.21)

and \( a \) is given as

\[ a = y_1^2 + y_2^2 + y_4^2 + 8y_3^2 + 2y_1y_2 - 2y_1y_4 - 2y_2y_4. \] (6.22)

From Eqs. (6.16), (6.17) and (6.18) we obtain

\[ \Delta m_{21}^2 = 4v_1^2(y_1 + y_2 + y_4)\sqrt{a}, \]
\[ \Delta m_{31}^2 = v_1^2(3y_1 - y_2 + y_4 - \sqrt{a})(y_1 - 3y_2 - y_4 + \sqrt{a}). \] (6.23)

The mixing angles can be seen from Eq. (6.19) to be

\[ \theta_{13}^{\nu} = 0, \]
\[ \tan \theta_{23}^{\nu} = 1, \]
\[ \tan \theta_{12}^{\nu} = \frac{(y_1 + y_2 - y_4 - \sqrt{a})b}{(y_1 + y_2 - y_4 + \sqrt{a})c}. \] (6.24)

In the above, neither the ratio of the two mass squared differences \( \Delta m_{21}^2/\Delta m_{31}^2 \), nor the mixing angles depend on the value of the triplet VEV \( v_1 \). They only depend on the Yukawa couplings. Only the absolute mass square differences \( \Delta m_{21}^2 \) and \( \Delta m_{31}^2 \) individually depend on the triplet VEV. The effective neutrino mass predicted for neutrino-less double beta decay is given as

\[ |m_{\nu_{ee}}| = |2v_1 y_4|, \] (6.25)

while the effective mass squared observable in beta decay \( m_{\beta}^2 \) and the total neutrino mass crucial for cosmology \( m_t \) are given as

\[ m_{\beta}^2 = \sum_i |m_i|^2 |U_{ei}|^2, \quad \text{and} \quad m_t = \sum_i |m_i|, \] (6.26)

respectively.
6.2.2 Charged Lepton Masses and Mixing

In this section we discuss the charged lepton masses. The Yukawa Lagrangian up to order $1/\Lambda^3$ giving the charged lepton mass is

$$-L_y^e = \frac{\gamma}{\Lambda} \overline{\tau} R H^\dagger (D_l \phi_e) + \frac{\gamma b}{\Lambda^3} \overline{\tau} R H^\dagger (D_l \phi_e)^2 (\xi \xi)^2 + \frac{\gamma'}{\Lambda^3} \overline{\tau} R H^\dagger l_1 (\phi_e \xi)$$

$$+ \frac{\beta'}{\Lambda^2} \overline{\tau} R H^\dagger (D_l \phi_e) + \frac{\beta''}{\Lambda^3} \overline{\tau} R H^\dagger l_1 (\phi_e \phi_e \xi) + \frac{\alpha''}{\Lambda^3} \overline{\tau} R H^\dagger l_1 (\phi_e \phi_e)$$

$$+ \frac{\alpha'}{\Lambda^3} \overline{\tau} R H^\dagger (D_l \phi_e') (\phi_e' \phi_e') \frac{1}{2} + \frac{\alpha}{\Lambda^3} \overline{\tau} R H^\dagger (D_l \phi_e')^2 (\phi_e' \phi_e')^2 + h.c + ... \quad (6.27)$$

While $Z_4$ symmetry was sufficient to get the desired $m_\nu$, the extra $Z_3$ symmetry had to be introduced in order to obtain the correct form for the charged lepton mass matrix. The presence of the $Z_4$ symmetry ensures that the flavon doublet $\phi_e$ couples to charged leptons only. This is a prerequisite since we wish to break $S_3$ such that the $\mu-\tau$ symmetry remains intact for the neutrinos while it gets maximally broken for the charged leptons.

For neutrinos the $\mu-\tau$ symmetry was kept intact by the choice of the vacuum alignments given in Eq. (6.11). To break it maximally for the charged leptons we choose the VEV alignment \[7\]

$$\langle \phi_e \rangle = \left( \begin{array}{c} v_c \\ 0 \end{array} \right). \quad (6.28)$$

Once $S_3$ is spontaneously broken by the VEVs of the flavons and SU(2)$_L \times$ U(1)$_Y$ by the VEV of the standard model doublet Higgs, we obtain the charged lepton mass matrix (leading terms only)$^3$

$$m_{cl} = \begin{pmatrix} \alpha'' \lambda^2 & 0 & 0 \\ \beta'' \lambda u_1' & \beta' \lambda & 0 \\ \gamma'' u_2' & \gamma' u_2'^2 & \gamma \end{pmatrix} v \lambda, \quad (6.29)$$

where $v = \langle H \rangle$ is the VEV of the standard Higgs, $\lambda = v_c/\Lambda$, $u_1' = u_1/\Lambda$ and $u_2' = u_2/\Lambda$. The charged lepton masses and mixing matrix are obtained from

$$m_{cl}^2 = U_l m_l m_l^\dagger U_l^\dagger, \quad (6.30)$$

giving the masses as

$$m_\tau \simeq \gamma \lambda v, \quad m_\mu \simeq \beta' \lambda^2 v, \quad m_e \simeq \alpha'' \lambda^3 v. \quad (6.31)$$

$^3$While this form of charged lepton mass matrix has been obtained using $Z_4 \times Z_3$ symmetry, similar viable forms can be obtained using other $Z_n$ symmetries. For example, we have explicitly checked that $Z_6 \times Z_2$ and $Z_8 \times Z_2$ symmetries also give viable structure for $m_{cl}$. 

