Chapter 7

Neutrino mass in SO(10) GUT

In a number of papers it has been shown that renormalisable $SO(10)$ – with and without supersymmetry (SUSY) – is quite predictive and powerful in constraining fermion mass patterns because of the underlying $SU(4)$, symmetry which relates the quark and lepton Yukawa couplings. In $SO(10)$, $16 \otimes 16 = 10 \oplus 120 \oplus 126$ and so Higgs fields giving mass to the $16_F$ can reside in the $10_H$, $120_H$ and $126_H$ representations. Obtaining correct masses for the quarks and the charged leptons requires at least two Higgs multiplets. It has been noted, for example in [1], that any one of the combinations $(10_H, 120_H)$, $(10_H, 126_H)$, or $(120_H, 126_H)$ can, in principle, be utilised. Among these the model with $10_H$ and $126_H$ has been extensively considered as the most successful candidate for the minimal $SO(10)$ GUT [2]. $126_H$ contains colour singlet submultiplets which transform as a triplet under $SU(2)_L$ and a singlet under $SU(2)_R$ or vice versa; these are the cornerstones of the seesaw mechanism [3]. Both type-I (mediated through singlets [3]) and type-II (mediated through scalar triplets [4]) seesaw have been examined for both supersymmetric [5] and non-supersymmetric [6] cases. The $126_H$ relates the Majorana mass of the neutrinos to the Dirac mass as well as other charged fermion masses making the model predictive. It is also possible and in some cases advantageous to include all the three Higgs representations [7,8]. The model with $10_H + 120_H$ [9, 10], on the other hand, does not have the requisite scalars to lead to neutrino masses through the seesaw mechanism. Here, neutrino mass can be obtained at two-loop through the radiative seesaw mechanism due to Witten [11] by adding $16_H + \overline{16}_H$ multiplets. This model has been studied in [12] and it was shown that under plausible assumptions it predicts $b - \tau$ unification, natural occurrence of large leptonic and small quark mixing and large value for the atmospheric mixing angle. However, the radiative seesaw runs into difficulty with
low-energy SUSY although it works well in the context of split SUSY [13]. Moreover, as has been shown in [10] the SUSY SO(10) model containing $10_H$ and $120_H$ cannot reproduce the charged fermion masses correctly. On the other hand in non-SUSY SO(10) the two-loop neutrino mass is very small.

In this chapter we consider the generation of neutrino masses in the $10_H + 120_H$ model embellished with a $\overline{10}_H$ by adding fermions belonging to the adjoint representation (45$F$) of SO(10). Such fermions couple to the usual sixteen-plet of quarks and leptons via the $\overline{10}_H$ and can give rise to neutrino masses through the ‘double seesaw’ mechanism. In models with $10_H + 120_H$ this can serve as an alternative option for generating small neutrino masses\(^1\). Fermions in the triplet adjoint representation of $SU(2)_L$ are also considered in the so called type-III [15] seesaw mechanism. Such models have become quite popular in the context of $SU(5)$ GUTs [16]. $SU(2)_L$ triplet fermions fit naturally into the 24-dimensional representation of $SU(5)$ and can cure two main problems of these theories, viz. generation of neutrino masses and unification of gauge couplings. The latter requires the mass of the fermionic triplets to be $\sim \mathcal{O}(1 \text{ TeV})$ making the model testable at the LHC [17]. Presence of adjoint fermions in the context of left-right symmetric models has been considered in [18], and generation of neutrino masses and possible collider signatures were discussed. From this point of view our model can also be considered as a generalization of type-III seesaw for SO(10). However as in LR symmetric models the mechanism of mass generation here is actually the ‘double seesaw’ mechanism.

