6

Operator-sum representation for Bosonic Gaussian channels

6.1 Introduction

Gaussian states are fully specified by their first and second moments. Since the first moments play no significant role in our study, we may assume that they vanish (this can indeed be ensured using the unitary Weyl-Heisenberg displacement operators), so that a Gaussian state for our purpose is fully described by its covariance matrix [192, 202, 226, 354]. The symplectic group of real linear canonical transformations (acting through its unitary metaplectic representation) and the Weyl-Heisenberg group of phase space translations are the only unitary evolutions which preserve Gaussianity, and these groups are generated by hermitian Hamiltonians which are respectively quadratic and linear in the creation and annihilation operators [192, 202, 226].

Any physical evolution that maps an input Gaussian state to a Gaussian state at the output is a Gaussian channel. In other words, Gaussian channels are those trace preserving completely positive (CP) maps which image every input Gaussian state into a Gaussian state at the output. The feasibility of processing information using Gaussian channels was originally explored in [355, 356]. More recently, the problem of evaluating the classical capacity of Gaussian channels was addressed in [141, 149, 282], and the quantum capacities in [157, 159, 281, 283, 284]. In particular, the classical capacity of the attenuator channel was evaluated in [282], and the quantum capacity of a class of channels was studied in [159]. A systematic study of the structure of the family of all Gaussian channels has been carried out in [158, 286–288, 357]; single-mode Gaussian channels have been classified in [158, 286], and the case of multimodes in [287, 288, 357].

Gaussian channels may be realized as Gaussianity preserving unitaries on a suitably
enlarged system:

\[ \rho_A \rightarrow \rho_A' = \text{Tr}_B \left( U_{AB} (\rho_A \otimes \rho_B) U_{AB}^\dagger \right). \]  

(6.1)

Here \( \rho_B \) is a Gaussian state of the ancilla B, and \( U_{AB} \) is a linear canonical transformation on the enlarged composite system consisting of the system of interest A and the ancilla B. That all Gaussian channels can indeed be realized in this manner has been shown by the work of Holevo and coauthors [158, 286, 288, 357].

It is clear that the most general trace-preserving linear map \( \Omega \) which takes Gaussian characteristic functions to Gaussian, taking states with vanishing first moments to ones with vanishing first moments, are necessarily of the form \( \Omega : \chi(\xi) \rightarrow \chi'(\xi) = \chi(X\xi) \exp\left[-\frac{1}{2} \xi^T Y \xi\right] \), where \( X, Y \) are real matrices with \( Y = Y^T \geq 0 \). And \( X, Y \) need to obey an appropriate matrix inequality to ensure that the trace-preserving map \( \Omega \) is completely positive [157, 159, 358, 359]. For a given \( X \), the minimal \( Y \), say \( Y_0 \), meeting this inequality represents the threshold Gaussian noise that needs to be added to \( \chi(X\xi) \) to make atonement for the failure of \( X \) to be a symplectic matrix, and thus rendering the map completely positive; if \( X \) happens to be a symplectic matrix, then the corresponding minimal \( Y_0 = 0 \).

Now, given a Gaussian channel \( \Omega \) we can construct, ‘quite cheaply’, an entire family of Gaussian channels by simply preceding and following \( \Omega \) with unitary (symplectic) Gaussian channels \( U(S_1), U(S_2) \) corresponding respectively to symplectic matrices \( S_1, S_2 \). Therefore in classifying Gaussian channels it is sufficient to classify these orbits or double cosets and, further, we may identify each orbit with the ‘simplest’ looking representative element of that orbit (the canonical form). Since

\[ U(S_1) \Omega U(S_2) : \chi(\xi) \rightarrow \chi(S_2 X S_1 \xi) \exp\left[-\frac{1}{2} \xi^T S_1^T Y S_1 \xi\right], \]  

(6.2)

the task actually reduces to enumeration of the orbits of \((X, Y)\) under the transformation \((X, Y) \rightarrow (X', Y') = (S_2 X S_1, S_1^T Y S_1)\).

The injection of an arbitrary amount of classical (Gaussian) noise into the state is obviously a Gaussian channel: \( \chi(\xi) \rightarrow \chi(\xi) \exp\left[-\frac{a}{2} \xi^T \xi\right], a > 0 \). It is called the classical noise channel. Now, given a Gaussian channel we may follow it up with a classical noise channel to obtain another Gaussian channel. A Gaussian channel will be said to be quantum-limited if it cannot be realized as another Gaussian channel followed by a classical noise channel. Conversely, the most general Gaussian channel is a quantum-limited Gaussian channel followed by a classical noise channel, and it follows that quantum-limited channels are the primary objects which need to be classified into orbits.

In the single-mode case where \( X, Y \) are \( 2 \times 2 \) matrices, \( S_1, S_2 \in Sp(2, R) \) can be so
chosen that $X'$ equals a multiple of identity, a multiple of $\sigma_3$, or $(I + \sigma_3)/2$ while $Y'$ equals a multiple of identity or $(I + \sigma_3)/2$. Thus the canonical form of a Gaussian channel $X,Y$ is fully determined by the rank and determinant of $X,Y$ and we have the following classification of quantum-limited bosonic Gaussian channels [158, 286]

\[
\begin{align*}
\mathcal{D}(\kappa;0) : & \quad X = -\kappa \sigma_3, & \quad Y_0 = (1 + \kappa^2)I, \quad \kappa > 0; \\
\mathcal{C}_1(\kappa;0) : & \quad X = \kappa I, & \quad Y_0 = (1 - \kappa^2)I, \quad 0 \leq \kappa \leq 1; \\
\mathcal{C}_2(\kappa;0) : & \quad X = \kappa I, & \quad Y_0 = (\kappa^2 - 1)I, \quad \kappa \geq 1; \\
\mathcal{A}_1(0) : & \quad X = 0, & \quad Y_0 = I; \\
\mathcal{A}_2(0) : & \quad X = (I + \sigma_3)/2, & \quad Y_0 = I; \\
\mathcal{B}_2(0) : & \quad X = I, & \quad Y_0 = 0; \\
\mathcal{B}_1(0) : & \quad X = I, & \quad Y_0 = 0.
\end{align*}
\]

It may be noted that the quantum-limited end of both the $\mathcal{B}_1$ and $\mathcal{B}_2$ families is the trivial identity channel.

By following the above listed quantum-limited channels by injection of classical noise of magnitude $a$ we get respectively $\mathcal{D}(\kappa;a)$, $\mathcal{C}_1(\kappa;a)$, $\mathcal{C}_2(\kappa;a)$, $\mathcal{A}_1(a)$, $\mathcal{A}_2(a)$, and $\mathcal{B}_2(a)$; the last case $\mathcal{B}_1(a)$ is special in that it is obtained from $\mathcal{B}_1(0)$ by injection of noise into just one quadrature: $\chi(\xi) \to \chi(\xi) \exp[-\xi^T \xi \sigma_3]$. It is clear in the case of $\mathcal{D}(\kappa;0)$ that $X = -\kappa \sigma_3$ corresponds to (scaled) phase conjugation or matrix transposition of the density operator. And the phase conjugation is the most famous among positive maps which are not CP [27, 28, 57]: it is the injection of additional classical noise of magnitude (not less than) $1 + \kappa^2$, represented by $Y_0$, that mends it into a CP map.