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For $\lambda \simeq 2 \times 10^{-2} \simeq \frac{\lambda^2}{2}$ where $\lambda_c$ is the Cabibbo angle, the correct mass hierarchy of $\tau$ to $\mu$ to $e$ as well as their exact numerical values can be obtained by choosing $\gamma = 0.36$, $\beta' = 1.01$ and $\alpha'' = 0.25$ and $v \sim 10^2$ GeV. In this present scenario the masses of the charged leptons do not depend on the VEV of $\xi$, however the mixing angles involved depend on this parameter. We have chosen $u'_1 = u'_2 \simeq \mathcal{O}(10^{-1})$, which is justified from large $y_1$ and $y_2$ in Fig.6.2 and large $y_4$ in Fig.6.3 and the perturbative nature of the Yukawa couplings. The large redefined $y_1$ and $y_2$ in Fig.6.2 and large redefined $y_4$ in Fig.6.3 suggest that $u'_{1,2}$ should not be much less than unity, which leads to the natural choice $u'_{1,2} \sim \mathcal{O}(1)$.

Since $U = U_l^\dagger U_\nu$, where $U$ is the observed lepton mixing matrix and $U_\nu$ is the matrix which diagonalizes $m_\nu$ given by Eq. (6.12), the contribution of charged lepton mixing matrix would be very tiny. For $\sin \theta_{13}$ the maximum contribution from the charged lepton is $\mathcal{O}(10^{-4})$. In any case, in what follows we show all results for $U = U_l^\dagger U_\nu$.

### 6.3 Phenomenology

#### 6.3.1 Exact $\mu - \tau$ Symmetry Limit

We have already presented in Eqs. (6.23) and (6.31) the expressions for the neutrino mass squared differences and charged lepton masses, while in Eqs. (6.24) and (6.32) we have given the mixing angles in the lepton sector in terms of model parameters. We argued that for $\lambda \simeq 2 \times 10^{-2} \simeq \frac{\lambda^2}{2}$, we obtain the charged lepton mass hierarchy in the right ballpark.

Since neutrino masses are directly proportional to $v_1$, it is phenomenologically demanded that the magnitude of this VEV should be small. In fact, since $\Delta m^2_{31} \propto v_1^2$, we take $v_1^2 \sim 10^{-4} - 10^{-3}$ eV$^2$, and find that all experimentally observed neutrino masses and mixing constraints are satisfied. It is not unnatural to expect such a small value for $v_1^2$. For instance, in the most generic left-right symmetric models,

$$v_1 \equiv v_L \sim v^2/v_R,$$

where $v$ is the electroweak scale and $v_R$ is the VEV of the SU(2)$_R$ Higgs triplet. It is natural to take $v_R \sim 10^{13} - 10^{15}$ GeV for which we get $v_1 \sim 1 - 10^{-2}$ eV.

Allowing $v_1^2$ to take any random value between $10^{-4} - 10^{-3}$ eV$^2$ we have checked that the neutrino mass spectrum obtained is hierarchical. For larger values of $v_1$ of course one would get larger values for the absolute neutrino mass scale and for $v_1 \sim 1$ eV, we expect a quasi-degenerate neutrino mass spectrum. In all our plots we keep $v_1^2$ between
Figure 6.1: Scatter plots showing the range of the solar mixing angle $\sin^2 \theta_{12}$ as a function of the model parameters in $m_\nu$ in the exact $\mu - \tau$ symmetry limit and for normal hierarchy. In each of the panels all other parameters except the one appearing in the $x$-axis are allowed to vary freely.

$10^{-4} - 10^{-3}$ eV$^2$. In Figure 6.1 we show the prediction for $\sin^2 \theta_{12}$ as a function of the model parameters $y_i$'s. In each panel we show the dependence of $\sin^2 \theta_{12}$ on a given $y_i$, allowing all the others to vary randomly. Here we have assumed normal mass hierarchy for the neutrinos. For the charged lepton sector we have assumed a fixed set of model parameters which give viable charged lepton masses and we took $\lambda \simeq 2 \times 10^{-2}$. We note that for normal hierarchy ($\Delta m^2_{31} > 0$):

- $y_1 = 0$ and $y_2 = 0$ are not allowed.
- There is almost negligible dependence of $\sin^2 \theta_{12}$ on $y_1$ and $y_2$ for $|y_1| > 1$ and $|y_2| > 1$ respectively.
- The range of $\sin^2 \theta_{12}$ decreases with $|y_3|$ and $|y_4|$.

Figure 6.2 gives the scatter plots showing allowed ranges for the model parameters in two-dimensional parameter spaces, taking two parameters at a time and allowing the
rest to vary freely. We have considered normal hierarchy in this figure. We note from the
figure that for normal hierarchy:

- $y_1 = 0$ and $y_2 = 0$ are not allowed as we had seen before. With $y_1 = 0$ we would
have obtained neutrino mass matrix Eq.(6.14) with two texture zeros in $\mu - \mu$ and
$\tau - \tau$ elements and with $e - \mu$ and $e - \tau$ entries same in the mass matrix it will not
be possible to get a normal-hierarchy [11]. However, as we will see from Figure 6.3
inverted hierarchy can occur in this case. With $y_2 = 0$ one gets neutrino mass matrix
with one texture zero in $\mu - \tau$ element which is not viable for normal ordering [12].
The allowed values of $y_1$ for normal hierarchy are highly correlated with the allowed
values of $y_2$ and they are necessarily of opposite signs. One obtains a rough linear
dependence between the allowed values of $y_1$ and $y_2$.

- $y_4 = 0$ is allowed and there is very little correlation of allowed values of $y_4$ with $y_1$
and $y_2$. For $y_4 = 0$, one gets a neutrino mass matrix Eq.(6.14) with one texture
zero in $e - e$ element which can produce normal hierarchy only [12]. This predicts
$|m_{\nu ee}| = 0$.

- $y_3$ and $y_4$ are strongly correlated.

