Chapter 5

$A_4$ Flavor Symmetry and Neutrino Phenomenology

5.1 Introduction

We have discussed in chapter 2 about neutrino oscillation and mass generation mechanism of the standard model neutrino. Other than small neutrino masses, the PMNS mixing shows a very drastic difference as compared to the Cabibo mixing angles in the quark sector. Recent analysis of neutrino oscillation data [1–6] suggest that the solar mixing angle is $\theta_{12} \sim 34.4^0$, while the atmospheric mixing angle is $\theta_{23} \sim 42.9^0$. We have given the present $1\sigma$ and $3\sigma$ range of the oscillation parameters in chapter 2. At present there is an upper bound on the CHOOZ mixing angle $\theta_{13}$ [1,7] as $\theta_{13} \leq 13.3^0$. The neutrino oscillation data suggest that unlike in the quark sector, in the leptonic sector two of the oscillation angles are large while the third one is small. The aesthetic belief of unification suggests that the standard model should be embedded in a higher ranked gauge group, for example SU(5) or SO(10). Although the quarks and leptons in a higher ranked gauge group belong to same representation, however the mixing in the leptonic sector is drastically different than the mixings in the quark sector. One very interesting scenario is when the mixing is tribimaximal one, which agrees very well with the experimentally allowed oscillation parameter ranges. The initial concept of tribimaximal mixing in the leptonic sector has been proposed by Harrison, Perkins, and Scott [8],

$$U_{TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}.$$  \hspace{1cm} (5.1)

The different mixing angles for a tribimaximal mixing will give $\sin^2 \theta_{12} = 0.33$, $\sin^2 \theta_{23} = 0.5$ and $\sin^2 \theta_{13} = 0$. For the TBM mixing to exist, the neutrino mass matrix should be
of the form

\[
m_\nu = \begin{pmatrix}
A & B \\
B & \frac{1}{2}(A + B + D) \\
B & \frac{1}{2}(A + B - D)
\end{pmatrix},
\]

(5.2)

where \( A = \frac{1}{3}(2m_1 + m_2) \), \( B = \frac{1}{3}(m_2 - m_1) \) and \( D = m_3 \), where \( m_1, m_2 \) and \( m_3 \) are the neutrino masses. The current neutrino data already give very good measurement of mass squared differences, while the best limit on the absolute neutrino mass scale comes from cosmological data, as given in chapter 2.

Maximal \( \theta_{23} \) and zero \( \theta_{13} \) of the TBM can be easily obtained if \( m_\nu \) possesses \( \mu - \tau \) exchange symmetry [9] or the \( L_{\mu} - L_{\tau} \) symmetry [10]. However, the solar mixing angle \( \sin^2 \theta_{12} \) is not so easily predicted to be exactly 1/3, as required for exact TBM mixing. Various symmetry groups explaining the flavor structure of the leptons have been invoked in the literature in order to accommodate the neutrino mass and mixing along with charged leptons. In particular, the study of the non-Abelian discrete symmetry group \( A_4 \) has received considerable interest in the recent past [11–18]. This group has been shown to successfully reproduce the TBM form of the neutrino mixing matrix, in the basis where the charged lepton mass matrix is diagonal [14]. However, the authors of [14] work in a very special framework where only one of the three possible one dimensional Higgs representations under \( A_4 \) is considered and for the \( A_4 \) triplet Higgs which contributes to the neutrino mass matrix, a particular vacuum alignment is taken. In this framework, the mixing matrix emerges as independent of the Yukawa couplings, the VEVs and the scale of \( A_4 \) breaking. Only the mass eigenstates depend on them.

In the present work, we have consider the most general scenario with all possible one dimensional Higgs representations that can be accommodated within this \( A_4 \) model. The Higgs fields which are charged under the symmetry group \( A_4 \), are standard model gauge singlets and they are known as flavons. In our model we have two flavon fields \( \phi_{S,T} \) which transform as three dimensional irreducible representation of the group \( A_4 \). In addition, we also have three other flavons \( \xi, \xi', \xi'' \) which transform as 1, 1' and 1'' respectively. The Lagrangian describing the Yukawa interaction between the different standard model leptons, Higgs and the flavons follows the effective field theoretical description. The different flavon fields take the vacuum expectation values, thereby resulting in a spontaneous breaking of the symmetry group \( A_4 \). We explore the conditions on VEVs and Yukawa couplings needed for obtaining exact TBM mixing in this present set-up. We have shown that in the model considered in [14], one gets TBM mixing simply through the alignment of the \( A_4 \) triplet Higgs VEV, however the experimentally allowed mass squared differences can be obtained if one has the vacuum expectation value of an additional singlet Higgs. To get the correct \( \Delta m^2_{21} \) and \( \Delta m^2_{31} \), the product of the VEV and Yukawa of this singlet is determined completely by the VEV and Yukawa of the triplet.

We have consider the presence of additional one dimensional Higgs representations under \( A_4 \), construct the neutrino mass matrices and study their phenomenology. In
Table 5.1: List of fermion and scalar fields used in this model. Two lower rows list additional one dimensional Higgs fields considered in the present work. In section 4, we also allow for a different VEV alignment for $\phi_S$.