It is well known that every trace-preserving completely positive map has an operator-sum representation of the form

\[
\rho \to \rho' = \sum_a W_a \rho W^\dagger_a, \quad \sum_a W^\dagger_a W_a = I,
\]

often called Kraus representation [7]. It may be noted, however, that this representation appears as Theorem 4 of a much earlier work of Sudarshan et al [31]. It has been presented also by Choi [6], apparently independently. Mathematicians seem to view it as a direct and immediate consequence of the dilation theorem of Stinespring [9].

In this Chapter we obtain the operator-sum representation of all the quantum limited single-mode Bosonic Gaussian channels. Our analysis lends insight into how unphysical processes such as the transposition map, or the scaling of Weyl-ordered characteristic function, or a combination of both can be rendered physical through a threshold Gaussian.
noise. The motive here is to bring out this aspect in a transparent manner through the operator-sum representation. We have that scaling of the diagonal weight function and scaling of the Husimi $Q$ function correspond to physical processes. As will be seen in the following Chapter, the fact that scaling of the $Q$ function is physical is of critical relevance when one defines a measure of non-Gaussianity for quantum states. This Chapter further explores the notion of nonclassicality breaking and the notion of entanglement breaking in light of the operator-sum representation.

We begin with the illustration a general scheme for computation of Kraus operators, and this scheme applies uniformly to all quantum-limited Gaussian channels. This scheme takes particular advantage of the fact that the symplectic two-mode transformation which realizes the channel in the sense of (6.1) does not couple, in the Holevo canonical form, the position variables with the momentum variables. With the ancilla mode assumed to be in its vacuum state initially, it turns out that the Kraus operators for each channel can be simply read off from the matrix elements of the appropriate two-mode metaplectic operator. Even though the single-quadrature classical noise channels $B_1(a)$, $a \neq 0$ [$B_1(0)$ is the identity channel] are not quantum-limited, we deal with them briefly just to bring out the fact that this case too is obedient to our general computational scheme.

### 6.2 Kraus representation: Some general considerations

Given density operator $\rho^{(a)}$ describing the state of a single-mode radiation field, the action of a quantum-limited Gaussian channel takes it to [158, 286]

$$\rho^{′(a)} = \text{Tr}_b (U^{′(ab)} \rho^{(a)} \otimes |0\rangle_b \langle 0|) U^{(ab)\dagger}.$$  (6.5)

Here $|0\rangle_b$ is the vacuum state of the ancilla mode $b$, and $U^{(ab)}$ is the unitary operator corresponding to a suitable two-mode linear canonical transformation. It is convenient to perform the partial trace in the Fock basis of mode $b$. We have

$$\rho^{′(a)} = \sum_\ell b_\ell \langle \ell | U^{(ab)} \rho^{(a)} \otimes |0\rangle_b \langle 0| U^{(ab)\dagger} |\ell\rangle_b.$$  (6.6)

Clearly, $b_\ell \langle \ell | U^{(ab)} | 0\rangle_b$ is an operator acting on the Hilbert space of mode $a$. The last expression thus leads us to the Kraus representation of the channel [7]:

$$\rho \rightarrow \rho^{′(a)} = \sum_\ell W_\ell \rho^{(a)} W_\ell^\dagger, \quad W_\ell = b_\ell \langle \ell | U^{(ab)} | 0\rangle_b.$$  (6.7)
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It follows that once the Fock basis matrix elements of $U^{(ab)}$ are known, the Kraus operators $W_\ell$ can be easily read off. Let $\langle m_1 m_2 | U^{(ab)} | n_1 n_2 \rangle \equiv C_{n_1 n_2}^{m_1 m_2}$ be the matrix elements of $U^{(ab)}$ in the two-mode Fock basis. Since the ancilla mode $b$ is assumed to be in the vacuum state, the $W_\ell$’s are obtained by setting $n_2 = 0$ and $m_2 = \ell$:

$$W_\ell = \sum_{n_1, m_1 = 0}^{\infty} C_{n_1 n_2}^{m_1 m_2} | m_1 \rangle \langle n_1 |.$$  

(6.8)

Now, in evaluating $C_{n_1 n_2}^{m_1 m_2}$ it proves useful to employ a resolution of identity in the position basis [205]:

$$C_{n_1 n_2}^{m_1 m_2} = \langle m_1 m_2 | U^{(ab)} | n_1 n_2 \rangle = \int_{-\infty}^{\infty} dx_1 dx_2 \langle m_1 m_2 | x_1 x_2 \rangle \langle x_1 x_2 | U^{(ab)} | n_1 n_2 \rangle.$$  

(6.9)

Under conjugation by $U^{(ab)}$ the quadrature variables $q_j, p_j$ $(j = 1, 2)$ undergo a linear canonical transformation $S \in Sp(4, R)$, of which $U^{(ab)}$ is the (metaplectic) unitary representation [192]. Let us assume that this canonical transformation does not mix the position variables with the momentum variables. That is,

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \rightarrow U^{(ab)\dagger} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} U^{(ab)} = \begin{pmatrix} q'_1 \\ q'_2 \end{pmatrix} = M \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \rightarrow U^{(ab)\dagger} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} U^{(ab)} = \begin{pmatrix} p'_1 \\ p'_2 \end{pmatrix} = (M^{-1})^T \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

(6.10)

where $M$ is a real non-singular $2 \times 2$ matrix. This assumption that our $S \in Sp(4, R)$ has the direct sum structure $S = M \oplus (M^{-1})^T$ will prove to be of much value in our analysis. We have

$$C_{n_1 n_2}^{m_1 m_2} = \int_{-\infty}^{\infty} dx_1 dx_2 \langle m_1 m_2 | x_1 x_2 \rangle \langle x_1 x_2 | U^{(ab)} | n_1 n_2 \rangle$$

$$= \int_{-\infty}^{\infty} dx_1 dx_2 \langle m_1 m_2 | x_1 x_2 \rangle \psi_{n_1} (x'_1) \psi_{n_2} (x'_2)$$

$$= \int_{-\infty}^{\infty} dx_1 dx_2 \psi_{n_1}^* (x'_1) \psi_{m_2}^* (x'_2) \psi_{n_1} (x'_1) \psi_{n_2} (x'_2),$$

(6.11)

where $(x'_1, x'_2)$ is linearly related to $(x_1, x_2)$ through $M$. These wavefunctions are the familiar Hermite functions, the Fock states in the position representation. The above
integral may be evaluated using the generating function for Hermite polynomials [205]:

\[ \psi_n(x) = \pi^{-1/4} \frac{e^{-x^2}}{\sqrt{2^n n!}} H_n(x) = \pi^{-1/4} \frac{\partial^n}{\partial z^n} \exp \left( -\frac{1}{2} ((x - z\sqrt{2})^2 - z^2) \right) \bigg|_{z=0}. \]  