In Figure 6.3 we show the corresponding allowed ranges for the model parameters for inverted hierarchy. In each of the panels, the parameters that do not appear on the $x$ and $y$-axes are allowed to vary randomly. From a comparison of Figures 6.2 and 6.3 we can observe that the allowed areas in the parameter space is almost complementary\(^4\). We find that for the inverted hierarchy:

- $y_1 = 0$ and $y_2 = 0$ *simultaneously* are still not allowed, though now we can have $y_1 = 0$ or $y_2 = 0$ separately when the other parameter is within a certain favorable (non-zero) range. As for normal hierarchy, allowed values of $y_1$ and $y_2$ are highly correlated. As before there is a linear dependence between them.

\(^4\)Of course the same set of model parameter values would never give both normal and inverted hierarchy simultaneously. However, we have shown two-dimensional projections of the model parameter space and hence their could be few overlapping points in the two figures.
• $y_3 = 0$ and $y_4 = 0$ are not allowed here.

In the left panel of Figure 6.4 we show the variation of the effective neutrino mass $m_{\nu_{ee}}$ with the model parameter $y_4$. The effective mass predicted for neutrino-less double beta decay in our model is $|m_{\nu_{ee}}| = |2v_1 y_4|$. We have allowed $v_1$ to vary freely in the range $10^{-1} - 10^{-2}$. From Figure 6.4 one can clearly see that our model predicts $|m_{\nu_{ee}}| \lesssim 0.07$ eV. The next generation of neutrino-less double experiments are expected to probe down to $|m_{\nu_{ee}}| = 0.01 - 0.05$ eV [13]. The middle panel of this figure shows the total predicted neutrino mass $m_t$ and right panel shows $m^2_\beta$. We find that the total neutrino mass $m_t$ (in eV) varies within the range $0.05 < m_t < 0.28$, while the effective mass squared observable in beta decay $m^2_\beta \simeq O(10^{-4} - 10^{-2}) eV^2$. The KATRIN experiment will be sensitive to $m_\beta > 0.3$ eV [14].

### 6.3.2 Mildly Broken $\mu - \tau$ Symmetry Limit

So far we have assumed that the $S_3$ breaking in the neutrino sector is such that the residual $\mu - \tau$ symmetry is exact. This was motivated by the fact that $S_2$ is a subgroup of $S_3$ and we took a particular VEV alignment given in Eq. (6.11). We will discuss about VEV alignment in section 6.4. In this subsection we will assume that the $\mu - \tau$ symmetry is mildly broken. This could come from explicit $\mu - \tau$ breaking terms in the Lagrangian. In the next section we will see that in our model this comes after the minimization of the scalar potential due to the deviation of the VEV alignments from that given in Eq. (6.11). Small breaking of the VEV alignments could also come from radiative corrections and/or higher order terms in the scalar part of the Lagrangian. Any breaking of $\mu - \tau$
symmetry will allow $\theta_{23}$ to deviate from maximal and $\theta_{13}$ from zero. Any non-zero $\theta_{13}$ will open up the possibility of low energy CP violation in the lepton sector. For the sake of illustration we consider a particular $\mu - \tau$ symmetry breaking for $m_\nu$ which results from the deviation of the VEV alignment from Eq. (6.11). We will see in the next section that this deviation is small and could come from $v_1 \neq v_2$ and/or $u_1 \neq u_2$. For the sake of illustration we consider only the breaking due to $v_1 \neq v_2$. We will see that from the minimization of the scalar potential one can take $v_1 = v_2(1 + \epsilon)$. As a result the neutrino mass matrix (6.10) becomes

$$
    m_\nu = 2v_2 \begin{pmatrix}
        y_4 u'(2 + \epsilon) & y_3 & y_3(1 + \epsilon) \\
        y_3 & y_1 u' & y_2 u'(2 + \epsilon) \\
        y_3(1 + \epsilon) & y_2 u'(2 + \epsilon) & y_1 u'(1 + \epsilon)
    \end{pmatrix}.
$$

(6.34)
We show the values of $|U_{e3}| \equiv \sin \theta_{13}$ predicted by the above $m_\nu$ as a function of the symmetry breaking parameter $|\epsilon|$ in the right panel of Figure 6.5. We have considered complex Yukawa coupling in this case. The left panel panel of this figure shows the Jarlskog invariant

$$J_{CP} = \text{Im} \left\{ U_{e1} U_{\nu2} U_{e2}^* U_{\mu1}^* \right\},$$

(6.35)

as a function of $|\epsilon|$. We note that the model predicts values of $\sin \theta_{13} \lesssim 10^{-1}$ and $J_{CP} \lesssim 10^{-2}$ with the exact value determined by the extent of symmetry breaking. This could give $\sin^2 2\theta_{13} \lesssim 0.04$, which is just within the sensitivity reach of the forthcoming reactor experiments like Double Chooz [15] and long baseline accelerator experiments like T2K [16] and NO$\nu$A [17]. These values of $\theta_{13}$ and $J_{CP}$ could give a large positive signal in the next generation high performance long baseline experiments using neutrino beams from Neutrino Factories, Superbeams and Beta-beams [18].

### 6.3.3 Collider Signature and Lepton Flavor Violation

Recent discussion on the analysis of the scalar potential and the Higgs mass spectrum for models with one triplet Higgs can be found in [19]. In our model with two Higgs triplets we get mixing between the two doubly charged Higgs $\Delta_1^{++}$ and $\Delta_2^{++}$. The physical Higgs fields can be obtained from the scalar potential and are given by

$$H_1^{++} = \Delta_1^{++} \cos \theta + \Delta_2^{++} \sin \theta,$$

$$H_2^{++} = -\Delta_1^{++} \sin \theta + \Delta_2^{++} \cos \theta,$$

(6.36)

where the mixing angle $\theta$ is

$$\tan 2\theta = \frac{e_2 (u_1^2 + u_2^2) + e'_2 u_1 u_2}{h_6 (u_2^2 - u_1^2) - h_6 |v_c|^2}.$$  

(6.37)

The parameters $e_2$, $e'_2$, $h_6$ and $h'_6$ are defined in Eq. (6.48). In the exact $\mu - \tau$ limit, which can be realized by setting $h_6 = 0$ and $h'_6 = 0$,

5 We will see this in the next section.

6 $v_c^2/\Lambda^2 \sim 10^{-4}$ from charged lepton masses and from neutrino phenomenology $u_1^2/\Lambda^2 \sim 10^{-2}$, hence

$v_c^2 \ll u_1^2, u_2^2$.