<table>
<thead>
<tr>
<th>Lepton</th>
<th>$SU(2)_L$</th>
<th>$A_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l$</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$\bar{e}_R$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\bar{\mu}_R$</td>
<td>1</td>
<td>$1''$</td>
</tr>
<tr>
<td>$\bar{\tau}_R$</td>
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<td>$1'$</td>
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</table>

<table>
<thead>
<tr>
<th>Scalar</th>
<th>VEV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_u$</td>
<td>$&lt;H_u^0&gt; = v_u$</td>
</tr>
<tr>
<td>$H_d$</td>
<td>$&lt;H_d^0&gt; = v_d$</td>
</tr>
<tr>
<td>$\phi_S$</td>
<td>$(v_S, v_S, v_S)$</td>
</tr>
<tr>
<td>$\phi_T$</td>
<td>$(v_T, 0, 0)$</td>
</tr>
<tr>
<td>$\xi$</td>
<td>$u$</td>
</tr>
<tr>
<td>$\xi'$</td>
<td>$u'$</td>
</tr>
<tr>
<td>$\xi''$</td>
<td>$u''$</td>
</tr>
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</table>

particular, we check which ones would produce TBM mixing and under what conditions. In this $A_4$ scenario, we have also studied the deviation from TBM and the corresponding effect on the neutrino masses. If just only one Higgs transforming as one dimensional Higgs representation under $A_4$ is allowed, the model of [14] is the only viable model. We further show that in the simplest version of this model, one necessarily gets the normal mass hierarchy. Inverted hierarchy can be possible if we have at least two or all three Higgs scalars with nonzero VEVs.

We proceed as follows. In section 5.2 we give a brief overview of the $A_4$ model considered. In section 5.3 we begin with detailed phenomenological analysis of the case where there is just one singlet Higgs under $A_4$. We next increase the number of contributing one dimensional Higgs representations, give analytical and numerical results. In section 5.4 we study the impact of the misalignment of the VEVs of the triplet Higgs. We end in section 5.5 with our conclusions.

### 5.2 Overview of the Model

The detail group theory of the discrete symmetry group $A_4$ has been discussed in chapter 2. The particle contents of our model has been shown in Table 5.1. There are five $SU(3)_C \times SU(2)_L \times U_Y(1)$ singlet Higgs, three ($\xi$, $\xi'$ and $\xi''$) of which transform as the different one dimensional representations under $A_4$ and two ($\phi_T$ and $\phi_S$) of which transform as triplets under the symmetry group $A_4$. The standard model lepton doublets are assigned to the
triplet representation of $A_4$,

$$ l = (L_e, L_\mu, L_\tau)^T, $$

where $L_\alpha$ denotes the standard model lepton doublets. The right handed charged leptons $e_R, \mu_R$ and $\tau_R$ are assumed to belong to the $1, 1'$ and $1''$ representation respectively. The standard Higgs doublets $H_u$ and $H_d$ remain invariant under $A_4$. The form of the $A_4$ invariant Yukawa part of the Lagrangian is

$$ \mathcal{L}_Y = y_e \bar{e}_R (\phi_T l) + y_\mu \bar{\mu}_R (\phi_T l)' + y_\tau \bar{\tau}_R (\phi_T l)'' + x_a \phi (ll') + x'_a \phi (ll'') + x''_a \phi (ll)' $$

where, following [14] we have used the compact notation, $y_e \bar{e}_R (\phi_T l) \equiv y_e \bar{e}_R (\phi_T l) H_d / \Lambda$, $x_a \phi (ll) \equiv x_a \phi (lH_u H_d) / \Lambda^2$ and so on, and $\Lambda$ is the cut-off scale of the theory. Here we adopt an effective field theory approach. We assume that $\phi_S$ does not couple to charged leptons and $\phi_T$ does not contribute to the Majorana mass matrix. These two additional features can be obtained by introducing extra abelian symmetries for example $Z_3$ [14]. After the spontaneous breaking of $A_4$ followed by $SU(2)_L \times U(1)_Y$, we get the mass terms for the charged leptons and neutrinos. Assuming the vacuum alignment

$$ \langle \phi_T \rangle = (v_T, 0, 0), \quad (5.4) $$

the charged lepton mass matrix is given as

$$ m_{cl} = \frac{v_d v_T}{\Lambda} \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix}, \quad (5.5) $$

Note that we could also obtain a diagonal charged lepton mass matrix even if we assume that $\bar{e}_R, \bar{\mu}_R$ and $\bar{\tau}_R$ transform as $1''$, $1'$ and $1$, and $\langle \phi_T \rangle = (0, v_T, 0)$ with appropriate change in the Yukawa Lagrangian. Similarly, $\bar{e}_R, \bar{\mu}_R$ and $\bar{\tau}_R$ transforming $1'$, $1$ and $1''$, and $\langle \phi_T \rangle = (0, 0, v_T)$ could give us the same $m_{cl}$.

In the most general case, where all three one dimensional $A_4$ Higgs as well as $\phi_S$ are present and we do not assume any particular vacuum alignment, the neutrino mass matrix looks like

$$ m_\nu = m_0 \begin{pmatrix} a + 2b_1/3 & c - b_3/3 & d - b_2/3 \\ c - b_3/3 & d + 2b_2/3 & a - b_1/3 \\ d - b_2/3 & a - b_1/3 & c + 2b_3/3 \end{pmatrix}, \quad (5.6) $$

where $m_0 = \frac{v_\nu^2}{\Lambda}$, $b_i = 2x_b v_S / \Lambda$, $a = 2x_a u / \Lambda$, $c = 2x_a'' u'' / \Lambda$ and $d = 2x_a u' / \Lambda$ and we have written the VEVs as

$$ \langle \phi_S \rangle = (v_s_1, v_s_2, v_s_3), \quad \langle \xi \rangle = u, \quad \langle \xi' \rangle = u', \quad \langle \xi'' \rangle = u'', \quad \langle H_{u,d} \rangle = v_{u,d}. \quad (5.7) $$

For simplicity, in this work we have considered all the Yukawa couplings as well as the parameters $a, b, c$ and $d$ as real.
5.3 Number of One Dimensional Higgs Representations and their VEVs

In this section we work under the assumption that the triplet Higgs $\phi_S$ has VEVs along the direction

$$\langle \phi_S \rangle = (v_S, v_S, v_S).$$  \hspace{1cm} (5.8)

This produces the neutrino mass matrix

$$m_\nu = m_0 \begin{pmatrix} a + 2b/3 & c - b/3 & d - b/3 \\ c - b/3 & d + 2b/3 & a - b/3 \\ d - b/3 & a - b/3 & c + 2b/3 \end{pmatrix},$$ \hspace{1cm} (5.9)

where $b = 2x_b \frac{v}{\Lambda}$. In the following we discuss the phenomenology of the different forms of $m_\nu$ possible as we change the number of one dimensional Higgs or put their VEVs to zero. We assume that $m_\nu$ is real.