(6.12)

Inserting in Eq. (6.11) the generating function for each of the four wavefunctions we have

\[ C_{n_1n_2}^{m_1m_2} = \frac{1}{\sqrt{n_1! n_2! m_1! m_2!}} \frac{\partial^{m_1}}{\partial \eta_1^{m_1}} \frac{\partial^{m_2}}{\partial \eta_2^{m_2}} F(z_1, z_2, \eta_1, \eta_2) \bigg|_{z_1, z_2, \eta_1, \eta_2 = 0}, \]  

(6.13)

where

\[ F(z_1, z_2, \eta_1, \eta_2) = \pi^{-1} \int_{-\infty}^{\infty} dx_1 dx_2 \exp \left\{ -\frac{1}{2} ((x_1 - \eta_1 \sqrt{2})^2 + (x_2 - \eta_2 \sqrt{2})^2 \right. \\
\left. + (x'_1 - z_1 \sqrt{2})^2 + (x'_2 - z_2 \sqrt{2})^2 - \eta_1^2 - \eta_2^2 - z_1^2 - z_2^2) \right\}. \]  

(6.14)

The Gaussian integration over the variables \( x_1 \) and \( x_2 \) can be easily carried out to obtain \( F(z_1, z_2, \eta_1, \eta_2) \), and from \( F(z_1, z_2, \eta_1, \eta_2) \) we may readily obtain \( C_{n_1n_2}^{m_1m_2} \), and hence the Kraus operators. This is the general scheme we will employ in what follows to obtain Kraus representation for quantum-limited Gaussian channels of the various families.

6.3 Phase conjugation or transposition channel \( \mathcal{D}(\kappa), \kappa \geq 0 \)

We now use the above scheme to evaluate a set of Kraus operators representing the phase conjugation channel. The metaplectic unitary operator \( U^{(ab)} \), appropriate for this case induces on the quadrature operators of the bipartite phase space a linear canonical transformation corresponding to the following \( S \in Sp(4, R)[158] \):

\[ S = \begin{pmatrix}
\sinh \mu & 0 & \cosh \mu & 0 \\
0 & -\sinh \mu & 0 & \cosh \mu \\
\cosh \mu & 0 & \sinh \mu & 0 \\
0 & \cosh \mu & 0 & -\sinh \mu
\end{pmatrix}. \]  

(6.15)
Written in detail, the phase space variables undergo, under the action of this channel, the transformation

\[
\begin{pmatrix}
q_1 \\
q_2
\end{pmatrix} \rightarrow \begin{pmatrix}
q'_1 \\
q'_2
\end{pmatrix} = M \begin{pmatrix}
q_1 \\
q_2
\end{pmatrix},
\]

\[
\begin{pmatrix}
p_1 \\
p_2
\end{pmatrix} \rightarrow \begin{pmatrix}
p'_1 \\
p'_2
\end{pmatrix} = (M^{-1})^T \begin{pmatrix}
p_1 \\
p_2
\end{pmatrix},
\]

\[
M = \begin{pmatrix}
-\sinh \mu & \cosh \mu \\
\cosh \mu & -\sinh \mu
\end{pmatrix}.
\]

(6.16)

It is seen that the above \( S \) is indeed of the form \( S = M \oplus (M^{-1})^T \in Sp(4, R) \), and does not mix the position variables with the momentum variables, and so our general scheme above readily applies.

It is clear from the structure of \( S \) that the parameter \( \mu \) is related to \( \kappa \) in \( D(\kappa) \) through \( \kappa = -\sinh \mu > 0 \), so that \( \cosh \mu = \sqrt{\kappa^2 + 1} \). Thus (6.14) translates, for the present case, to the following expression:

\[
F(z_1, z_2, \eta_1, \eta_2) = \pi^{-1} \int_{-\infty}^{\infty} dx_1 dx_2 \exp \left\{ -\frac{1}{2} \left[ (x_1 - \eta_1 \sqrt{2})^2 + (x_2 - \eta_2 \sqrt{2})^2 \\
+ (-\kappa x_1 + \sqrt{1 + \kappa^2} x_2 - z_1 \sqrt{2})^2 + (\sqrt{1 + \kappa^2} x_1 - \kappa x_2 - z_2 \sqrt{2})^2 \\
- \eta_1^2 - \eta_2^2 - z_1^2 - z_2^2 \right] \right\}.
\]

(6.17)

Performing the Gaussian integrals in \( x_1 \) and \( x_2 \) we obtain

\[
F(z_1, z_2, \eta_1, \eta_2) = (\sqrt{1 + \kappa^2})^{-1} \exp \left\{ (\sqrt{1 + \kappa^2})^{-1} (\eta_1 \eta_2 - z_1 z_2) \\
+ (\sqrt{1 + \kappa^2})^{-1} (\eta_1 z_2 + \eta_2 z_1) \right\}.
\]

(6.18)

To obtain the matrix elements \( C_{m_1 m_2}^{n_1 n_2} \) we need to carry out the procedure indicated in Eq. (6.13). This may be done in two steps. We begin by rewriting the function \( F(z_1, z_2, \eta_1, \eta_2) \) as

\[
F(z_1, z_2, \eta_1, \eta_2) = (\sqrt{\kappa^2 + 1})^{-1} \exp \left\{ z_2 [(\sqrt{1 + \kappa^2})^{-1} \eta_1 - (\sqrt{1 + \kappa^2})^{-1} z_1] \\
+ \eta_2 [(\sqrt{1 + \kappa^2})^{-1} \eta_1 + (\sqrt{1 + \kappa^2})^{-1} z_1] \right\}.
\]

(6.19)

Performing the \( z_2 \) and \( \eta_2 \) differentiations respectively \( n_2 \) and \( m_2 \) times on \( F(z_1, z_2, \eta_1, \eta_2) \),
we obtain
\[
[(\sqrt{1 + \kappa^2})^{-1}\eta_1 - (\sqrt{1 + \kappa^{-2}})^{-1}z_1]^{m_2} \times
[(\sqrt{1 + \kappa^{-2}})^{-1}\eta_1 + (\sqrt{1 + \kappa^2})^{-1}z_1]^{m_2} F \equiv GF. \quad (6.20)
\]

The remaining differentiations can be carried out using the Leibniz rule. Since we finally set \( z_1, z_2, \eta_1, \eta_2 = 0 \), and since \( F(0) = 1 \), the only terms that could possibly survive are necessarily of the form
\[
\frac{\partial^{m_1}}{\partial \eta_1^{m_1}} \frac{\partial^{m_1}}{\partial z_1^{m_1}} [(\sqrt{1 + \kappa^2})^{-1}\eta_1 - (\sqrt{1 + \kappa^{-2}})^{-1}z_1]^{m_2} \times
[(\sqrt{1 + \kappa^{-2}})^{-1}\eta_1 + (\sqrt{1 + \kappa^2})^{-1}z_1]^{m_2}. \quad (6.21)
\]