7 from neutrino phenomenology. Also see next section.

In the approximate limit where we take $u_1 \simeq u_2 = u$ and neglect $|v_1|^2$, $|v_c|^2$ and $v^2$ in comparison to $u^2$, the square of the masses of the doubly charged Higgs are given by

$$M_{H_1^{++}}^2 \simeq -a + u^2 \left[ (4e_1 + 2e'_1) - (2e_2 + e'_2) \right],$$

(6.38)

$$M_{H_2^{++}}^2 \simeq -a + u^2 \left[ (4e_1 + 2e'_1) + (2e_2 + e'_2) \right].$$

(6.39)
The quantities \( e_1, e'_1, e_2 \) and \( e'_2 \) are dimensionless coefficients in the scalar potential and will be explained in the next section. We will see in the next section that \( a \) in Eqs. (6.38) and (6.39) has a mass dimension 2 and comes as the co-efficient of the \( \Delta^{++}_{1,2} \Delta^{--}_{1,2} \) term in the scalar potential. We note that the masses are modified due the mixing between \( \Delta^{++}_{1} \) and \( \Delta^{++}_{2} \), and depend on the VEV \( u \). The difference between the square of the masses of the two doubly charged Higgs depends only on \( u \) and the coefficients \( e_2 \) and \( e'_2 \). Measuring this mass squared difference at a collider experiment will provide a handle on the VEV \( u \), which could then be used in conjunction with the lepton mass and mixing data to constrain the new scale \( \Lambda \).

In the most natural limit where we take all coupling constants \( e_1, e'_1, e_2 \) and \( e'_2 \) to be of the same order, then \( M_{H^{++}_{1,2}}^2 \simeq -a \) and \( M_{H^{++}_{2}}^2 \simeq -a + 9 e_1 u^2 \).

The most distinctive signature of the existence of triplet Higgs can be obtained in collider experiments, through the production and subsequent decay of the doubly charged Higgs particle(s) [20–22]. The doubly charged Higgs, if produced, would decay through the following possible channels:

\[
H^{++} \to H^+ H^+, \\
H^{++} \to H^+ W^+, \\
H^{++} \to l^+ l^+, \\
H^{++} \to W^+ W^+.
\]

(6.40)

In our model we have two doubly charged and two singly charged Higgs, however we have suppressed the corresponding indices in Eq. (6.40). Likewise, we have suppressed the flavor indices of the leptons. The first two decay modes depend on the mass difference between the singly and doubly charged Higgs and hence might be kinematically suppressed compared to the last two channels. We therefore do not consider them any further. The decay rate \( H^{++}_{1,2} \to W^+ W^+ \) is proportional to the square of triplet Higgs VEVs \( v_{1,2} \), while the decay rate to dileptons is inversely proportional to them. As a result the ratio of the decay rates for the two channels is proportional to \( v_1^4 \) and is given as [21]

\[
\frac{\Gamma(H^{++}_{1,2} \to l^+_a l^+_b)}{\Gamma(H^{++}_{1,2} \to W^+ W^+)} \approx \left( \frac{m_\nu}{M_{H^{++}_{1,2}}} \right)^2 \left( \frac{v}{v_{1,2}} \right)^4,
\]

(6.41)

where \( M_{H^{++}_{1,2}} \) is the mass of the doubly charged Higgs and \( m_\nu \) is the scale of neutrino mass. It has been shown [21] from a detailed calculation that for \( M_{H^{++}_{1,2}} \simeq 300 \text{ GeV} \) and \( v_{1,2} \lesssim 10^{-4} \text{ GeV} \), decay to dileptons will dominate. For our model \( v_1^2 \approx v_2^2 \approx 10^{-3} - 10^{-4} \text{ eV}^2 \) and hence we can safely neglect decays to \( W^+ W^+ \). The decay rate to dileptons is given as [20–22]

\[
\Gamma(H^{++}_{1,2} \to l^+_a l^+_b) = \frac{1}{4\pi(1 + \delta_{ab})} |F_{ab}|^2 M_{H^{++}_{1,2}}^2,
\]

(6.42)
while the branching ratio for this decay mode is

$$BR_{ab} = BR(H_{1,2}^{++} \rightarrow l_a^+ l_b^+) = \frac{2}{(1 + \delta_{ab}) \Sigma_{ab} |F_{ab}|^2},$$

(6.43)

where $F_{ab}$ are the relevant vertex factors which directly depend on the form of the neutrino mass matrix. Using Eq. (6.7), we have tabulated in Table 6.2 the vertex factors for all possible interaction channels in our model. We see that apart from $e-\mu$ or $e-\tau$ combinations given in the table, all vertices have extra suppression factor of $\frac{u_{1,2}}{\Lambda}$. All other vertices arising from Eqs. (6.7) and (6.27) will involve the flavon fields $\xi$ and/or $\phi$ and will be suppressed by higher orders in $\Lambda$. We therefore do not give them here. We had argued from Eq. (6.37) that in the exact $\mu-\tau$ symmetric limit $\theta = \pi/4$. One can then immediately see from Table 6.2 that $H_{2}^{++}ee$ and $H_{2}^{++}\mu\tau$ couplings are zero. Therefore, in the exact $\mu-\tau$ symmetric limit, the decay of $H_{2}^{++}$ to $ee$ and $\mu\tau$ is strictly forbidden. We have noted above that even when we do not impose exact $\mu-\tau$ symmetry, $\theta \approx \pi/4$ and hence these decay channels will be suppressed. The branching ratio of all the other decay modes are determined by the corresponding Yukawa couplings. Generally speaking, since all the vertices other than $H_{1,2}^{++}e\mu$ and $H_{1,2}^{++}e\tau$ are $\frac{u_{1,2}}{\Lambda}$ suppressed, branching ratio of these channels will be larger than all others, assuming equal values of $y_1, y_2, y_3$ and $y_4$. However, $y_3$ and $y_4$ could be small for normal hierarchy while inverted hierarchy could be produced for very small $y_1$ and $y_2$. This will give a handle on determining the neutrino parameters in general and the neutrino mass hierarchy in particular [22]. For instance, if the decay modes of doubly charged Higgs to $e\mu$ and $e\tau$ are not observed at a collider experiment, then it would imply small $y_3$, which would disfavor the inverted hierarchy.