5.3.1 No One Dimensional $A_4$ Higgs

If there were no Higgs which transforms as one dimensional irreducible representation under $A_4$, or if the VEV of all three of them ($\xi, \xi', \xi''$) were zero, one would get the neutrino mass matrix

$$m_\nu = m_0 \begin{pmatrix} 2b/3 & -b/3 & -b/3 \\ -b/3 & 2b/3 & -b/3 \\ -b/3 & -b/3 & 2b/3 \end{pmatrix}.$$ \hspace{1cm} (5.10)

On diagonalizing this one obtains the eigenvalues

$$m_1 = m_0 b, \hspace{0.5cm} m_2 = 0, \hspace{0.5cm} m_3 = m_0 b$$ \hspace{1cm} (5.11)

and the mixing matrix

$$U = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}.$$ \hspace{1cm} (5.12)

Therefore, we can see that the tribimaximal pattern of the mixing matrix is coming directly from the term containing the triplet Higgs $\phi_S$ and does not depend on the terms containing the one Higgs scalars $\xi, \xi', \xi''$, which are charged under different one dimensional Higgs representations of the group $A_4$. However, in the absence of the one dimensional Higgs representations of the group $A_4$, the mixing pattern does not depend explicitly even on the magnitude of the VEV of $\phi_S$.\footnote{The mixing pattern does not depend explicitly even on the magnitude of the VEV of $\phi_S$.}
Figure 5.1: Contour plot of $\Delta m^2_{21}$, $\Delta m^2_{31}$ and sum of absolute neutrino masses $m_t$ in the $a - b$ plane, for three different values of $m_0$ ($m_0=0.016$, $m_0=0.024$ and $m_0=0.032$) for the case with $\xi$ and $\phi_S$. The first row shows contour plots for different values of $\Delta m^2_{21}$ (in $10^{-5}$ eV$^2$), with red dashed lines for 6.5 blue solid lines for 7.1, green dashed lines for 7.7, green solid lines for 8.3, and orange dotted lines for 8.9. The second row shows contour plots for different values of $\Delta m^2_{31}$ (in $10^{-3}$ eV$^2$), with red dashed lines for 1.6, blue solid lines for 2.0, green dashed lines for 2.4, green solid lines for 2.8, and orange dotted lines for 3.2. The third row shows contour plots for $m_t$ (in eV), with red dashed lines for 0.05, blue solid lines for 0.07, green dashed lines for 0.09, green solid lines for 1.1, orange dotted lines for 1.3, turquoise solid lines for 1.4, magenta solid lines for 1.6, and brown solid lines for 1.8.
Figure 5.2: Scatter plot showing regions in $a - b$ parameter space for the model considered in [14], which are compatible with the $3\sigma$ allowed range of values of the mass squared differences given in [20]. The parameter $m_0$ is allowed to vary freely.

dimensional Higgs contributions to $m_\nu$, the predicted neutrino masses turn out to be very wrong. In this case, $\Delta m_{21}^2 = -b^2 m_0^2$ and $\Delta m_{31}^2 = 0$, in strong disagreement with the oscillation data.

### 5.3.2 Only One One-Dimensional $A_4$ Higgs

If we take only one Higgs at a time, which transforms as one dimensional irreducible representation under $A_4$, then there are three possibilities. The resulting mass matrices are shown in column 2 of Table 5.2. One could get exactly the same situation with three one dimensional Higgs $\xi$, $\xi'$ and $\xi''$ and demanding that the VEV of two of them are zero while that of the third is nonzero. The $m_\nu$ given in Table 5.2 can be exactly diagonalized and the eigenvalues and eigenvectors are shown in column 3 and 4 of the Table, respectively. It is evident that only the case where $\xi$ is present gives rise to a viable mixing matrix, which is exactly tribimaximal [14]. Only the first case has the form for $m_\nu$ given in Eq. (5.2). Each of the $m_\nu$ given in Table 5.2 possesses an $S_2$ symmetry. While the case with $\xi$ exhibits the $\mu - \tau$ exchange symmetry, the one with $\xi''$ remains invariant under $e - \mu$ permutation and the one with $\xi'$ under $e - \tau$ permutation. This would necessarily demand that while for the first case $\theta_{23}$ would be maximal and $\theta_{13} = 0$, for the second and third cases $\theta_{23}$ would be either $90^\circ$ or $0$ respectively, and $\theta_{13}$ maximal, and hence disallowed by the neutrino oscillation data.

Since only the case with $\xi$ reproduces the correct form for the mixing matrix, we do not discuss the remaining two cases any further. The mass squared differences in this
Table 5.2: The mass matrix taking one Higgs at a time, its mass eigenvalues, and its mixing matrix.

