Abstract

Nonclassicality and entanglement are two important features exhibited by continuous variable quantum states. This thesis is centered on the connection between nonclassicality and entanglement in the context of continuous variable quantum systems. Evidently, nonclassicality is a prerequisite for entanglement. The connection between the two has been well explored in the context of Gaussian states, namely in the context of squeezing nonclassicality. We study the connection in the context of other well known nonclassicalities, namely nonclassical photon number statistics and antibunching. By definition, every classical state is a convex sum of coherent states, and hence is separable. Nonclassicality does not imply entanglement, but every entangled state is nonclassical. Negativity under Partial Transpose (NPT) implies nonclassicality, but Positivity under Partial Transpose (PPT) by itself does not indicate that the state is classical or separable. A PPT state can be separable or entangled, can be classical or nonclassical.

Chapter 1 is primarily introductory in nature, bringing forth the various concepts involved in the theory of entanglement, both in the finite dimensional situation as well as in the infinite dimensional case of continuous variable systems. It is expository in nature and collects some of the techniques useful later in the thesis.

In Chapter 2 we bring forth a relationship between nonclassicality and entanglement. The problem of studying the interrelationship between nonclassicality and entanglement is tied to the fact that there is no simple test which can conclude in a definite manner if a given generic mixed state is classical or not, and there is no single test which can answer with certainty if a mixed state is entangled or separable. However, in very special or specific cases one can make definitive statements. For states of a single mode of radiation which are diagonal in the Fock basis, the issue of classicality/nonclassicality has been settled. This is possible thanks to the result of the classical Stieltjes moment problem [170]. We bring out the possibility of using such nonclassical (non-Gaussian) resources to generate useful entanglement. With a product state of the form $\hat{\rho}_{in}^{(ab)} = \hat{\rho}^{(a)} \otimes |0\rangle_{bb}\langle 0|$ as input, the output two-mode state $\hat{\rho}_{out}^{(ab)}$ of a beamsplitter is shown to be NPT whenever the photon number distribution (PND) statistics $\{p(n_a)\}$ associated with the mixed state $\hat{\rho}^{(a)}$ of the input $a$-mode is antibunched or otherwise nonclassical, i.e., if $\{p(n_a)\}$ fails to respect any one of an infinite sequence of necessary and sufficient classicality conditions. We establish the equivalence of classicality and PPT of $\hat{\rho}_{out}^{(ab)}$ in this kind of situations. Thus NPT is a necessary and sufficient test of entanglement of $\hat{\rho}_{out}^{(ab)}$. Furthermore $\hat{\rho}_{out}^{(ab)}$ is shown to be distillable if $\hat{\rho}^{(a)}$ is antibunched or violates any one of an infinite sequence of three term classicality conditions. We also discuss the issue of distillability arising from an intrinsically higher order violation of classicality. This is the only second instance in
continuous variable entanglement theory where NPT has turned out to be a necessary and sufficient criterion for entanglement, the earlier instance being that of two-mode Gaussian states. A preliminary version of these results is found in [194]. We attempt to estimate the entanglement of formation (EOF) of entangled states generated in the above manner. We evaluate both upper and lower bounds on EOF for very special examples. Our principal tool in this scheme is the fact that average entanglement does not increase under local operations and classical communications (LOCC). The general idea used has been to project out the state into 2 × 2 subspaces, and then use Wootters’s formula for the entanglement of formation of a two-qubit system to estimate the entanglement; such a process is clearly an LOCC. However, a drawback with such a scheme is the fact that one cannot estimate more than one ebit of entanglement even from a highly entangled state. For the simple example of an entangled state generated by passing through a 50:50 beamsplitter an arbitrary mixture of the ground state and $n^{\text{th}}$ Fock state on Alice’s side, with Bob’s side in the ground state, we give a distillation procedure whereby we distill more entanglement than given by lower bound for EOF in [76]. We extend these ideas to entangled states generated from PND’s which correspond to a very special superposition of coherent states, and we demonstrate distillation procedures which distill well above one ebit of entanglement. We also indicate the possibility of using the Terhal-Vollbrecht formula [69,76] in estimating entanglement, in a more general context, using a truncation scheme.