Signature of doubly charged Higgs could in principle also be seen in lepton flavor violating processes. However, in the framework of our model the additional contribution to $l_i \rightarrow l_j \gamma$ are smaller than what is expected in the standard model. One can check from Table 6.2 that the only additional diagram which does not have any $\Lambda$ (or $u_{1,2}/\Lambda$) suppression contributes to $\tau \rightarrow \mu\gamma$. However, even this diagram will be suppressed due to $M_{H_{1,2}^{++}} \gg M_W$ [23]. The presence of $H_{1,2}^{++}$ will allow the decay modes of the form $l_i \rightarrow l_i j k$ at the tree level, where $l_i, l_j$ and $l_k$ are leptons of any flavor. The branching ratios for $\mu \rightarrow eee$ and $\tau \rightarrow eee$ in our model for exact $\mu-\tau$ symmetry is given by [24]

$$BR(\mu \rightarrow eee) \approx \frac{1}{16 G_F^2 \Lambda^2} \frac{u_{1}^2 |y_2^* y_3|^2}{M_{H_{1}^{++}}^4}.$$  

(6.44)

Thus we see that even this process is suppressed by $u_{1}^2/\Lambda^2$ compared to other models with triplet Higgs. Branching ratio for all other lepton flavor violating decay modes such as $\tau \rightarrow \mu\mu\mu$ are further suppressed. The only decay mode which comes unsuppressed is $\tau \rightarrow e\mu$, for which the branching ratio is given by

$$BR(\tau \rightarrow e\mu) \approx \frac{1}{4 G_F^2} \frac{|y_3|^4}{M_{H_{1,2}^{++}}^4}.$$  

(6.45)
Table 6.2: Doubly charged Higgs triplet and lepton vertices and the corresponding vertex factors $F_{ab}$, where $a$ and $b$ are generation indices. The charged lepton mass matrix is almost diagonal in our model. In this analysis we have considered that mass basis and flavor basis of the charged leptons are the same.

The current experimental constraint on this decay mode is $BR(\tau \rightarrow ee\mu) < 2 \times 10^{-7}$ [25], which constrains our model parameter $y_3$ as (assuming $M_{H^{++1,2}} \sim 300$ GeV)

$$|y_3| \lesssim 10^{-1}.$$  (6.46)

In our model $y_3$ is predicted to be large for the inverted hierarchy while it could be tiny for the normal hierarchy. On the face of it then it appears that the bound given by Eq. (6.46) disfavors the inverted hierarchy for our model. However, recall that the allowed values of $y_3$ shown in Figs. 6.2 and 6.3 were presented assuming $v_1^2$ to lie between $10^{-3} - 10^{-4}$ eV$^2$. However, $v_1^2$ could be higher and since what determines the mass squared differences $\Delta m_{21}^2$ and $\Delta m_{31}^2$ is the product of $v_1^2$ and the Yukawas, higher $v_1^2$ would imply smaller values of the latter. For instance, we could have taken $v_1^2 \sim 10^{-1} - 10^{-2}$ eV$^2$ and in that case inverted hierarchy would still be allowed. The bound given by Eq. (6.46) has been obtained assuming $M_{H^{++1,2}} \sim 300$ GeV. For more massive doubly charged Higgs the branching ratio would go down. On the other hand, if one uses bounds from lepton flavor violating decays to constrain the Yukawas, then one would obtain corresponding limits on the value of $v_1$. We conclude that with improved bounds on lepton flavor violating decay modes, one could test our model and/or the neutrino mass hierarchy predicted by our model.
6.4 The Vacuum Expectation Values

In this section we discuss about the necessary conditions which have to be satisfied, in order to achieve the VEV alignments required for \( \mu - \tau \) symmetry. Up to terms of dimension four, the \( S_3 \times Z_4 \times Z_3 \) invariant scalar potential (cf. Table 6.1) is given by

\[
V = \sum_i V_i
\]  

(6.47)

where

\[
V_1 = -a Tr[\Delta' \Delta] + b (Tr[\Delta' \Delta])^2
\]

\[
V_2^a = [-c(\xi \xi) + h.c] + \xi' (\xi \xi)
\]

\[
V_2^b = [d(\xi' \xi')(\xi' \xi') + h.c] + d'(\xi' \xi')^2(\xi' \xi') + [d''(\xi' \xi')^2(\xi' \xi') + h.c]
\]

\[
V_3^a = [e_1 Tr[(\Delta' \Delta)\xi] + h.c] + e'_1 Tr[(\Delta' \Delta)\xi'] + e''_1 Tr[(\Delta' \Delta)\xi]\]

\[
V_3^b = [e_2 Tr[(\Delta' \Delta)\xi] + h.c] + e'_2 Tr[(\Delta' \Delta)\xi'] + e''_2 Tr[(\Delta' \Delta)\xi]\]

\[
V_3^c = h_6 Tr[\Delta' \Delta|^2 (\xi' \xi') + h_7 (\xi' \xi')^2 (\phi_e \phi_e)]
\]