We discuss the conditions which the Yukawa coupling matrices should satisfy for the model to have predictive power. This requires ascribing some additional flavour symmetry to the model which we choose to be the generalized $\mu - \tau$ symmetry that has been considered widely for explaining the neutrino mixing angles [19]. It predicts $\theta_{23}$ to be $\pi/4$ which is the best-fit value of this angle from global fits. In addition it implies $\theta_{13} = 0$ which is also consistent with the data. Small deviation from these exact values may be generated by breaking the $\mu - \tau$ symmetry by a small amount. Combining $\mu - \tau$ flavour symmetry with GUTs has been considered in the case of $SU(5)$ in [20] and also for SO(10) [8]. Here we impose $\mu - \tau$ symmetry on the Yukawa matrix for the $10_H$ and $\overline{10}_H$ whereas the one for $120_H$ is taken to be antisymmetric. We also impose a parity symmetry leading to Hermitian Yukawa matrices. Thus we consider the model

\(^1\)It is also possible to get a double seesaw type mass matrix using singlet fields [14].
SO(10) $\otimes Z_2^{(\mu-\tau)} \otimes Z_2^\nu$ [8]. Imposition of these two symmetries help in reducing the number of unknown parameters in the Yukawa sector. In addition, we make an ansatz relating the effective $\nu_R$ mass matrix arising due to the inclusion of adjoint fermions with the Yukawa matrix for $10_H$. As a result the light neutrino mass matrix after seesaw mechanism obtains a simple form and can be written as a sum of two contributions. It turns out that with the above choice the neutrino mass matrix is $\mu-\tau$ symmetric so that one immediately gets $\theta_{13} = 0$ and $\theta_{23} = \pi/4$. It is straightforward to get the consequence for the neutrino masses and $\theta_{12}$ and obtain the conditions on the parameters such that tri-bimaximal mixing is obtained.

7.1 The Model

We explore an $SO(10)$ model where the three fermion families acquire mass through the $10_H$ and/or $120_H$. The model also includes additional fermion multiplets in the $SO(10)$ adjoint representation, $45_F$, and a $16_H$.

In this model the Yukawa terms for the fermions can be expressed as:

$$L = Y_{10}16_F16_F10_H + Y_{120}16_F16_F120_H.$$ (7.1)

In general, $Y_{10}$ is a complex symmetric matrix while $Y_{120}$ is complex antisymmetric. When the $10_H$ and $120_H$ scalars obtain their vacuum expectation values (vevs) quarks and leptons obtain masses which can be represented as:

$$m_d = M_0 + iM_2, \quad m_u = c_0M_0 + ic_2M_2,$$

$$m_l = M_0 + ic_3M_2, \quad m_D = c_0M_0 + ic_4M_2.$$ (7.2)

Above, $m_d$ ($m_u$) denotes the mass matrix for the down-type (up-type) quarks, $m_l$ is the charged lepton mass matrix, whereas $m_D$ is the Dirac mass matrix of the neutrinos. The matrices $M_0$ and $M_2$ are proportional to $Y_{10}$ and $Y_{120}$ respectively.

$$M_0 = M_0^T, \quad M_2 = -M_2^T.$$ (7.3)

$c_0, c_2, c_3,$ and $c_4$ are constants fixed by Clebsch-Gordan (CG) coefficients and vev ratios which are taken to be real. We impose a generalized parity symmetry and make appropriate choices of the vevs [21] which make $M_0$ and $M_2$ real thereby reducing the number of free parameters and ensuring the hermiticity of the mass matrices in eq. 7.2.
For neutrinos the above implies the presence of only the Dirac mass term which cannot reproduce the correct neutrino mass pattern [12]. Since the $\overline{120}_H$ field is not present the type-I and type-II seesaw mass terms are absent in this model. One can of course generate the neutrino mass through the Witten mechanism of radiative seesaw [11] but then for non-SUSY $SO(10)$ such contributions are too small [12].

In this chapter we propose a new mechanism to generate a neutrino mass in a non-SUSY $SO(10)$ with $10_H$ and $120_H$. We introduce additional matter multiplets $(45_F)$ which belong to the adjoint representation of $SO(10)$. Note that this is similar to the so called type-III seesaw mechanism where one adds additional matter fields in the adjoint representation. However, as we will see, the neutrino mass is generated here through the ‘double seesaw’ mechanism.