The study undertaken in Chapter 2 is continued in Chapter 3 from a more general perspective. We describe a single test which, if successful, is able to simultaneously establish both the nonclassicality and NPT entanglement of a given two-mode state. We extend the notion of antibunching to two-mode systems through the Mandel matrix construct, and show that nonclassicality at this level naturally separates into two distinct kinds, Type I and Type II, depending on whether the sub Poissonian statistics is visible or not at a single-mode level. The “Type” of a nonclassical state is invariant under the action of every $U(2)$ beamsplitter. A state could go from separable to entangled under beamsplitter action, but its Type is invariant. Type II states are special in the sense that one may pass such states through any $U(2)$ beamsplitter, even then can never detect antibunching locally i.e., in a single-mode. We construct examples of both types. We introduce a beamsplitter invariant definition for the Mandel parameter, extended to the case of two-mode systems through the nonpositivity of the Mandel matrix. That we are able to do so is because the Mandel matrix transforms covariantly under beamsplitter action. However, we find that the two-mode Mandel parameter can take values less than −1, as compared to the Mandel parameter in the single-mode case. This feature seems to expose the limitation of the beamsplitter as an entangling device, as there are en-
tangled states that the beamsplitter cannot produce. The two-mode Mandel parameter is relevant only within the Type under consideration. We explore the production of bipartite entanglement from separable nonclassical states by beamsplitters, we trace back the entanglement to the nonclassicality involved, and we illustrate this aspect through several examples. We demonstrate distillable entanglement in this context. We extend these ideas to the case of generating tripartite entanglement through generalised beamsplitters, and examine their detection through simple moment-based tests which trace back the entanglement to a particular type of nonclassicality. We also demonstrate the possibility of generating genuine tripartite entanglement from two-mode Mandel type nonclassicality.

In Chapter 4 the EOF of an arbitrary two-mode Gaussian state is computed. In this context, we bring out the intimate connection between the two-mode squeeze parameter as a measure of the strength of nonclassicality and alternatively as a measure of entanglement. Apart from a conjecture, our analysis rests on two main ingredients. One of them is a four-parameter canonical form we develop for the covariance matrix, one of these parameters, the squeeze parameter, acting as a measure of EOF. The other is the generalisation of the EPR correlation used in the work of Giedke et al [70] to noncommuting variables. The conjecture is in respect of an extremal property of this correlation [327].

In Chapter 5 we study the compatibility conditions between the (global) spectrum and the spectra of the individual modes of a general $n$-mode Gaussian state. We present an elementary proof for the compatibility conditions, making optimal use of beamsplitter and two-mode squeezing transformations. An unexpected by-product of our elementary approach is the result that every two-mode Gaussian state is uniquely determined, modulo local transformations, by its global spectrum and local spectra, a property shared not even by a pair of qubits [18].

In Chapter 6 we obtain the operator-sum representation of all the quantum limited single-mode Bosonic Gaussian channels. The analysis lends insight into how certain unphysical processes such as the transposition map, or scaling of the Weyl-ordered characteristic function, or a combination of both can be rendered physical through a threshold Gaussian noise. The motive here is to bring out this aspect in a transparent manner through the operator-sum representation. We have that the scaling of the diagonal weight function and scaling of the Husimi $Q$ function correspond to physical processes. As will be seen in the following Chapter, the fact that scaling of the $Q$ function is physical is of critical relevance when one defines a measure of non-Gaussianity for quantum states. This Chapter further explores the notion of nonclassicality breaking and the notion of entanglement breaking in light of the operator-sum representation.
Having brought out the connection between nonclassicality and entanglement, and having exposed nonclassicality as a resource, it is useful to understand this resource as being Gaussian and non-Gaussian. Chapters 2 and 3 primarily dealt with non-Gaussian states and the nonclassicality associated with them, but Chapters 4, 5, and 6, dealt with Gaussian states and issues regarding them. In Chapter 7 we bring out the essential difference between these two very different resources through the consideration of cumulants. Since the higher order cumulants defined through an $s$-ordered quasi-probability is independent of the ordering parameter $s$ and hence is intrinsic to the state, every nonvanishing cumulant of order greater than two serves as an indicator of non-Gaussianity. We introduce a new measure for non-Gaussianity based on the negentropy of the $Q$ function. We show that our measure satisfies some of the requirements that a good non-Gaussianity measure should satisfy, especially the invariance of the measure under uniform scaling of the $Q$ function. The scale invariance of the measure is demanded by the fact that scaling of the $Q$ function is a valid physical transformation as shown in Chapter 6. The measure is well supported by the fact that the Marcinkiewicz theorem holds for phase space distributions too [358]. We analytically evaluate this non-Gaussianity measure for mixed entangled states generated by passing the photon-added thermal state through a $U(2)$ beamsplitter, the ancilla being in the ground state. We find for these examples that the non-Gaussianity as evaluated by our measure, is independent of temperature, which is a direct manifestation of scale invariance. That we are able to evaluate the non-Gaussianity for these mixed entangled states is because of the invariance of the measure under passive transformations. We also evaluate the measure for the phase-averaged coherent state. In a recent work [361, 362], Genoni et al introduced distance based measures of non-Gaussianity of a state through the Hilbert-Schmidt distance and relative entropy defined at the density operator level. We compare their measure with ours for the simple example of the photon-added thermal state [216].

Finally we conclude with some remarks and discuss possible future directions of research, particularly in the context of the use of non-Gaussian resources in quantum information processing.