\[
V_4 = f_1 Tr[(\Delta' \Delta)[\Delta' \Delta] + f_2 Tr[(\Delta' \Delta)[\Delta' \Delta] + f_3 Tr[(\Delta' \Delta)[\Delta' \Delta] + f_5 Tr[(\Delta' \Delta)[\Delta' \Delta] + f_6 Tr[(\Delta' \Delta)[\Delta' \Delta] + f_7 Tr[(\Delta' \Delta)[\Delta' \Delta] + f_8 Tr[(\Delta' \Delta)[\Delta' \Delta]
\]

\[
V_5 = -h_1(\phi_1' \phi_1') + h_2(\phi_1' \phi_1') + h_3(\phi_1' \phi_1')^2
\]

\[
V_6 = h_4 Tr[\Delta' \Delta]\xi(\xi \xi) + h_5 Tr[\Delta' \Delta]\xi(\xi \xi) + h_6 Tr[\Delta' \Delta]\xi(\xi \xi)
\]

\[
V_7^a = [l_1(\xi \xi)(\phi_e \phi_e) + h.c] + l'_1(\xi \xi)(\phi_e \phi_e)
\]

\[
V_7^b = [l_2(\xi' \xi')(\phi_e \phi_e) + h.c] + l'_2(\xi' \xi')(\phi_e \phi_e)
\]

\[
V_8 = a_1 Tr[\Delta' \Delta](H^1 H) + [a_2 (H^1 H)(\xi \xi) + h.c] - \mu^2 (H^1 H) + \lambda (H^1 H)^2
\]

\[
+ r (H^1 \tau H) Tr[\Delta' \tau \Delta] + a^*_2 (H^1 H)(\xi \xi)
\]

(6.48)

The underline sign in the superscript represents the particular \( S_3 \) representation from the tensor product of the two \( S_3 \) doublets. The superscripts “2” without the underline represent the square of the term. The quantities with primes are obtained following Eq. (2.55)

\[
\xi' = \sigma_1(\xi)^\dagger = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}, \quad \phi_e' = \sigma_1(\phi_e)^\dagger = \begin{pmatrix} \phi_e^1 \\ \phi_e^2 \end{pmatrix}, \quad \Delta' = \sigma_1(\Delta)^\dagger = \begin{pmatrix} \Delta^1 \\ \Delta^2 \end{pmatrix}
\]

(6.49)

The potential given by Eqs. (6.47) and (6.48) has to be minimized. The singlets \( \xi \) and \( \phi_e \) pick up VEVs which spontaneously breaks the \( S_3 \) symmetry at some high scale, while \( \Delta \) picks up a VEV when \( SU(2)_L \times U(1)_Y \) is broken at the electroweak scale. The VEVs have already been given in Eqs. (6.8), (6.9), and (6.28). For the sake of keeping the algebra simple we take the VEVs of \( \Delta \) and \( \phi_e \) to be complex but the VEVs of \( \xi \) to be real.

We denote \( v_1 = |v_1| e^{i\alpha_1}, v_2 = |v_2| e^{i\alpha_2}, \) where \( v_1 \) and \( v_2 \) are the VEVs of \( \Delta_1 \) and \( \Delta_2 \). Substituting this in Eqs. (6.47) and (6.48) we obtain

\[
V = (-a + 4e_1 u_1 u_2 + e_1'(u_1^2 + u_2^2) + h_4|v_e|^2 + a_1 v^2)(|v_2|^2 + |v_1|^2) + (b + f_1 + f_2)(|v_2|^2 + |v_1|^2)^2
\]

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From Eq. (6.51) we obtain the condition,

\[-4cu_1u_2 + c'(u_1^2 + u_2^2) + 8du_1^2u_2^2 + d'(u_4^2 + u_4^3) + 4d''u_1u_2(u_1^2 + u_2^2) + 2(f_3 - 2f_2)|v_1|^2v_2|^2 + 2|v_1||v_2|[e_2(u_1^2 + u_2^2) + e'_2u_1u_2] \cos(\alpha_2 - \alpha_1) + (-h_1 + 4t_1u_1u_2)|v_1|^2 + t''_1|v_1|^2(u_1^2 + u_2^2) + h_3|v_1|^4 + 4a_2v^2u_1u_2 + [-h_6|v_1|^2 + h'_6(u_1^2 - u_1^3)](|v_2|^2 - |v_1|^2) - h''_6(u_1^2 - u_1^3)|v_1|^2 + a_2''v^2(u_1^2 + u_2^2) + t_4v^2|v_1|^2 - \mu^2 v^2 + \lambda v^4\]  

(6.50)

where we have absorbed \(r\) in the redefined \(a_1\). The minimization conditions are:

\[\frac{\partial V}{\partial (\alpha_2 - \alpha_1)} = 0 ,\]  

(6.51)

\[\frac{\partial V}{\partial (|v_1|)} = 0 ,\]  

(6.52)

\[\frac{\partial V}{\partial (|v_2|)} = 0 ,\]  

(6.53)

\[\frac{\partial V}{\partial u_1} = 0 ,\]  

(6.54)

\[\frac{\partial V}{\partial u_2} = 0 ,\]  

(6.55)

\[\frac{\partial V}{\partial |v_1|} = 0 .\]  

(6.56)

From Eq. (6.51) we obtain the condition,

\[2|v_1||v_2|[e_2(u_1^2 + u_2^2) + e'_2u_1u_2] \sin(\alpha_2 - \alpha_1) = 0 .\]  

(6.57)

Hence

\[\alpha_2 = \alpha_1 ,\]  