$SO(10)$ breaks to the SM through two intermediate steps:

$$
SO(10) \xrightarrow{M_X} SU(4)_c \otimes SU(2)_L \otimes SU(2)_R
\xrightarrow{M_C} SU(3)_c \otimes SU(2)_L \otimes U(1)_R \otimes U(1)_{(B-L)}
\xrightarrow{M_R} G_{SM}.
$$

The Pati-Salam ($G_{422} \equiv SU(4)_c \otimes SU(2)_L \otimes SU(2)_R$) decomposition gives:

$$
45 = (\Sigma_{3R}, \Sigma_{3C}, \Sigma_{LRC}) = (1, 3, 1) \oplus (1, 1, 3) \oplus (15, 1, 1) \oplus (6, 2, 2).
$$

It is useful to note the $SU(3)_c \otimes SU(2)_L \otimes U(1)_R \otimes U(1)_{B-L}$ decompositions

$$
(15, 1, 1) \equiv (1, 1, 0, 0) + (3, 1, 0, -4/3) + (3, 1, 0, 4/3) + (8, 1, 0, 0),
\quad
(4, 1, 2) \equiv (1, 1, \pm \frac{1}{2}, 1) + (3, 1, \pm \frac{1}{2}, -1/3).
$$

The colour, $U(1)_R$, and $U(1)_{(B-L)}$ singlet members of $\Sigma_{3R}$ and $\Sigma_{4c}$ couple to $\nu_R$ when $\overline{10}_H$ gets a vev along $(1, 1, -\frac{1}{2}, 1) \subset (4, 1, 2)$ that breaks $U(1)_R \otimes U(1)_{B-L}$.

The relevant Yukawa coupling is:

$$
Y_{161645F}45F \overline{10}_H \supset Y_{16} \left[ a_1(1, 1, \frac{1}{2}, -1)_F(1, 1, 0, 0)_{\overline{\Sigma}^{3R}} + a_2(1, 1, \frac{1}{2}, -1)_F(1, 1, 0, 0)_{\Sigma^{4c}} \right]
(1, 1, -\frac{1}{2}, 1)_H.
$$

$a_{1,2}$ are CG coefficients. The vev $v_R \equiv <(1, 1, -\frac{1}{2}, 1)_{\overline{10}}> \text{sets the scale } M_R$.

The masses of the adjoint matter fields are generated from

$$
M Tr(45^2_F) + \lambda Tr(45^2_F210_H).
$$
Once $210_H$ acquires a vev along the $(1,1,1)$ direction, $SO(10)$ is broken to $SU(4)_c \otimes SU(2)_L \otimes SU(2)_R$. In the mass term $M_N$ of $(1,1,0,0)_F \subset (15,1,1)_F$ and $M_{\Sigma_R}$ of $(1,1,0,0)_F \subset (1,1,3)_F$, an extra contribution (from the second term of eq. 7.8) is added, i.e., $M_{\Sigma_R} = M_N = M + \lambda < (1,1,0,0)_H >$. There is no symmetry that protects the masses of these adjoint fermions. So naturally these are very heavy ($\sim M_X$).

### 7.2 Constraints from gauge coupling unification

In this section, we discuss the Renormalisation Group (RG) evolution of the gauge couplings at the one-loop level, check for the scale of unification and determine the possible intermediate scales. The symmetry breaks in two stages following the steps given in (7.4). The contributions in the RG running from scalars at the different scales are included according to the ‘extended survival hypothesis’ (ESH) [22] which amounts to minimal fine tuning of the parameters of the potential. Our model contains extra adjoint fermions. But these fermions are very heavy $\sim O(M_X)$, so they do not contribute in the renormalisation group evolution of the gauge couplings.

When the $SO(10)$ symmetry is broken to the Pati-Salam group [23] $G_{422}$ by a $210_H$ multiplet through the vev in the $< (1,1,1) >$ direction, D-parity [24] is spontaneously broken at this scale ($M_C$).