<table>
<thead>
<tr>
<th>Higgs</th>
<th>Neutrino mass matrix</th>
<th>Eigenvalues</th>
<th>Mixing Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi$</td>
<td>$m_0 \begin{pmatrix} a + 2b/3 &amp; -b/3 &amp; -b/3 \ -b/3 &amp; a - b/3 &amp; 2b/3 \ -b/3 &amp; 2b/3 &amp; a - b/3 \end{pmatrix}$</td>
<td>$m_0(a + b), m_0a, m_0(b - a)$</td>
<td>$\begin{pmatrix} \sqrt{2}/3 &amp; 1/\sqrt{3} &amp; 0 \ -1/\sqrt{6} &amp; 1/\sqrt{3} &amp; -1/\sqrt{2} \ -1/\sqrt{6} &amp; 1/\sqrt{3} &amp; 1/\sqrt{2} \end{pmatrix}$</td>
</tr>
<tr>
<td>$\xi''$</td>
<td>$m_0 \begin{pmatrix} 2b/3 &amp; c - b/3 &amp; -b/3 \ c - b/3 &amp; 2b/3 &amp; -b/3 \ -b/3 &amp; -b/3 &amp; c + 2b/3 \end{pmatrix}$</td>
<td>$m_0(c + b), m_0c, m_0(b - c)$</td>
<td>$\begin{pmatrix} -\sqrt{6}/3 &amp; -1/\sqrt{3} &amp; -\sqrt{2} \ -\sqrt{6}/3 &amp; 1/\sqrt{3} &amp; -\sqrt{2} \ \sqrt{2}/3 &amp; 1/\sqrt{3} &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$\xi'$</td>
<td>$m_0 \begin{pmatrix} 2b/3 &amp; -b/3 &amp; d - b/3 \ -b/3 &amp; d + 2b/3 &amp; -b/3 \ d - b/3 &amp; -b/3 &amp; 2b/3 \end{pmatrix}$</td>
<td>$m_0(d + b), m_0d, m_0(b - d)$</td>
<td>$\begin{pmatrix} -\sqrt{6}/3 &amp; -1/\sqrt{3} &amp; -\sqrt{2} \ -\sqrt{6}/3 &amp; 1/\sqrt{3} &amp; -\sqrt{2} \ \sqrt{2}/3 &amp; 1/\sqrt{3} &amp; 0 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

The case are

$$\Delta m^2_{21} = (-b^2 - 2ab)m_0^2, \quad \Delta m^2_{31} = -4abm_0^2,$$

where $m_0 = v_u^2/\Lambda$. Since it is now known at more than $6\sigma$ C.L. that $\Delta m^2_{21} > 0$ [19,20], we have the condition that $-2ab > b^2$. We have considered the parameters $a$ and $b$ as real. Since $b^2$ is a positive definite quantity the above relation implies that $-2ab > 0$, which can happen only if $\text{sgn}(a) \neq \text{sgn}(b)$. Inserting this condition into the expression for $\Delta m^2_{31}$ gives us $\Delta m^2_{31} > 0$ necessarily in this model. Therefore inverted neutrino mass hierarchy is impossible to get in the effective $A_4$ model originally proposed by Altarelli and Feruglio [14]. The conclusion is also valid for complex parameter $a$ and $b$ with a relative phase $\phi$. However, going to the seesaw-realization given in [14], normal as well as inverted hierarchy is possible to realize.

The sum of the absolute neutrino masses, effective mass in neutrinoless double beta decay and prediction for tritium beta decay are given respectively as

$$m_t = |m_1| + |m_2| + |m_3|, \quad \langle m_{ee} \rangle = m_0(a + 2b/3), \quad m^2_{23} = m_0^2 \left( a^2 + \frac{4ab}{3} + \frac{2b^2}{3} \right).$$

In Fig. 5.1 we show the contours for the observables $\Delta m^2_{21}$, $\Delta m^2_{31}$ and $m_t$ in the $a - b$ plane, for three different fixed values of $m_0$. The details of the figure and description of the different lines can be found in the caption of the figure.
In Fig. 5.2 we present a scatter plot showing the points in the $a-b$ parameter space which are compatible with the $3\sigma$ [20] allowed range of the mass squared differences. We have allowed $m_0$ to vary freely and taken a projection of all allowed points in the $a-b$ plane. While $a$ is related to the VEV of the singlet $\xi$, $b$ is given in terms of the VEVs of the triplet $\phi_S$. The TBM form for the mixing matrix comes solely from the vacuum alignment of $\phi_S$ and $\xi$ is not needed for that. The singlet $\xi$ is necessary only for producing the correct values of $\Delta m^2_{21}$ and $\Delta m^2_{31}$. However, it is evident from Fig. 5.2 that for a given value of $a$ needed to obtain the right mass squared differences, the value of $b$ is almost fixed. The relation between $a$ and $b$ can be obtained by looking at the ratio \[
abla m^2_{21} = \frac{b^2 - 2ab}{-4ab} \approx 0.03,
\] where 0.03 on the right-hand-side (RHS) is the current experimental value. This gives us the relation $b \approx -1.88a$. Therefore, in addition to the alignment $\langle \phi_S \rangle = (v_S, v_S, v_S)$ to get TBM, one also needs a particular relation between the product of Yukawa couplings and VEVs of $\phi_S$ and $\xi$ in order to reproduce the correct phenomenology. Even if one includes the $3\sigma$ uncertainties on $\Delta m^2_{21}$ and $\Delta m^2_{31}$, $|b|$ is fine tuned to $|a|$ within a factor of about $10^{-2}$.