To evaluate the above expression we set \( x = (\sqrt{\kappa^2 + 1})^{-1}\eta_1 - (\sqrt{1 + \kappa^{-2}})^{-1}z_1 \) and \( y = (\sqrt{1 + \kappa^2})^{-1}\eta_1 + (\sqrt{1 + \kappa^{-2}})^{-1}z_1 \), and compute
\[
[(\sqrt{1 + \kappa^2})^{-1}\partial_x + (\sqrt{1 + \kappa^{-2}})^{-1}\partial_y]^{m_1} \times
[-(\sqrt{1 + \kappa^{-2}})^{-1}\partial_x + (\sqrt{1 + \kappa^2})^{-1}\partial_y]^{m_1} x^{n_2} y^{m_2} |_{x,y=0}. \quad (6.22)
\]

Straight forward algebra leads, in view of Eq. (6.13), to
\[
C_{n_1n_2}^{m_1m_2} = \frac{(\sqrt{1 + \kappa^2})^{-1}}{\sqrt{n_1!n_2!m_1!m_2!}} \sum_{j=0}^{m_1} \sum_{r=0}^{m_2} C_j^{m_1} C_r^{m_1} (-\sqrt{1 + \kappa^{-2}})^{-m_1+j-r} (\sqrt{1 + \kappa^2})^{-n_1-j+r} \times (-1)^{m_1-r} n_2!m_2! \delta_{n_2,r+j} \delta_{m_2,n_1-j+m_1-r}. \quad (6.23)
\]

The Kraus operators \( W_\ell \), denoted \( T_\ell(\kappa) \) in this case, are obtained from these matrix elements by setting \( n_2 = 0 \) and \( m_2 = \ell \). Since \( n_2 = 0 \Rightarrow r, j = 0 \), we have,
\[
T_\ell(\kappa) = (\sqrt{1 + \kappa^2})^{-1} \sum_{n_1,m_1=0}^{\infty} (\sqrt{1 + \kappa^{-2}})^{-n_1} (-\sqrt{1 + \kappa^{-2}})^{-m_1} \sqrt{\ell!} \times
\delta_{\ell,n_1+m_1} (-1)^{m_1} n_1! m_1! \langle n_1 | \langle n_1 |. \quad (6.24)
\]

We set \( n_1 + m_1 = \ell \) and denote \( n_1 = n \), leading to
\[
T_\ell(\kappa) = (\sqrt{1 + \kappa^2})^{-1} \sum_{n=0}^{\ell} (\sqrt{1 + \kappa^2})^{-n} (\sqrt{1 + \kappa^{-2}})^{-(\ell-n)} \times
\sqrt{\ell C_n} \delta_{\ell-n} \langle n |, \quad \ell = 0, 1, 2, \ldots \quad (6.25)
\]
as our final form for the Kraus operators of the phase conjugation channel. We note that the $T_\ell(\kappa)$’s are real and manifestly trace-orthogonal: $\text{tr}(T_\ell(\kappa)T_\ell'(\kappa)) = 0$ if $\ell \neq \ell'$.

### 6.3.1 The dual of $\mathcal{D}(\kappa)$

As is well known (and also obvious), if a set of Kraus operators $\{W_\ell\}$ describes the completely positive map $\Phi: \rho \rightarrow \rho' = \sum_\ell W_\ell \rho W_\ell^\dagger$, then the dual map $\bar{\Phi}: \rho \rightarrow \rho' = \sum_\ell W_\ell^\dagger \rho W_\ell$, described by the dual or adjoint set of operators $\{W_\ell^\dagger\}$, is also completely positive. It is clear that the dual map $\bar{\Phi}$ is unital or trace-preserving according as $\Phi$ is trace-preserving or unital.

For the present case of $\mathcal{D}(\kappa)$, it is readily verified that the Kraus operators $\{T_\ell(\kappa)\}$ presented in (6.25) meet $\sum_\ell T_\ell(\kappa)T_\ell(\kappa) = \mathbb{I}$, consistent with the expected trace-preserving nature of $\rho \rightarrow \rho' = \sum_\ell T_\ell(\kappa) \rho T_\ell(\kappa)$. But the phase conjugation channel is not unital in general, for we have

$$\sum_\ell T_\ell(\kappa)T_\ell^\dagger(\kappa) = \kappa^{-2} \mathbb{I}. \quad (6.26)$$

We may say that it is ‘almost unital’ to emphasise the minimal nature of the failure: the unit element is taken by the channel into a scalar multiple of itself. However, the scale factor $\kappa^{-2}$ can not be transformed away by absorbing $\kappa^{-1}$ into the Kraus operators, for the Kraus operators so modified would not then respect the trace-preserving property of the map.

It is thus of interest to understand the nature of the unital channel described by the set of Kraus operators $\{T_\ell(\kappa)^\dagger\}$. We have

$$T_\ell(\kappa)^\dagger = (\sqrt{1 + \kappa^2})^{-1} \sum_{n=0}^{\ell} (\sqrt{1 + \kappa^2})^{-n} (\sqrt{1 + \kappa^{-2}})^{-(\ell-n)} \sqrt{C_n|\ell-n|} \ell \ell - n|n\rangle$$

$$= (\sqrt{1 + \kappa^2})^{-1} \sum_{n'=\ell}^{\ell} (\sqrt{1 + \kappa^2})^{-(\ell-n')}(\sqrt{1 + \kappa^{-2}})^{-(n')} \sqrt{C_{\ell-n'}|\ell-n'|} \ell \ell - n'|n'|$$

$$= (\sqrt{1 + \kappa^2})^{-1} \sum_{n=0}^{\ell} (\kappa^2 + 1)^{-(\ell-n)} (\sqrt{1 + \kappa^{-2}})^{-n} \sqrt{C_n|\ell-n|} \ell \ell - n|n\rangle$$

$$= \kappa^{-1} T_\ell(\kappa^{-1}). \quad (6.27)$$

Thus the dual $\{T_\ell(\kappa)^\dagger\}$ differs from the original $\{T_\ell(\kappa)\}$ in two elementary aspects. The multiplicative factor $\kappa^{-1}$ is the same for all Kraus operators, independent of $\ell$. Thus the only significant difference is change in the argument of $T_\ell$, from $\kappa$ to $\kappa^{-1}$. We conclude that the ‘dual’ channel whose Kraus operators are $\kappa T_\ell(\kappa)^\dagger$ is the (trace-preserving) phase conjugation channel $\mathcal{D}(\kappa^{-1})$. We have thus proved
Theorem 6.1 While the Kraus operators \( \{T_\ell(\kappa)\} \) describe \( D(\kappa) \), the ‘dual’ channel described by Kraus operators \( \{\kappa T_\ell(\kappa)^\dagger\} \) is the trace-preserving phase conjugation channel \( D(\kappa^{-1}) \) with reciprocal scale parameter.