The gauge coupling evolution is usually stated as [25]:

$$\mu \frac{dg}{d\mu} = \beta_i(g_i, g_j), \quad (i, j = 1, \ldots, n), \quad (7.9)$$

where $n$ is the number of couplings in the theory and at one-loop order

$$\beta_i(g_i, g_j) = (16\pi^2)^{-1} b_i g_i^3. \quad (7.10)$$

There is, however, a subtlety which must be taken into account since the gauge symmetry in the energy range $M_R$ to $M_C$ includes two $U(1)$ factors. According to the ESH the $SO(10)$ multiplets are split in mass with some submultiplets having mass above and some below this range. The incomplete scalar and fermion multiplets that contribute to the RG evolution at this stage lead to a mixing between these two $U(1)$ gauge groups. Thus even at the one-loop level one cannot treat the evolution of these $U(1)$ couplings in separation and in a generic scenario one
Table 7.1: Higgs submultiplets contributing to the RG evolution as per the extended survival hypothesis when symmetry breaking of $SO(10)$ takes place with two intermediate stages – see (7.4).

<table>
<thead>
<tr>
<th>$SO(10)$ representation</th>
<th>Symmetry breaking</th>
<th>Scalars contributing to RG evolution $M_Z \rightarrow M_R$</th>
<th>$M_R \rightarrow M_C$</th>
<th>$M_C \rightarrow M_X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$G_{SM} \rightarrow EM$</td>
<td>$(1,2,\pm 1)$</td>
<td>$(1,2,\pm \frac{1}{2},0)$</td>
<td>$(1,2,2)$</td>
</tr>
<tr>
<td>120</td>
<td>$G_{3211} \rightarrow G_{SM}$</td>
<td>...</td>
<td>$(1,1,\pm \frac{1}{2},1)$</td>
<td>$(4,1,2)$</td>
</tr>
<tr>
<td>$\overline{16}$</td>
<td>$G_{422} \rightarrow G_{3211}$</td>
<td>...</td>
<td>...</td>
<td>$(15,1,3)$</td>
</tr>
</tbody>
</table>

Taking all this into account, the gauge couplings evolve as follows:

i) From $M_C$ to $M_X$:

\[ b_{2L} = 7/3; \quad b_{2R} = 13; \quad b_{4c} = -1. \] (7.11)

ii) From $M_R$ to $M_C$:

\[ b_{2L} = -3; \quad b_{RR} = 53/12; \quad b_{3c} = -7; \quad b_{(B-L)(B-L)} = 33/8; \]

\[ \tilde{b}_{R(B-L)} = \tilde{b}_{(B-L)R} = -1/4\sqrt{6}. \] (7.12)

iii) From $M_Z$ to $M_R$:

\[ b_{1Y} = 21/5; \quad b_{2L} = -3; \quad b_{3c} = -7. \] (7.13)
Figure 7.1: The allowed ranges of the unification ($M_X$, pale, green) and intermediate Pati-Salam ($M_C$, dark, red) scales as a function of the $U(1)_{B-L}$ breaking scale ($M_R$) for $SO(10)$ with two intermediate scales. The inset is a zoom of the region of interest for generating neutrino masses of the right magnitude.

The mixing of the two $U(1)$ groups adds flexibility to the model. With this, we find for every $M_R$ a range of consistent solutions for $M_C$ and $M_X$ (see Fig. 7.1). In the plot we have exhibited the maximum and minimum values of both $M_C$ and $M_X$ consistent with unification. In a Grand Unified Theory low intermediate scales are always perceived with extra interest. These low intermediate scale scenarios keep alive the hope that signals of the GUT may be identified at accessible energies. In Fig. 7.1, we have shown that $M_R$ and $M_C$ can be quite low ($\sim 10$ TeV) which is within the reach of recent colliders, such as the LHC; this is an artifact of the inclusion of the $U(1)$ mixings. The vev $v_R$ of the scalar $(1, 1, -\frac{1}{2}, 1) \subset \overline{16}$, sets the scale $M_R$. In the next section we have shown that $v_R$ needs to be very high ($\sim 10^{14}$ GeV) to yield the correct neutrino mass with the Yukawa couplings $\sim O(1)$. In the inset of Fig. 7.1 we magnify this range of $M_R$. It is to be noted that this establishes that the proposed model of ‘double-seesaw’ mechanism is compatible with gauge coupling unification at a scale which is not in conflict with the present bound on the proton lifetime.
7.3 Neutrino Mass