### 5.3.3 Two One-Dimensional $A_4$ Higgs

If we take two Higgs fields at a time, belonging to different one dimensional $A_4$ representations and allow for nonzero VEVs for them, then the $m_\nu$ obtained for the three possible cases are shown in the first three rows of Table 5.3. One can again see that of the three possible combinations, only the $\xi', \xi''$ combination gives a viable TBM matrix. The other two mass matrices exhibit $e-\tau$ ($\xi, \xi''$) and $e-\mu$ ($\xi, \xi'$) symmetry respectively and are ruled out. Note that we have chosen $a = c$ for the $\xi, \xi'$ combination, $a = d$ for the $\xi, \xi'$ combination and $c = d$ for the $\xi', \xi''$ combination for the results given in Table 5.3. This is a reasonable assumption to make since the phenomenology of the three cases does not change drastically unless the VEVs of the singlet Higgs vary by a huge amount. In particular, by changing the relative magnitude of the VEVs, we do not expect the structure of the mixing matrix for the first two rows of Table 5.3 to change so much so that they could be allowed by the current data. In the limit that $c = d$, it is not hard to appreciate that the resultant matrix with $\xi'$ and $\xi''$ would exhibit $\mu - \tau$ symmetry, though the $\xi'$ and $\xi''$ terms alone have $e - \tau$ and $e - \mu$ symmetry respectively.

Since the $\xi', \xi''$ combination is the only one which gives exact TBM mixing in the approximation that $c = d$, we perform a detailed analysis only for this case. Putting $c = d$ is again contrived and would also lead to a certain fixed relation between them and $b$, as in the only $\xi$ case. This would mean additional fine tuning of the parameters, unless explained by symmetry arguments. Hence, we allow the two VEVs to differ from each other so that $c = d + \epsilon$. If $\epsilon$ is small we can solve the eigenvalue problem keeping only the first order terms in $\epsilon$. The results for this case are shown in the final row of the Table.

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5.3. The deviation of the mixing angles from their TBM values can be seen to be

\[ D_{12} \simeq 0, \quad D_{23} \simeq -\frac{\epsilon}{4d}, \quad U_{e3} \simeq -\frac{\epsilon}{2\sqrt{2d}}, \]

(5.15)

where \( D_{12} = \sin^2 \theta_{12} - 1/3 \) and \( D_{23} = \sin^2 \theta_{23} - 1/2 \). We show in the left hand panels of Fig. 5.3 the mixing angles \( \sin^2 \theta_{12} \) (upper panel), \( \sin^2 \theta_{13} \) (middle panel) and \( \sin^2 \theta_{23} \) (lower panel) as a function of \( \epsilon \). We vary \( \epsilon \) from large negative to large positive values and solve the exact eigenvalue problem numerically allowing the other parameters, \( m_0, b \) and \( d \), to vary freely. For \( \epsilon = 0 \) of course we get TBM mixing as expected. For very small values of \( \epsilon \), the deviation of the mixing angles from their TBM values is reproduced well by the approximate expressions given in Eq. (5.15). For large \( \epsilon \) of course the approximate expressions fail and we have significant deviation from TBM. The values of \( \sin^2 \theta_{12} \) and \( \sin^2 \theta_{13} \) increase very fast with \( \epsilon \), while \( \sin^2 \theta_{23} \) decreases with it. We have only showed the scatter plots up to the 3\( \sigma \) allowed ranges for the mixing angles given in [20]. The mass dependent observables can be calculated upto first order in \( \epsilon \) as

\[ \Delta m^2_{21} \simeq m_0^2 (b + d + \frac{\epsilon}{2})(3d - b + \frac{3\epsilon}{2}), \quad \Delta m^2_{31} \simeq (4bd + 2bc)m_0^2, \]

(5.16)

\[ \langle m_{ee} \rangle = m_0 \frac{2b}{3}, \quad m_t = \sum_i |m_i|, \quad m^2_3 \simeq m_0^2 \left( \frac{2b^2}{3} + 2d^2 - \frac{4bd}{3} + 2de - \frac{2be}{3} \right). \]

(5.17)

Of course, epsilon can be larger and we show numerical results for those cases.

**Normal Hierarchy**

The right panels of Fig. 5.3 show \( \Delta m^2_{21} \) (upper panel), \( \Delta m^2_{31} \) (middle panel) and \( m_t \) (lower panel) as a function of \( \epsilon \) assuming \( m_1 < m_2 \ll m_3 \). We show \( \Delta m^2_{21} \) and \( \Delta m^2_{31} \) only within their 3\( \sigma \) [20] allowed range. We notice that while \( \Delta m^2_{21} \) and \( \Delta m^2_{31} \) are hardly constrained by \( \epsilon \), there appears to some mild dependence of \( m_t \) on it.

Fig. 5.4 gives the scatter plots showing the allowed parameter regions for this case. The upper, middle and lower panels show the allowed points projected on the \( b - c, d - c \) and \( b - d \) plane, respectively.