(6.58)

as long as \(|v_1|, |v_2|\) and \([e_2(u_1^2 + u_2^2) + e'_2u_1u_2]\) are \(\neq 0\). Eq. (6.52) leads to the condition,

\[-2a|v_1| + 4B(|v_2|^2 + |v_1|^2)|v_1| + 2|v_1|[4e_1u_1u_2 + e'_1(u_1^2 + u_2^2)] + 2|v_2|[e_2(u_2^2 + u_1^2) + e'_2u_1u_2] + 4F|v_1||v_2|^2 + 2h_4|v_1||v_2|^2 + 2a_1|v_1|^2 + 2h_6|v_1|^2|v_1| - 2h'_6|v_1|(u_2^2 - u_1^2) = 0 ,\]  

(6.59)
where we have defined \( B = (b + f_1 + f_2) \), \( F = f_3 - 2f_2 \) and we have used \( \alpha_2 = \alpha_1 \). Using Eq. (6.53) we obtain,

\[
\begin{align*}
-2u|v_2| + 4B|v_2|^2 + |v_1|^2|v_2| + 2|v_2|[4e_1u_1u_2 + e_1'(u_1^2 + u_2^2)] + 2|v_1|\left[e_2(u_2^2 + u_1^2) + e_2' u_1u_2\right] \\
+ 4F|v_2||v_1|^2 + 2h_4|v_2||v_c|^2 + 2u_1|v_2|v_2 - 2h_6|v_c|^2|v_2| + 2h_6'|v_1|(u_2^2 - u_1^2) = 0.
\end{align*}
\]

(6.60)

Multiplying Eq. (6.59) by \(|v_2|\) and Eq. (6.60) by \(|v_1|\) and subtracting one from the other we obtain,

\[
(2e_2(u_1^2 + u_2^2) + 2e_2' u_1u_2 + 4F|v_1||v_2|)(|v_1|^2 - |v_2|^2) = 4|v_1||v_2|[h_6|v_c|^2 - h_6'(u_2^2 - u_1^2)]
\]

(6.61)

In the limit \( h_6 = 0 \) and \( h_6' = 0 \) we get \(|v_2| = |v_1|\) (if \( e_2', e_2 \) and \( F \neq 0 \) simultaneously), which is required for exact \( \mu-\tau \) symmetry in the neutrino sector. However, there is no a priori reason to assume that \( h_6 \), and \( h_6' \) are zero. In the most general case keeping non-zero \( h_6 \) and \( h_6' \), we obtain

\[
|v_1|^2 = |v_2|^2 + \frac{4|v_1||v_2|[h_6|v_c|^2 - h_6'(u_2^2 - u_1^2)]}{2e_2(u_1^2 + u_2^2) + 2e_2' u_1u_2 + 4F|v_1||v_2|}.
\]

(6.62)

Since \(|v_1||v_2| \ll u_1u_2 \) and \((u_1^2 + u_2^2)\) we neglect the \( 4F|v_1||v_2| \) term from the denominator. For \( u_1 \simeq u_2 = u \) and \( e_2 \simeq e_2' \), one obtains

\[
|v_1|^2 = |v_2|^2 + \frac{2|v_1||v_2|h_6|v_c|^2}{3e_2 u^2}.
\]

(6.63)

For a fixed \( v_2 \), this is a quadratic equation in \( v_1 \) which allows the solution \( v_1 \simeq v_2(1 + \epsilon) \) where \( \epsilon = \frac{h_6|v_c|^2}{3e_2 u^2} \). For \( h_6 \) and \( e_2 \) of the same order and \( \frac{\Lambda}{\mu} = 10^{-1}, \frac{\Lambda}{\Lambda} = 10^{-2} \) we obtain \( \epsilon \simeq 10^{-2} \ll 1 \). This would give rise to a very mild breaking of the \( \mu-\tau \) symmetry. We have discussed this case in section 6.3.2.

Using Eqs. (6.54) and (6.55) and repeating the same exercise we get the deviation from \( u_1 = u_2 \) as

\[
u^2_1 = u^2_2 + \frac{A}{B}
\]

(6.64)

where \( A \) and \( B \) are

\[
A = 4u_1u_2[h_6''|v_c|^2 - h_6'(|v_2|^2 - |v_1|^2)]
\]

(6.65)

\[B = (-4c + 16du_1u_2 - 4d'u_1u_2 + 4d''(u_1^2 + u_2^2) + 4e_1(|v_1|^2 + |v_2|^2) + 2e_2'|v_1||v_2| + 4h_1|v_c|^2 + 4a_2u^2)\) \]

(6.66)

\(^8u_{1,2}^2 \sim 10^{-3}/10^{-4} V^2, \) \( u_{1,2}/\Lambda \sim 10^{-1} \) from neutrino phenomenology. Hence the hierarchy between the VEV’s is justified.
Using the same arguments as above, it is not hard to see that the deviation from $u_1 = u_2$ is also mild. Again, $u_1 = u_2$ is satisfied when $h_6' = 0$ and $h_6'' = 0$. Since $h_6 = 0$ is also required for $|v_1| = |v_2|$ to be satisfied, we conclude that exact $\mu - \tau$ symmetry for neutrinos demands that $h_6 = 0$, $h_6' = 0$ and $h_6'' = 0$ simultaneously.