The neutrino mass matrix in the basis \((v_L)^c, v_R, \Sigma_0^R, N)\) is:

\[
M_\nu = \begin{pmatrix}
0 & m_D & 0 & 0 \\
0 & m_D^T & 0 & a_1 Y_{16} v_R \\
0 & a_1 Y_{16}^T v_R & M_N & 0 \\
0 & a_2 Y_{16}^T v_R & 0 & M_N
\end{pmatrix}.
\] (7.14)

The left-handed fermionic triplets, \(\Sigma_{3L}\), having a mass matrix identical to \(M_N\), do not mix with other fermions since the left-handed analogue of \(v_R\) is chosen to be zero. From the mass matrix (7.14) it is seen that the masses of the light neutrinos are obtained by integrating out the heavy triplet and singlet fermions. Thus we can have type-III and type-I seesaw mechanism in succession.

The right-handed neutrino mass term is generated once the heavy triplet fermion \(\Sigma_3^R\) and \(N\) are integrated out – an effective type-I + type-III seesaw. Assuming \(M_N \gg v_R Y_{16} \gg m_D\), the right-handed neutrino mass matrix is:

\[
M_R = v_R^2 Y_{16} M_M^{-1} Y_{16}^T.
\] (7.15)

where,

\[
M_M^{-1} = (a_1^2 + a_2^2) M_N^{-1},
\] (7.16)

and the light neutrino mass matrix after an effective type-I seesaw becomes:

\[
m_\nu = M_D M_R^{-1} m_D^T.
\] (7.17)

Substituting for \(m_D\) from eq. 7.2 one arrives at the general expression of \(m_\nu\) as

\[
m_\nu = c_0^2 M_0 M_R^{-1} M_0 - c_0 c_4 M_0 M_R^{-1} M_0 + c_4 M_0 M_2 M_R^{-1} M_0 + c_4^2 M_2 M_R^{-1} M_2.
\] (7.18)

Typical values for the various parameters are \(v_R \sim 10^{14}\) GeV, \(M_N \sim 10^{15}\) GeV, and \(c_i \sim O(1), Y_i \sim O(1)\) which gives \(M_R \sim 10^{12}\) GeV. Then with \(m_D \sim 100\) GeV one gets \(m_\nu \sim 1\) eV.

With three neutrino generations, the model has 6 real parameters in \(M_0\) and 3 in \(M_2\). In addition there are 5 \(v\)evs \((c_0, c_2, c_3, c_4, v_R)\). Besides, there are additional parameters in \(Y_{16}\) and \(M_N\). However the low energy neutrino mass matrix is characterized by 9 parameters. Neutrino oscillation experiments have so far determined and/or bounded 5 of these. The general case is obviously not sufficiently constrained. One way to address this lacuna requires invoking some flavour symmetry. We consider this to be the \(\mu - \tau\) symmetry.
7.4 $\mu - \tau$ symmetry and allowed textures

$\mu - \tau$ symmetry has been considered widely for explaining the large atmospheric mixing angle in the neutrino sector [19]. In addition it gives $\theta_{13} = 0$ which is also consistent with the current global fits\(^2\). We impose the condition of a generalized $\mu - \tau$ symmetry on the Yukawa matrices stemming from \(10_H\) and \(\overline{16}_H\). This implies that these matrices are invariant under the exchange of the second and third rows and columns. This reduces the number of unknown parameters in the Yukawa sector. However, this symmetry cannot be exact in the quark and lepton sector. This is accomplished by the term \(M_2\) in the fermion mass matrices which originates from the \(120_H\) which is taken to be antisymmetric under the exchange of 2 \leftrightarrow 3 and breaks $\mu - \tau$ symmetry spontaneously.