**Inverted Hierarchy**

For this case it is possible to obtain even inverted hierarchy. We show in Fig. 5.4 the scatter plots showing the allowed parameter regions for inverted hierarchy. We have allowed \( m_0 \) to vary freely and show the allowed points projected on the \( c - b, e - d \) and \( d - b \) planes. One can check that only for the points appearing in this plot, \( m_3 < m_1 < m_2 \).
<table>
<thead>
<tr>
<th>Higgs</th>
<th>Neutrino mass matrix</th>
<th>Eigenvalues</th>
<th>Eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi, \xi''$</td>
<td>$m_0 \begin{pmatrix} a + \frac{2b}{3} &amp; c - \frac{b}{3} &amp; -\frac{b}{3} \ c - \frac{b}{3} &amp; a - \frac{b}{3} &amp; c + \frac{2b}{3} \ -\frac{b}{3} &amp; a - \frac{b}{3} &amp; -\frac{b}{3} \end{pmatrix}$</td>
<td>$a = c$;</td>
<td>$\left( \begin{array}{ccc} -\frac{1}{\sqrt{6}} &amp; \frac{1}{\sqrt{3}} &amp; -\frac{1}{\sqrt{2}} \ \frac{\sqrt{2}}{3} &amp; \frac{1}{\sqrt{3}} &amp; 0 \ -\frac{1}{\sqrt{6}} &amp; \frac{1}{\sqrt{3}} &amp; \frac{1}{\sqrt{2}} \end{array} \right)$</td>
</tr>
<tr>
<td>$\xi, \xi'$</td>
<td>$m_0 \begin{pmatrix} a + \frac{2b}{3} &amp; -\frac{b}{3} &amp; d - \frac{b}{3} \ -\frac{b}{3} &amp; d + \frac{2b}{3} &amp; a - \frac{2b}{3} \ d - \frac{b}{3} &amp; a - \frac{2b}{3} &amp; -\frac{b}{3} \end{pmatrix}$</td>
<td>$a = d$;</td>
<td>$\left( \begin{array}{ccc} -\frac{1}{\sqrt{6}} &amp; \frac{1}{\sqrt{3}} &amp; -\frac{1}{\sqrt{2}} \ -\frac{\sqrt{2}}{3} &amp; \frac{1}{\sqrt{3}} &amp; \frac{1}{\sqrt{2}} \ \frac{\sqrt{2}}{3} &amp; \frac{1}{\sqrt{3}} &amp; 0 \end{array} \right)$</td>
</tr>
<tr>
<td>$\xi', \xi''$</td>
<td>$m_0 \begin{pmatrix} \frac{2b}{3} &amp; c - \frac{b}{3} &amp; d - \frac{b}{3} \ c - \frac{b}{3} &amp; d + \frac{2b}{3} &amp; -\frac{b}{3} \ d - \frac{b}{3} &amp; -\frac{b}{3} &amp; c + \frac{2b}{3} \end{pmatrix}$</td>
<td>$c = d + \epsilon$;</td>
<td>$\left( \begin{array}{ccc} \frac{\sqrt{2}}{3} &amp; \frac{1}{\sqrt{3}} &amp; 0 \ \frac{1}{\sqrt{6}} &amp; \frac{1}{\sqrt{3}} &amp; -\frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{6}} &amp; \frac{1}{\sqrt{3}} &amp; \frac{1}{\sqrt{2}} \end{array} \right)$</td>
</tr>
</tbody>
</table>

Table 5.3: Here we take two one dimensional $A_4$ Higgs at a time, and analytically display eigenvalues and eigenvectors of the neutrino mass matrix.
Figure 5.3: The left panels show $\sin^2 \theta_{12}$, $\sin^2 \theta_{13}$ and $\sin^2 \theta_{23}$ vs $\epsilon$ and the right panels show $\Delta m^2_{21}$, $\Delta m^2_{31}$ and $m_t$ vs $\epsilon$ respectively. Here $\xi'$ and $\xi''$ acquire VEVs. The other parameters $c$, $b$ and $m_0$ are allowed to vary freely.

5.3.4 Three One-Dimensional $A_4$ Higgs

Finally, we let all three Higgs which transform as different one dimensional irreducible representations under the symmetry group $A_4$ contribute to $m_\nu$. In this case one has to diagonalize the most general mass matrix given in Eq. (5.9). This matrix has four independent parameters. If we assume that the VEVs of $\xi$, $\xi'$ and $\xi''$ are such that $a = c = d$, then the eigenvalues and mixing matrix are given in the first row of Table 5.4. This gives us a mass matrix, which gives us two of the mass eigenstates as degenerate. To get the correct mass splitting in association with TBM mixing, it is essential that (i) we should have contribution from the VEVs of the one dimensional Higgs and (ii) the contribution from the the three one dimensional Higgs $\xi$, $\xi'$ and $\xi''$ in $m_\nu$ should not be identical. If we assume that $a = c \neq d$, then one can easily check that $m_\nu$ has $e - \tau$ exchange symmetry, and hence the resulting mass matrix is disallowed. This is because for $a = c$, as discussed before we get $e - \tau$ exchange symmetry and the $\xi'$ term has an in-built $e - \tau$ symmetry. Similarly for $a = d \neq c$, one gets $e - \mu$ symmetry in $m_\nu$ and is hence disfavored. Only when we impose the condition $c = d$, we have $\mu - \tau$ symmetry in $m_\nu$, since the $\xi$ term and the sum of the $\xi'$ and $\xi''$ terms are now separately $\mu - \tau$ symmetric.
Figure 5.4: In the left panel, scatter plot showing the $3\sigma$ [20] allowed regions for the $b-c-d$ parameters for the case where $\xi'$ and $\xi''$ acquire VEVs. The top, middle and lower panels show the allowed points projected on the $c-b$, $c-d$ and $d-b$ plane, respectively. The parameter $m_0$ was allowed to take any value. Here we have assumed normal hierarchy. In the right panel, the scatter plot for inverted hierarchy.

Therefore, the case $a \neq c = d$ gives us the TBM matrix and the mass eigenvalues are shown in Table 5.4.