6.3.2 Action of the Kraus operators

The expected or defining action of the phase conjugation channel on the characteristic function is [158]:

\[
\chi_W(\xi) \rightarrow \chi'_W(\xi) = \chi_W(-\kappa \xi^*) \exp[-(1 + \kappa^2)|\xi|^2/2].
\]

(6.28)

It is of interest to understand how the ‘antilinear’ phase conjugation \( (\xi \rightarrow \xi^*) \) action of this channel on the characteristic function emerges from the linear action of the Kraus operators. To this end, it is sufficient to establish such an action on the ‘characteristic function’ corresponding to the operators \( |n\rangle \langle m| \), for arbitrary pairs of integers \( n, m \geq 0 \). The ‘characteristic function’ of \( |n\rangle \langle m| \) is given by [169]

\[
\chi_W[n\rangle \langle m| \equiv \langle m|D(\xi)|n\rangle \\
= \sqrt{\frac{m!}{n!}}(\xi^*)^{n-m}L_{n-m}^{m-n}(|\xi|^2) \exp[-|\xi|^2/2] \text{ for } n \geq m, \\
= \sqrt{\frac{n!}{m!}}(\xi)^{m-n}L_{m-n}^{m-n}(|\xi|^2) \exp[-|\xi|^2/2] \text{ for } n \leq m.
\]

(6.29)

Assuming \( n \geq m \), the action of the phase conjugation channel on the operator \( |n\rangle \langle m| \) is

\[
\sum_{\ell=0}^{\infty} T_\ell(\kappa)|n\rangle \langle m|T_\ell(\kappa)^\dagger = (1 + \kappa^2)^{-1} \sum_{\ell=n}^{\infty} (\sqrt{1 + \kappa^2})^{-(n+m)} (\sqrt{1 + \kappa^2})^{-(2\ell-n-m)} \times \\
\sqrt{\ell C_{n+m}} \langle \ell - n| \langle \ell - m|.
\]

(6.30)

Denoting \( n = m + \delta \) and \( \ell - n = \lambda \), we have

\[
\sum_{\ell=0}^{\infty} T_\ell(\kappa)|m + \delta\rangle \langle m|T_\ell(\kappa)^\dagger = (1 + \kappa^2)^{-1} (\sqrt{1 + \kappa^2})^{-(2m+\delta)} (\sqrt{1 + \kappa^2})^{-\delta} \\
\times \sum_{\lambda=0}^{\infty} \frac{1}{\sqrt{(m + \delta)!m!\lambda!}} \frac{1}{(1 + \kappa^2)^{\lambda + \delta}} |\lambda\rangle \langle \lambda + \delta|.
\]

(6.31)

The manner in which \( D(\kappa) \), matrix transposition accompanied by threshold Gaussian noise \( \exp[-(1 + \kappa^2)|\xi|^2/2] \), acts as a channel may now be appreciated. Every operator \( M \) can be written in the Kronecker delta basis \( \{j\rangle \langle \ell| \} \) as \( M = \sum_{j,\ell} c_{j\ell}|j\rangle \langle \ell| \). The coefficient
matrix $C$ associated with $|5\rangle\langle 3|$, for instance, is $c_{i,k} = \delta_{ij}\delta_{k3}$, with non-zero entry only at the lower-diagonal location $(5, 3)$ marked $\otimes$ in the matrix below.

$$
\begin{pmatrix}
0 & 0 & \times & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \times & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \oplus & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \times & 0 & 0 & \cdots \\
0 & 0 & \otimes & 0 & 0 & 0 & \times & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \times \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
$$

On transposition this entry moves to the upper-diagonal location $(3, 5)$ marked $\oplus$, and the threshold noise then spreads it along the parallel upper diagonal $(3+r, 5+r)$, $-3 \leq r < \infty$ marked $\times$.

Let the Weyl-ordered characteristic function $\text{tr}(D(\xi)|m + \delta\rangle\langle m|)$ where $D(\xi) = \exp[\xi a^\dagger - \xi^* a]$ is the displacement operator, be denoted $\chi_{W|m + \delta\rangle\langle m|}(\xi)$, and that of the output $\sum_{\ell=0}^{\infty} T_{\ell}(\kappa)|m + \delta\rangle\langle m|T_{\ell}(\kappa)^\dagger$ be denoted $\chi'_{W|m + \delta\rangle\langle m|}(\xi)$. Then we have from Eq. (6.31)

$$
\chi'_{W|m + \delta\rangle\langle m|}(\xi) = (1 + \kappa^2)^{-1} (\sqrt{1 + \kappa^2})^{2m + \delta} (\sqrt{1 + \kappa^{-2}})^{-\delta} \\
\times \sum_{\lambda=0}^{\infty} \frac{(\lambda + m + \delta)! (1 + \kappa^{-2})^{-\lambda}}{\sqrt{(m + \delta)!m!\lambda!(\lambda + \delta)!}} (\lambda + \delta|D(\xi))|\lambda) \\
= \frac{(1 + \kappa^2)^{-1} e^{-|\xi|^2/2}}{\sqrt{(m + \delta)!m!}} (\sqrt{1 + \kappa^2})^{2m + \delta} (\sqrt{1 + \kappa^{-2}})^{-\delta} \\
\times \sum_{\lambda=0}^{\infty} (1 + \kappa^{-2})^{-\lambda} \frac{(\lambda + m + \delta)!}{(\lambda + \delta)!} \xi^\delta L^\delta_{\lambda}(|\xi|^2),
$$

(6.32)

where we used (6.29), the Fock basis representation of the displacement operator. While no ‘phase conjugation’ is manifest as yet, we expect from Eq. (6.28) that the channel should take the characteristic function of $|m + \delta\rangle\langle m|$ to

$$
\chi''_{W|m + \delta\rangle\langle m|}(\xi) = \langle m|D(-\kappa^* \xi^*)|m + \delta\rangle \exp \left[ -\frac{1}{2} (1 + \kappa^2)|\xi|^2 \right] \\
= \langle m + \delta|D(\kappa \xi^*)|m\rangle^* \exp \left[ -\frac{1}{2} (1 + \kappa^2)|\xi|^2 \right] \\
= \sqrt{\frac{m!}{m + \delta!}} (\kappa \xi^*)^\delta L^\delta_m(\kappa^2|\xi|^2) \exp \left[ -\left(\frac{1}{2} + \kappa^2\right)|\xi|^2 \right].
$$

(6.33)
Thus the problem reduces to one of establishing equality of $\chi'_W(m+\delta)(\xi)$ in (6.32) and $\chi''_W(m+\delta)(\xi)$ in (6.33). That is, it remains to prove
\[
\sqrt{\frac{m!}{m+\delta!}}(\kappa\xi^\delta I_m(\kappa^2|\xi|^2)|\lambda\rangle\langle\lambda|^{2m}\exp\left[-\frac{1}{2} + \kappa^2\right]|\xi|^2]
= \frac{1+\kappa^2}{\sqrt{(m+\delta)!}}\sum_{\lambda=0}^{\infty}(1+\kappa^{-2})^{-\lambda}\sqrt{1+\kappa^2} - (2m+\delta)\sqrt{1+\kappa^{-2} - \delta}
\times \frac{(\lambda+m+\delta)!}{(\lambda+\delta)!}\xi^\delta L^\delta_n(|\xi|^2),
\tag{6.34}
\]
for all $m, \delta \geq 0$ [the case of $|m\rangle\langle m+\delta|$ can be handled similarly].