In addition we had imposed a generalized parity symmetry [21] which makes the complex matrices \(M_0\) and \(M_2\) real thereby reducing the number of free parameters. Thus the model that we consider is \(SO(10) \otimes Z_2^{\mu-\tau} \otimes Z_2^P\) [8]. However it is to be mentioned that if we assume exact $\mu - \tau$ (anti)symmetry in \((M_2)\) \(M_0\) then a generalized CP-invariance holds [8] and the CKM matrix comes out as real. This can be rectified either by assuming some of the \(v\)evs to be complex or by allowing a small explicit breaking of $\mu - \tau$ symmetry in \(M_0\). This induces CP-violation phases in both \(U_{\text{CKM}}\) and \(U_{\text{PMNS}}\) [8]. We work in the basis where the charged lepton mass matrix is diagonal and the PMNS matrix is solely determined by the mixing in the neutrino sector.

The structures for \(M_0\) and \(M_2\) under the above symmetries are given by

\[
M_0 = \begin{pmatrix}
a' & b' & b' \\
b' & c' & d' \\
b' & d' & c'
\end{pmatrix}, \quad M_2 = \begin{pmatrix}
0 & x' & -x' \\
-x' & 0 & y' \\
x' & -y' & 0
\end{pmatrix}.
\]

We consider a model with three adjoint fermion multiplets, i.e., the model consists of \((3\nu_L + 3\nu_R + 3N + 3\Sigma_R)\). Thus, \(Y_{16}\) and \(M_N\) are also \(3 \times 3\) matrices which we take to be $\mu - \tau$ symmetric. It follows from eq. 7.15 that \(M_R\) also respects this symmetry. Thus we have both \(M_0\) and \(M_R\) to be $\mu - \tau$ symmetric. In order to make the model predictive we make the further assumption that \(M_R\) and \(M_0\) are proportional, i.e.,

\[
KM_R = M_0.
\]

\(^2\)Recent global fits have found indication for non-zero $\theta_{13}$ although this is only a 1$\sigma$ effect. A small non-zero value of $\theta_{13}$ can be induced by breaking the $\mu - \tau$ symmetry.
where $K$ is a constant. $m_\nu$ in eq. 7.18 then takes the form

$$m_\nu = Kc_0^2M_0 + Kc_2^2M_2M_0^{-1}M_2 = M_1 + M_1'$ \quad (7.21)$$

The number of free real parameters in the theory are now 4 from $M_0$, 2 in $M_2$, and 4 real vevs. Because of eq. 7.20 $M_R$ adds just one further parameter. Thus in total we have 11 real parameters. The vev ratios $c_2$ and $c_3$ do not affect eq. 7.21 and thus we have 9 parameters involved in the neutrino sector. Some of these appear only as overall scale factors.

We note that although $M_2$ is $\mu - \tau$ antisymmetric the product $M_2M_0^{-1}M_2$ possesses $\mu - \tau$ symmetry. Thus $m_\nu$ is $\mu - \tau$ symmetric. This immediately implies $\theta_{13} = 0$ and $\theta_{23} = \pi/4$. Therefore the mixing matrix in the basis where the charged lepton mass matrix is diagonal is given as,

$$U_{PMNS} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12}/\sqrt{2} & c_{12}/\sqrt{2} & 1/\sqrt{2} \\ -s_{12}/\sqrt{2} & c_{12}/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}, \quad (7.22)$$

which can be brought to the standard $U_{PMNS}$ form by a suitable redefinition of fermion phases. We have