Since $a \neq c = d$ is the only allowed case for the three one dimensional Higgs case, we find the eigenvalues and the mixing matrix for the case where $c$ and $d$ are not equal, but differ by $\epsilon$. We take $c = d + \epsilon$ and for small values of $\epsilon$ give the results in the last row of Table 5.4, keeping just the first order terms in $\epsilon$. The deviation from TBM is given as follows

$$D_{12} \simeq 0, \quad D_{23} \simeq \frac{\epsilon}{4(a-d)}, \quad U_{e3} \simeq \frac{\epsilon}{2\sqrt{2}(a-d)} . \quad (5.18)$$

The mass squared differences are

$$\Delta m_{21}^2 \simeq m_0^2 (2a + b + d + \frac{\epsilon}{2})(3d - b + \frac{3\epsilon}{2}), \quad \Delta m_{31}^2 \simeq 2 m_0^2 b(2d - 2a + \epsilon). \quad (5.19)$$

From the expression of the mass eigenvalues given in the Table 5.4, one can calculate the
Figure 5.5: Scatter plot showing the $3\sigma$ [20] allowed regions in the model parameter space for the case where $\xi$, $\xi'$ and $\xi''$ all acquire VEVs. The upper panels show allowed regions projected onto the $a - c$, $a - d$, $a - b$ planes. The lower panels show allowed regions projected onto the $c - d$, $c - b$, $d - b$ planes. Normal hierarchy is assumed.

observables $m_t$, $m_\beta^2$ and $\langle m_{ee} \rangle$

\[
\langle m_{ee} \rangle = m_0 (a + \frac{2b}{3}), \quad m_t = \sum_i |m_i|, \quad m_\beta^2 \simeq m_0^2 (a^2 + \frac{4ab}{3} + \frac{2b^2}{3} + 2d^2 - \frac{4bd}{3} + 2de - \frac{2be}{3})
\] (5.20)

Normal Hierarchy

Let us begin by restricting the neutrino masses to obey the condition $m_1 < m_2 \ll m_3$ and allow $a$, $b$, $c$ and $d$ to take any random value and find the regions in the $a$, $b$, $c$ and $d$ space that give $\Delta m^2_{21}$, $\Delta m^2_{31}$ and the mixing angles within their $3\sigma$ [20] allowed ranges. This is done by numerically diagonalizing $m_\nu$. The results are shown as scatter plots in
<table>
<thead>
<tr>
<th>Higgs</th>
<th>Neutrino mass matrix</th>
<th>Eigenvalues</th>
<th>Eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>ξ, ξ',</td>
<td>( m_0 \begin{pmatrix} a + \frac{2b}{3} &amp; c - \frac{b}{3} &amp; d - \frac{b}{3} \ c - \frac{b}{3} &amp; d + \frac{2b}{3} &amp; a - \frac{b}{3} \ d - \frac{2b}{3} &amp; a - \frac{2b}{3} &amp; c + \frac{2b}{3} \end{pmatrix} )</td>
<td>( a = c = d; )</td>
<td>( \begin{pmatrix} \sqrt{\frac{2}{3}} \ -\frac{1}{\sqrt{6}} \ -\frac{1}{\sqrt{2}} \end{pmatrix} )</td>
</tr>
<tr>
<td>ξ''</td>
<td></td>
<td></td>
<td>( -\frac{1}{\sqrt{6}} \sqrt{\frac{2}{3}} \quad \frac{1}{\sqrt{3}} \quad -\frac{1}{\sqrt{2}} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( a \neq c = d; )</td>
<td>( \begin{pmatrix} \sqrt{\frac{2}{3}} \ -\frac{1}{\sqrt{6}} \ -\frac{1}{\sqrt{2}} \end{pmatrix} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( -\frac{1}{\sqrt{6}} \sqrt{\frac{2}{3}} \quad \frac{1}{\sqrt{3}} \quad -\frac{1}{\sqrt{2}} )</td>
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<td></td>
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<tr>
<td></td>
<td></td>
<td>( a = c \neq d; )</td>
<td>( \begin{pmatrix} -\frac{1}{\sqrt{6}} \ \sqrt{\frac{2}{3}} \ -\frac{1}{\sqrt{2}} \end{pmatrix} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( -\frac{1}{\sqrt{6}} \sqrt{\frac{2}{3}} \quad \frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{2}} )</td>
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<td></td>
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<td></td>
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<tr>
<td></td>
<td></td>
<td>( a = d \neq c; )</td>
<td>( \begin{pmatrix} -\frac{1}{\sqrt{6}} \ \sqrt{\frac{2}{3}} \ -\frac{1}{\sqrt{2}} \end{pmatrix} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( -\frac{1}{\sqrt{6}} \sqrt{\frac{2}{3}} \quad \frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{2}} )</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( a \neq c = d + \epsilon; )</td>
<td>( \begin{pmatrix} \sqrt{\frac{2}{3}} \ -\frac{1}{\sqrt{6}} + a_2 \ -\frac{1}{\sqrt{6}} + a_3 \end{pmatrix} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( -\frac{1}{\sqrt{6}} + a_2 \sqrt{\frac{2}{3}} \quad -\frac{1}{\sqrt{6}} + a_3 \sqrt{\frac{2}{3}} \quad -\frac{1}{\sqrt{2}} - a_5 )</td>
</tr>
</tbody>
</table>

Table 5.4: The mass matrix taking all three one dimensional \( A_4 \) Higgs, its mass eigenvalues and its mixing matrix. The correction factors in the last row are: \( a_2 = \frac{\sqrt{\frac{2}{3}}}{4\sqrt{2}(a-d)} \), \( a_3 = \frac{\epsilon}{4\sqrt{2}(a-d)} \), \( a_4 = \frac{1}{2\sqrt{2}(a-d)} \), \( a_5 = \frac{1}{4\sqrt{2}(a-d)} \), \( a_6 = \frac{\epsilon}{4\sqrt{2}(a-d)} \).
Fig. 5.5. To help see the allowed zones better, we have projected the allowed points on the $a - c$, $a - d$ and $a - b$ plane shown in the upper panels, and $c - d$, $c - b$ and $d - b$ plane in the lower panels. There are several things one can note about the VEVs and hence the structure of the resultant $m_{\nu}$

- $a = 0$ is allowed, since this gives a $m_{\nu}$ which has contributions from $\xi'$ and $\xi''$, discussed in section 3.3,
- $b = 0$ is never allowed since $b$ is needed for TBM mixing as pointed out before,
- $a = b$, $a = c$ and $a = d$ are never allowed,
- $c = d$ is allowed and we can see from the lower left-hand panel how much deviation of $c$ from $d$ can be tolerated,
- $c = 0$ and $d = 0$ can also be tolerated when $a \neq 0$. 