Finally, from the last minimization condition (6.56) we get the solution,

$$|v_2|^2 = \frac{1}{4h_3} \left[ 2h_6(|v_2|^2 - |v_1|^2) + 2h_6''(u_2^2 - u_1^2) - 2l_1''(u_1^2 + u_2^2) + 2h_1 
- 2h_4(|v_1|^2 + |v_2|^2) - 8u_1u_2l_1 - 2l_4v^2 \right].$$

(6.67)

We next use use the condition (6.67) to estimate the cut-off scale $\Lambda$. Since $h_1$ define in Eq. (6.48) gives the square of the mass of the $\phi_e$ fields, it could be large. The other couplings $h_3, l_1, l_4, h_6$, $l_1''$ and $h_6$ are dimensionless and can be assumed to have roughly the same order of magnitude which should be much much smaller than $h_1$. Dividing both sides of Eq. (6.67) by $\Lambda^2$ and using $|v_1|^2 \simeq |v_2|^2 = 10^{-3}$ eV$^2$, $\frac{v_1}{\Lambda} \simeq 2 \times 10^{-2}$, $\frac{u_2}{\Lambda} \simeq 10^{-1}$ and hence $(\frac{u_1}{\Lambda})^2 \ll (\frac{u_1}{\Lambda}) < (\frac{u_1}{\Lambda})^2$, we get $^9$

$$\Lambda^2 \simeq \frac{h_1}{4l_1'' + 2l_1''} \times 10^2 \text{ GeV}^2.$$  

(6.68)

The coupling $h_1$ has mass dimension 2 and in principle could be large. As an example, if we take $h_1$ in TeV range, for example if we take $\sqrt{h_1} = 10$ TeV, then the cut-off scale of the theory is fixed as $10^2$ TeV, where we have taken $l_1$ and $l_1'' \sim O(1)$. From $\frac{u_1}{\Lambda} = \frac{u_2}{\Lambda} \sim 10^{-1}$ and $\frac{l_1''}{\Lambda} \sim 10^{-2}$, we then obtain $u_{1,2} = 10$ TeV and $v_e = 1$ TeV. The constraints from the lepton masses themselves do not impose any restriction on the cut-off scale and the VEVs. One can obtain estimates on them only through limits on the masses of the Higgs. For instance, from Eqs. (6.38) and (6.39) one could in principle estimate $u_1$ by measuring the difference between doubly charged Higgs masses. This could then be combined with the neutrino data to get $\Lambda$, and finally use the charged lepton masses to get $v_e$.

Since we consider a model with triplet Higgs to generate Majorana neutrino masses, it is pertinent to make some comments regarding breaking of lepton number and possible creation of a massless goldstone called Majoron [26]. It is possible to break lepton number explicitly by giving a lepton number to the fields $\xi$. In that case one would not break lepton number spontaneously and there would be no Majoron.

These VEV alignments have been obtained by assuming no effect of renormalization group running. However, it is understood that the running from the high scale where $S_3$ is broken to the electroweak scale where the masses are generated, will modify the

$^9$ $h_6, h_6'' = 0$ is motivated from $\mu - \tau$ symmetry, which together with $(\frac{u_1}{\Lambda})^2 \ll (\frac{u_2}{\Lambda})^2 \ll (\frac{l_1}{\Lambda})^2$ can also lead to Eq. 6.68. Even for a mild breaking of $\mu - \tau$ which leads to $(u_2^2 - u_1^2) \ll u_1^2$, the equation is valid.
VEV alignments. Another way the VEV alignments could get modified is through higher dimensional terms in the scalar potential. Due to the $Z_4$ as well as $Z_3$ symmetry that we have imposed, one cannot get terms of dimension five in the scalar potential. The possible next order terms in $V$ would therefore be terms of dimension six. These terms would be suppressed by $\Lambda^2$ and are therefore expected to be much less important in $V$.

6.5 Conclusions

In this work, we have attempted to provide a viable model for the lepton masses and mixing by imposing a $S_3 \times Z_4 \times Z_3$ family symmetry. Our model has two SU(2)$_L$ Higgs triplets arranged in the doublet representation of $S_3$. In addition we also have two sets of $S_3$ flavon doublets which are singlets with respect to the standard model. We have assigned the standard model fermions in suitable representation of $S_3$. We have obtained viable neutrino mass and mixing as well as the charged lepton mass hierarchy due to the VEV alignments. We have analyzed the normal and inverted mass hierarchy in detail. We gave predictions for $\sin^2 \theta_{12}$, $\Delta m^2_{21}$, $\Delta m^2_{31}$, effective mass in neutrino-less double beta decay, the observed mass squared in direct beta decay and total mass of the neutrinos relevant to cosmological data. We have analyzed the potential and have shown in the most general case, one would obtain mild deviation from $\theta_{23} = \frac{\pi}{4}$ and $\theta_{13} = 0$. This will open up the possibility of CP violation in the leptonic sector. Production and subsequent decay of the doubly charged Higgs at particle colliders is a smoking gun signal for the existence of triplet Higgs. We relate the $\mu - \tau$ or mildly broken $\mu - \tau$ symmetry in the neutrino sector with the doubly charged Higgs decay modes in Colliders. We showed that in our model since the triplet VEV is required to be very small, decay to dileptons would predominate. In our model the mixing between the two doubly charged Higgs is very closely related with the extent of the $\mu - \tau$ symmetry in the neutrino sector. In the exact $\mu - \tau$ limit the mixing angle $\theta$ between the two doubly charged Higgs is $\theta = \frac{\pi}{4}$, whereas mild breaking of the $\mu - \tau$ symmetry results in a mild deviation $\theta \sim \frac{\pi}{4}$. This close connection between the $\mu - \tau$ symmetry and the doubly charged mixing angle significantly effects the doubly charged Higgs-dileptonic vertices. In the exact $\mu - \tau$ limit, the vertex factors $H_2^{++} - \mu - \tau$ and $H_2^{++} - e - e$ are zero, hence the doubly charged Higgs $H_2^{++}$ never decays to $\mu^+ + \tau^+$ or to $2e^+$ states. Other than this, the non observation of the $e\mu$ and $e\tau$ states in the dileptonic decay of the doubly charged Higgs would possibly disfavor the inverted mass hierarchy of the standard model neutrino. Hence, the lepton flavors involved in the final lepton pair could be used to distinguish this model from the other models with triplet Higgs as well as to distinguish the inverted and normal hierarchy. Our model predicts lepton flavor violating processes such at $\tau \rightarrow e\mu\mu$ at the tree level. This and other lepton flavor violating processes could therefore be used to constrain the model as well as the neutrino mass hierarchy.
Bibliography