$$m_\nu = U_{PMNS}M_{dia}U_{PMNS}^T, \quad (7.23)$$

where $M_{dia} = \text{Diag}(m_1,m_2,m_3)$. $m_1,m_2,m_3$, the mass eigenvalues are real \(^3\), and are given as

$$m_1 = \frac{X - \sqrt{X^2 - 4(d - c)Y}}{2(d - c)},$$

$$m_2 = \frac{X + \sqrt{X^2 - 4(d - c)Y}}{2(d - c)},$$

$$m_3 = \frac{Y}{2b^2 - ac - ad}. \quad (7.24)$$

Here

$$X = -ac - c^2 + ad + d^2 + 2x^2 + y^2;$$

$$Y = 2b^2c - ac^2 - 2b^2d + ad^2 + 2cx^2 + 2dx^2 + 4bxy + ay^2. \quad (7.25)$$

\(^3\)Since the mass matrices have real entries, complex roots can appear only in conjugate pairs leading to unacceptable degenerate neutrinos. We take the eigenvalues to be all non-negative.
and

\[ a = Kc_0 d', \quad b = Kc_0^2 b', \quad c = Kc_0^2 c', \quad d = Kc_0^2 d', \quad x = Kc_4^2 x', \quad y = Kc_4^2 y'. \]  \hspace{1cm} (7.26)

Note that the eigenstate \( m_3 \) is determined to be the one associated with the eigenvector \((0, 1/\sqrt{2}, -1/\sqrt{2})\). Whether this is the highest mass state or the lowest mass state, i.e., whether the hierarchy is normal or inverted will depend on the values of the parameters. We further require \( \Delta m^2_{21} > 0 \) from the solar data. This implies that for our choice of \( m_2 \) and \( m_1 \)

\[ \frac{X}{(d - c)^2} \sqrt{X^2 - 4(d - c)Y} > 0 \]  \hspace{1cm} (7.27)

Using eqs. 7.22 and 7.23 we obtain,

\[ \tan \theta_{12} = \frac{1}{\sqrt{2}} \frac{(a - m_1)(c - d) - 2x^2}{b(c - d) + xy}. \]  \hspace{1cm} (7.28)

The condition for tri-bimaximal mixing implies

\[ (a - m_1 - b)(c - d) = 2x^2 + xy. \]  \hspace{1cm} (7.29)

### 7.4.1 \( 10_H \) dominance

In this case, \( a, b, c, d \gg x, y \). The light neutrino mass matrix \( m_\nu \) is approximated as \( Kc_0^2 M_0 \) with \( M_0 \) defined in eq. 7.19. In this limit the mass eigenvalues are given as,

\[ m_1 = \frac{1}{2}(f_1 - R), \quad m_2 = \frac{1}{2}(f_1 + R), \quad m_3 = c - d, \]  \hspace{1cm} (7.30)

with

\[ R = \sqrt{8b^2 + f_2^2}, \]  \hspace{1cm} (7.31)

where,

\[ f_1 = a + c + d, \quad f_2 = -a + c + d. \]  \hspace{1cm} (7.32)

Again, \( m_3 \) is identified as the eigenvalue for the state eigenvector \((0, 1/\sqrt{2}, -1/\sqrt{2})\). Since the solar data has determined the ordering of the 1 and 2 mass states to Then the mass squared differences can be expressed as,

\[ \Delta m^2_{21} = f_1 R \quad \Delta m^2_{31} = (f_1 R - a^2 - 4b^2 + c^2 + d^2 - 6cd)/2. \]  \hspace{1cm} (7.33)

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Again, the mass ordering will depend on the values of the parameters. In general both normal and inverted hierarchy are possible. In addition, the solar neutrino data require $\Delta m^2_{21} > 0$ which implies $f_1 R > 0$ for the above selection of states.