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Inverted Hierarchy

In this case too it’s possible to get inverted hierarchy. The corresponding values of the parameters of $m_\nu$ which allow this are shown as scatter plots in Fig. 5.6. Here $m_0$ has been allowed to take any value, and we show the points projected on the $a - c$, $a - b$, $a - d$ plane in the upper panels and $c - b$, $c - d$, $b - d$ plane in the lower panels. Each of these points also satisfy the $3\sigma$ experimental bounds given in [20]. Note that for $a = 0$ we get the same regions in $b$, $c$ and $d$, as in the right-panel of Fig. 5.4.

5.4 Vacuum Alignment of the Triplet Higgs

In case we do not confine ourselves to $\langle \phi_S \rangle = (v_S, v_S, v_S)$, we would have the general $m_\nu$ given in Eq. (5.6). Since we have argued in the previous section that the only viable scenario where one allows for all three Higgs singlet is when $a \neq c \simeq d$, we will assume that this condition for the singlet terms holds. We further realize that to reproduce a mixing matrix with $\theta_{13} \sim 0$ and $\theta_{23} \sim 45^\circ$, it might be desirable to keep $\mu - \tau$ symmetry in the mass matrix. Therefore, we show our results for the case $\langle \phi_S \rangle = (v_{S1}, v_S, v_S)$. The
mass matrix is then given as

\[ m_\nu = m_0 \begin{pmatrix} a + 2b_1/3 & d - b/3 & d - b/3 \\ d - b/3 & d + 2b/3 & a - b_1/3 \\ d - b/3 & a - b_1/3 & d + 2b/3 \end{pmatrix}. \] (5.21)

Of course for \( b_1 = b \) one would recover the case considered in the previous section and TBM mixing would result. The possibility of \( b_1 \neq b \) gives rise to deviation from TBM mixing. In order to solve this matrix analytically we assume that \( b_1 = b + \eta \) and keep only the first order terms in \( \eta \). The mass eigenvalues obtained are

\[ m_1 = m_0 (a + b - d + \frac{\eta}{3}), \quad m_2 = m_0 (a + 2d), \quad m_3 = m_0 (-a + b + d + \frac{\eta}{3}), \] (5.22)

and the mixing matrix is

\[
\begin{pmatrix}
\sqrt{\frac{2}{3}} \left( 1 - \frac{\eta}{3(3d-b)} \right) & \sqrt{\frac{1}{3}} \left( 1 + \frac{2\eta}{3(3d-b)} \right) & 0 \\
-\sqrt{\frac{1}{6}} \left( 1 + \frac{2\eta}{3(3d-b)} \right) & \sqrt{\frac{1}{3}} \left( 1 - \frac{\eta}{3(3d-b)} \right) & -\sqrt{\frac{1}{2}} \\
-\sqrt{\frac{1}{6}} \left( 1 + \frac{2\eta}{3(3d-b)} \right) & \sqrt{\frac{1}{3}} \left( 1 - \frac{\eta}{3(3d-b)} \right) & \sqrt{\frac{1}{2}}
\end{pmatrix}. \] (5.23)

Therefore, the only deviation from TBM comes in \( \theta_{12} \) and we have

\[ D_{12} \simeq \frac{4\eta}{9(3d-b)} . \] (5.24)

We show in Fig. 5.7 variation of \( \sin^2 \theta_{12} \) with \( \eta \). As expected, \( \sin^2 \theta_{12} \) is seen to deviate further and further from its TBM value of 1/3 as we increase the difference between \( v_S \) and \( v_{S1} \). The other two mixing angles are predicted to be exactly at their TBM values due to the presence of \( \mu - \tau \) symmetry in \( m_\nu \). They would also deviate from TBM once we allow for either \( v_{S2} \neq v_{S3} \) or \( c \neq d \), and in the most general case, both.

5.5 Conclusions

In this work we have seen the implication of \( A_4 \) flavor symmetry on the neutrino oscillation data while emphasizing mostly on tribimaximal mixing pattern. We stick to a most generic setup and we consider all the possibilities, where the one dimensional Higgses \( \xi, \xi', \xi'' \) belong to \( 1, 1', 1'' \) representation under the symmetry group \( A_4 \). Other than this, we also have two \( A_4 \) triplets \( \phi_{T,S} \). We analyze the different cases by taking one, two and all three one dimensional \( A_4 \) Higgs fields at a time and show only few of them can give viable neutrino mass and mixing. To get tribimaximal mixing one would require VEV alignments of the triplet and also specific relations between different VEV’s and Yukawa couplings.
Also we have shown, while the triplet alone is sufficient to provide the tribimaximal mixing, to get the viable neutrino mass splitting the one dimensional representation must be present. We have analyzed the possibilities to get normal and/or inverted hierarchy for these different viable scenarios. The most simple case with $A_4$ singlet $\xi$ will lead to normal hierarchy, while inclusion of other one dimensional representation allows for inverted hierarchy as well. Finally we address the deviation from the tribimaximal mixing and give one illustrative example where the deviation is realized due to deviation from the triplet vacuum alignments.
Bibliography