Since the associated Laguerre functions form a complete orthonormal set, we may expand the LHS of Eq. (6.34) in the Laguerre basis. That is, we multiply both sides of Eq. (6.34) by $L^\delta_n(|\xi|^2) e^{-\frac{|\xi|^2}{2}}$ and evaluate the overlap integrals. We use the following two standard results: (i) orthogonality relation among Laguerres, and (ii) the overlap between a Laguerre and a scaled Laguerre function [360]:
\[
\int_{0}^{\infty} e^{-|\xi|^2} |\xi|^{2\delta} L^\delta_n(|\xi|^2) L^\delta_m(|\xi|^2) d|\xi|^2 = \frac{(n+\delta)!}{n!}\delta_{n,m},
\]
\[
\int_{0}^{\infty} e^{-|\xi|^2} |\xi|^{2\delta} L^\delta_n(n^2|\xi|^2) L^\delta_n(|\xi|^2) d|\xi|^2 = \frac{(m+\ell+\delta)!}{m!}\frac{(t-\eta^2)^m(t-1)^\ell}{t^{m+\ell+\delta+1}} \times F\left[-m,-\ell; -m-\ell-\delta,\frac{t(t-\eta^2-1)}{(t-1)(t-\eta^2)}\right].
\tag{6.35}
\]

Here $F[\cdot]$ is the hypergeometric function. In our case $t = \eta^2 + 1$, which implies that the last argument of $F[\cdot]$ in Eq. (6.35) is zero, and thereby $F[\cdot] = 1$. Performing the overlap integrals, we obtain for the left and right hand sides of (6.34)
\[
\text{LHS} = \frac{(m+\ell+\delta)!}{\ell!\sqrt{(m+\delta)!m!}} \frac{\kappa^{2\ell+\delta}}{(1+\kappa^2)^{m+\ell+\delta+1}},
\]
\[
\text{RHS} = \frac{(m+\ell+\delta)!}{\ell!\sqrt{(m+\delta)!m!}} \frac{\kappa^{2\ell+\delta}}{(1+\kappa^2)^{(2m+\delta+1)}(1+\kappa^{-2})^{-(2m+\delta)}}. \tag{6.36}
\]
These two expressions obviously equal one another for all $\ell$. We have thus established Eq. (6.34), and the fact that the Kraus operators indeed effect the ‘completely positive phase conjugation’ operation, transforming the characteristic function as expected in (6.28).

**Theorem 6.2** The scaled phase conjugation transformation $\chi'_W(\xi) \to \chi''_W(\xi) = \chi_W(-\kappa\xi^\ast) \exp[-(1 + \kappa^2)]|\xi|^2/2\rangle\langle\xi|$ is, in view of the threshold noise $\exp[-(1 + \kappa^2)|\xi|^2/2]$ a completely positive map, and is implemented linearly by the Kraus operators $\{T_\ell(\kappa)\}$ in
Eq. (6.25).

The phase conjugation channel has an interesting property in respect of classicality/nonclassicality of the output states. We may say a channel is nonclassicality breaking if the output of the channel is classical for every input state. That is, if the normal-ordered characteristic function $\chi'_N(\xi)$ of the output, related to the Weyl-ordered characteristic function $\chi'_W(\xi)$ of (6.28) through $\chi'_N(\xi) = \chi'_W(\xi) \exp[|\xi|^2/2]$, is such that its Fourier transform, called the diagonal ‘weight’ function $\phi(\alpha)$ [112], is a genuine probability density.

Now, Eq. (6.28) written in terms of the normal-ordered characteristic function reads

$$\chi_N(\xi) \rightarrow \chi'_N(\xi) = \chi_W(-\kappa \xi^*) \exp[-\kappa^2 |\xi^*|^2/2]$$

$$= \chi_A(-\kappa \xi^*), \quad (6.37)$$

where $\chi_A(\xi) = \chi_N(\xi) \exp[-|\xi|^2]$ is the antinormal-ordered characteristic function corresponding to the $Q$ or Husimi distribution.

Under Fourier transformation this important relation (6.37), namely $\chi'_N(\xi) = \chi_A(-\kappa \xi^*)$, reads that the output diagonal weight function $\phi'(\alpha)$ evaluated at $\alpha$ equals the input $Q(\alpha)$ evaluated at $\kappa^{-1} \alpha^*$. Thus $\phi'(\alpha)$ is a genuine probability density for every input state, and we have

$$\mathcal{D}(\kappa) : \phi_{in}(\alpha) \rightarrow \phi_{out}(\alpha) = \kappa^{-2} Q_{in}(\kappa^{-1} \alpha^*). \quad (6.38)$$

Since the $Q$-distribution of a density operator is given by $Q(\alpha) = \langle \alpha | \rho | \alpha \rangle$, it is a genuine probability distribution for all states including nonclassical states. We have thus proved

Theorem 6.3 The phase conjugation channel is a nonclassicality breaking channel.

6.3.3 Entanglement breaking property

It is known that the phase conjugating channel is entanglement breaking [361, 362]. It is also known that every entanglement breaking channel has a description in terms of rank one Kraus operators [363]. We demonstrate these aspects using our Kraus operators $\{T_{\ell}(\kappa)\}$.

The Kraus operators $T_{\ell}(\kappa)$ presented in (6.25) are not of unit rank; indeed, rank $T_{\ell}(\kappa) = \ell + 1$, $\ell = 0, 1, 2, \cdots$. We noted immediately following (6.25) that $T_{\ell}(\kappa)$ are trace-orthogonal. In the generic case, trace-orthogonality requirement would render the Kraus operators unique, but this is not true with the present situation. The reason is that all these trace-orthogonal $T_{\ell}(\kappa)$’s have the same Frobenius norm: $\text{tr} \left(T_{\ell}(\kappa)T_{\ell}(\kappa)^\dagger\right) =$
$(1 + \kappa^2)^{-1}$, independent of $\ell$. Thus the set $\{T'_\ell\}$ defined through $T'_\ell(\kappa) = \sum_\ell U_{\ell r} T_\ell(\kappa)$, for any unitary matrix $(U_{\ell r})$ will be a set of trace-orthogonal Kraus operators describing the same channel as the original trace-orthogonal set $\{T_\ell(\kappa)\}$.