The mixing angles are given as,

$$\theta_{13} = 0, \quad \theta_{23} = \pi/4, \quad \tan \theta_{12} = \frac{(R - f_2)}{2\sqrt{2}b}. \quad (7.34)$$

Tri-bimaximal mixing implies $\theta_{13} = 0, \theta_{23} = \pi/4$ and $\tan^2 \theta_{12} = 1/2$. We see that the requirements for $\theta_{13}$ and $\theta_{23}$ are already satisfied. If in addition we impose

$$f_2 = b \quad \Rightarrow \quad R = 3b, \quad f_1 = (2a + b), \quad (7.35)$$

tri-bimaximal mixing is obtained. In this limit

$$\Delta m^2_{21} = 3b (2a + b) \quad \Delta m^2_{31} = (c - d)^2 - (a - b)^2. \quad (7.36)$$

### 7.4.2 $120_H$ dominance

In this limit $a, b, c, d \ll x, y$ and the low energy neutrino mass matrix is given as

$$m_\nu = M_4 = K c_4^2 M_2 M_0^{-1} M_2. \quad (7.37)$$

The $U_{PMNS}$ continues to be given by eq. 7.22. The eigenvalues, in terms of the parameters defined in eq. 7.26, are given as,

$$m_1 = 0, \quad m_2 = \frac{2x^2 + y^2}{d - c}, \quad m_3 = \frac{2cx^2 + 2d x^2 + 4bxy + ay^2}{2b^2 - ac - ad}. \quad (7.38)$$

Since the eigenvector $\left(0, 1/\sqrt{2}, -1/\sqrt{2}\right)$ belongs to the eigenvalue $m_3$ so that the zero eigenvalue has to be associated with the eigenstate $m_1$. Therefore this case corresponds to the normal hierarchy. Since $m_1 = 0$, $\Delta m^2_{21} = m^2_2$ and $\Delta m^2_{31} = m^2_3$. Then, using eqs. 7.22 and 7.23 one obtains the 1-2 mixing angle as,

$$\tan \theta_{12} = \frac{-\sqrt{2}x}{y}. \quad (7.39)$$

Thus, the mixing matrix in this case is completely determined by the parameters of $M_2$. The condition for obtaining exact tri-bimaximal mixing is $y = -2x$.

In summary in this chapter we have considered a non-SUSY $SO(10)$ model in which the fermion masses originate from Yukawa couplings to $10_H$ and $120_H$. 

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In such a model the usual type-I and type-II seesaw mass terms which arise from $\overline{126}_H$ are not present. It is possible to generate the neutrino mass at two-loops by the radiative seesaw mechanism [11]. But for non-SUSY $SO(10)$ the contribution is very small.

Here we suggest a new possibility to generate neutrino masses in a non-SUSY $SO(10)$ model with $10_H + 120_H$ using fermions in the $45_F$ representation and an additional $\overline{16}_H$ scalar multiplet. Constraints from gauge coupling unification requires the vev $< \overline{16}_H >$ to be in the range $\sim 10^4 - 10^{16}$ GeV. However from the standpoint of generation of naturally small neutrinos masses the range $\sim 10^{13} - 10^{15}$ GeV is preferred. We show that in this case one can generate small neutrino masses through the ‘double seesaw’ mechanism. Predictions for mixing angles require further imposition of a flavour symmetry which we chose to be the $\mu - \tau$ symmetry for the Yukawa matrices due to $10_H$ and $\overline{16}_H$ whereas for the one originating from $120_H$ we take the matrix to be $\mu - \tau$ antisymmetric. We further assume the right-handed matrix ($M_R$) due to the heavy fields to be proportional to the one ($M_0$) originating from $10_H$. With this the light neutrino mass matrix is given by the sum of two terms which are both $\mu - \tau$ symmetric. This automatically satisfies $\theta_{13} = 0$ and $\theta_{23} = \pi/4$. We present the neutrino masses and $\theta_{12}$ obtained from this model and determine the condition for satisfying tri-bimaximality. We also discuss the limiting values when one of the terms dominate. For the $10_H$-dominance case both hierarchies are possible whereas if the $120_H$ dominates the hierarchy can only be normal.