More generally, and independent of trace-orthogonality, the map $\rho \rightarrow \rho' = \sum_{\alpha} T'_\alpha(\kappa) \rho T'^\dagger_{\alpha}(\kappa)$ describes the same channel as $\rho \rightarrow \rho' = \sum_\ell T_\ell(\kappa) \rho T^\dagger_\ell(\kappa)$ if the matrix $U$ connecting the sets $\{T_\ell(\kappa)\}$ and $\{T'_\alpha(\kappa)\}$ is an isometry [6, 364]:

$$T'_\alpha(\kappa) = \sum_{\alpha} U_{\ell \alpha} T_\ell(\kappa), \quad \sum_{\alpha} U_{\ell \alpha} U^*_{r \alpha} = \delta_{\ell r},$$

$$\Rightarrow \sum_\ell T_\ell(\kappa) \rho T^\dagger_\ell(\kappa) = \sum_{\alpha} T'_\alpha(\kappa) \rho T'^\dagger_{\alpha}(\kappa). \quad (6.39)$$

If the index set $\alpha$ is continuous, as in the case below, then $\sum_{\alpha}$ is to be understood, of course, as an integral. Now, the matrix elements between coherent states $|\alpha\rangle$ and Fock states $|k\rangle$ define such an isometry

$$U_{\ell \alpha} \equiv \langle \ell | \alpha \rangle = \exp[-|\alpha|^2/2] \frac{\alpha^\ell}{\sqrt{\ell!}}. \quad (6.40)$$

The resulting new Kraus operators $T'_\alpha(\kappa)$ are

$$T'_\alpha(\kappa) = e^{-|\alpha|^2/2} \sum_{\ell=0}^\infty \frac{\alpha^\ell}{\sqrt{\ell!}} T_\ell(\kappa)$$

$$= e^{-|\alpha|^2/2} \sum_{\ell=0}^\infty \frac{\alpha^\ell}{\sqrt{\ell!}} (\sqrt{1 + \kappa^2})^{-1} \sum_{n=0}^\ell \sqrt{\ell C_n} (\sqrt{1 + \kappa^2})^{-n} (\sqrt{1 + \kappa^{-2}})^{-(\ell-n)} |\ell - n\rangle \langle n|$$

$$= e^{-|\alpha|^2/2} \sum_{\ell=0}^\infty (\sqrt{1 + \kappa^2})^{-1} \sum_{n=0}^\ell \frac{[(\sqrt{1 + \kappa^2})^{-1}\alpha]^n [((\sqrt{1 + \kappa^{-2}})^{-1}\alpha)^{\ell-n}]}{\sqrt{(\ell-n)!n!}} |\ell - n\rangle \langle n|$$

$$= \frac{1}{\sqrt{1 + \kappa^2}} \langle 0| (\sqrt{1 + \kappa^{-2}}) (\alpha^*/\sqrt{1 + \kappa^2}), \forall \alpha \in \mathcal{C}. \quad (6.41)$$

It is manifest that rank $T'_\alpha(\kappa) = 1$ for all $\alpha \in \mathcal{C}$, the complex plane, showing that the phase conjugation channel is indeed entanglement breaking. However $\{T'_\alpha(\kappa)\}$ are not trace-orthogonal even though $\{T_\ell(\kappa)\}$ from which the former are constructed were trace-orthogonal. This is due to the fact that the isometry $U$ defined in (6.40) is not an unitary, which in turn is a consequence of the overcompleteness of the coherent states.

This brings us to another aspect of $\mathcal{D}(\kappa)$. In terms of these new Kraus operators the
phase conjugation channel $\mathcal{D}(\kappa)$ reads
\[
\rho \rightarrow \rho' = \pi^{-1} \int d^2 \alpha T^\dagger_\alpha(\kappa) \rho T_\alpha(\kappa)
= \pi^{-1}(1 + \kappa^2)^{-1} \int d^2 \alpha Q((\sqrt{1 + \kappa^2})^{-1}\alpha^*)|\alpha/\sqrt{1 + \kappa^2}\rangle\langle\alpha/\sqrt{1 + \kappa^2}|. \quad (6.42)
\]
Thus the diagonal weight function of the output state of the channel is the $Q$-distribution of the input state $\rho$: $\phi_{\text{out}} = \kappa^{-2}Q_{\text{in}}(\kappa^{-1}\alpha^*)$. We may combine this result with the earlier one on rank one Kraus operators to state

**Theorem 6.4** The diagonal weight of the output of the quantum-limited phase conjugation channel is essentially the $Q$-distribution of the input state. The channel $\mathcal{D}(\kappa)$ is not only classicality breaking, but also entanglement breaking.

The diagonal weight of the output state at $\alpha$ is the $Q$-distribution of the input state evaluated at $\kappa^{-1}\alpha^*$. Since $Q(\alpha) \geq 0$ for all $\alpha$ and for any $\rho$, the channel is nonclassicality breaking. The intimate relationship between this result and the earlier one on nonclassicality breaking may be noted. While the former followed directly from the behaviour of the characteristic function, the present one required consideration of the Kraus operators.

### 6.4 Beamsplitter/attenuator channel $\mathcal{C}_1(\kappa)$, $0 < \kappa < 1$

The two-mode unitary operator corresponding to the beamsplitter channel induces the following symplectic transformation on the quadrature operators of the bipartite phase space [158]:

\[
S = \begin{pmatrix}
\cos \theta & 0 & -\sin \theta & 0 \\
0 & \cos \theta & 0 & -\sin \theta \\
\sin \theta & 0 & \cos \theta & 0 \\
0 & \sin \theta & 0 & \cos \theta
\end{pmatrix}.
\quad (6.43)
\]

Note that $S$ is a direct sum of identical two-dimensional rotations: as in the case of $\mathcal{D}(\kappa)$, the position and momentum operators are not mixed by this transformation. The position variables transform as
\[
\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \rightarrow \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} = M \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.
\quad (6.44)
\]
and, consequently, the momentum variables as
\[
\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \rightarrow \begin{pmatrix} p'_1 \\ p'_2 \end{pmatrix} = (M^{-1})^T \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = M \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.
\]
(6.45)

It is evident from $S$ that the parameter $\kappa$ in $C_1(\kappa)$ is related to $\theta$ through $\cos \theta = \kappa$, $\sin \theta = \sqrt{1 - \kappa^2}$. The function $F(z_1, z_2, \eta_1, \eta_2)$ of (6.14) for the present case is given by
\[
F(z_1, z_2, \eta_1, \eta_2) = \exp \left[ \eta_2 (\sqrt{1 - \kappa^2} z_1 + \kappa z_2) + \eta_1 (\kappa z_1 - \sqrt{1 - \kappa^2} z_2) \right].
\]
(6.46)

As in the previous case of $\mathcal{D}(\kappa)$, the differentiation on $F(z_1, z_2, \eta_1, \eta_2)$ can be performed in a straightforward manner to obtain the matrix elements of the unitary operator [326], leading to
\[
C_{n_1 | n_2}^{m_1 | m_2} = \frac{1}{\sqrt{n_1! n_2! m_1! m_2!}} \sum_{r=0}^{n_1} \sum_{j=0}^{n_2} n_1^r n_2^j C_r^j ( -1)^{n_2-j} \kappa^{n_1-r+j} (\sqrt{1 - \kappa^2})^{r+n_2-j} \\
\times m_1! m_2! \delta_{m_2, r+j} \delta_{m_1, n_1+n_2-r-j}.
\]
(6.47)

Now, to obtain the Kraus operators from these matrix elements we set, as in the case of $\mathcal{D}(\kappa)$, $n_2 = 0$ and $m_2 = \ell$. Setting $n_2 = 0 \Rightarrow j = 0$, and we have
\[
B_\ell(\kappa) = \sum_{m=0}^{\infty} \sqrt{m+\ell} C_\ell^m (\sqrt{1 - \kappa^2})^\ell \kappa^m |m\rangle \langle m+\ell|, \quad \ell = 0, 1, 2, \cdots
\]
(6.48)
as the Kraus operators of the beamsplitter or quantum-limited attenuator channel. It is easy to see that the Kraus operators are real and pairwise trace-orthogonal, as in the case of $\mathcal{D}(\kappa)$.

### 6.4.1 Action of the Kraus operators

Recall that the beamsplitter channel induces the following transformation on the characteristic function [158]:
\[
\chi_W(\xi) \rightarrow \chi_W'(\xi) = \chi_W(\kappa \xi) \exp[-(1 - \kappa^2)|\xi|^2/2] = \chi_W(\kappa \xi) \exp[\kappa^2|\xi|^2/2] \exp[-|\xi|^2/2].
\]
(6.49)

Thus the normal ordered characteristic function $\chi_N(\xi)$ transforms as
\[
\chi_N(\xi) \equiv \chi_W(\xi) \exp(|\xi|^2/2) \rightarrow \chi_N'(\xi) = \chi_N(\kappa \xi).
\]
(6.50)
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Since $\chi_N(\xi)$ and the diagonal weight $\phi(\alpha)$ form a Fourier transform pair, it is immediately seen that $\phi(\alpha)$ gets simply scaled under the action of the $\mathcal{C}_1(\kappa)$ channel: $\phi(\alpha) \rightarrow \phi'(\alpha) = \kappa^{-2}\phi(\kappa^{-1}\alpha)$ [365].

It is instructive to bring out this fact from the perspective of the Kraus operators. Since every state $\rho$ can be expressed through a diagonal ‘weight’ $\phi(\alpha)$ as [112]

$$\rho = \pi^{-1} \int d^2\alpha \phi(\alpha) |\alpha\rangle \langle \alpha|,$$  

(6.51)

to exhibit the action of the channel on an arbitrary state it is sufficient to consider its action on a generic coherent state. We have

$$|\alpha\rangle \langle \alpha| \rightarrow \sum_{\ell=0}^{\infty} B_{\ell}(\kappa) |\alpha\rangle \langle \alpha| B^{\dagger}_{\ell}(\kappa)$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1 - \kappa^2)|\alpha|^2 \ell}{\ell!} (\kappa \alpha^*)^m (\kappa \alpha)^n \frac{\epsilon^{-|\alpha|^2}}{\sqrt{m!n!}} |m\rangle \langle n|,$$  

(6.52)

where we used the fact that the operator

$$|m\rangle \langle n| \rightarrow \sum_{\ell=0}^{\infty} B_{\ell}(\kappa) |m\rangle \langle n| B^{\dagger}_{\ell}(\kappa)$$

$$= \sum_{\ell=0}^{\min\{m,n\}} \sqrt{mC_{\ell}^nC_{\ell}} (1 - \kappa^2)\ell \kappa^{m+n-2\ell} |m - \ell\rangle \langle n - \ell|.$$  

(6.53)

Carrying out the summations in Eq. (6.52), one finds [366]

$$\sum_{\ell=0}^{\infty} B_{\ell}(\kappa) |\alpha\rangle \langle \alpha| B^{\dagger}_{\ell}(\kappa) = |\kappa \alpha\rangle \langle \kappa \alpha|.$$  

(6.54)

With this the action of the channel $\mathcal{C}_1(\kappa)$ reads

$$\rho \rightarrow \rho' = \pi^{-1} \int d^2\alpha \phi(\alpha) |\kappa \alpha\rangle \langle \kappa \alpha|$$

$$= \pi^{-1} \kappa^{-2} \int d^2\alpha \phi(\kappa^{-1} \alpha) |\alpha\rangle \langle \alpha|,$$  

(6.55)

which means

$$\mathcal{C}_1(\kappa) : \phi(\alpha) \rightarrow \kappa^{-2} \phi(\kappa^{-1} \alpha).$$  

(6.56)

We have thus proved in the Kraus representation
Theorem 6.5 The scaling \( \phi_\rho(\alpha) \to \phi_\rho^\prime(\alpha) = \kappa^{-2}\phi_\rho(\kappa^{-1}\alpha) \), \( 0 < \kappa < 1 \), is a completely positive map whose Kraus decomposition is given by \( \{ B_\ell(\kappa) \} \) of (6.48).

As an immediate consequence we have

Corollary 6.1 The beamsplitter channel cannot generate or destroy nonclassicality.

Proof: By definition a state is classical if and only if its diagonal weight function \( \phi(\alpha) \) is pointwise nonnegative everywhere in the complex plane \( |12| \). Since a pointwise positive function goes to a pointwise positive function under the above scaling transformation, it follows that a classical state (and a classical state alone) is taken to a classical state under the action of the (quantum-limited) attenuator channel.

6.4.2 The issue of Entanglement breaking

It is known that the beamsplitter channel is not entanglement breaking \([361]\). It should thus be possible, as it is obligatory, to demonstrate that this channel cannot be represented using a set of rank one Kraus operators. We begin by noting that in the limiting case \( \kappa = 0 \), all our Kraus operators \( B_\ell(0) \) are of rank one. Indeed, \( (B_\ell(0))_{mn} = \delta_{m0}\delta_{n\ell} \).

This singular limit corresponds to the quantum-limited \( A_1 \) channel which is known to be entanglement breaking. We consider therefore the nontrivial case \( \kappa \neq 0 \). It is manifestly clear that rank \( B_\ell(\kappa) = \infty \) for all \( \ell \) (for \( \kappa \neq 0 \)). If we represent this channel using another set of Kraus operators \( \{ B^\prime_\ell(\kappa) \} \), then these new operators should necessarily be in the support of the set of operators \( \{ B_\ell(\kappa) \} \). Thus a necessary condition that one is able to represent the channel \( \{ B_\ell(\kappa) \} \) using rank one Kraus operators is that there be (sufficient number of) rank one operators in the support of \( \{ B_\ell(\kappa) \} \). It turns out that there is not even one rank one operator in this support. Indeed, a much stronger result is true.

Theorem 6.6: There exists no finite rank operator in the support of the set \( \{ B_\ell(\kappa) \}, \kappa \neq 0 \).

Proof follows immediately from the structure of the \( B_\ell(\kappa) \)'s: \( B_0(\kappa) \) is diagonal, and the \( mn \)th entry of \( B_\ell(\kappa) \) is nonzero iff \( n = m + \ell \). Any matrix in the linear span of \( \{ B_\ell(\kappa) \} \) is of the form \( M = \sum_\ell c_\ell B_\ell(\kappa) \), and is upper diagonal. Let \( N \) be the smallest \( \ell \) for which the \( \ell \)-number coefficient \( c_\ell \neq 0 \). Let \( \tilde{M} \) be the matrix obtained from the upper-diagonal \( M \) by deleting the first \( N \) columns. Clearly, rank \( \tilde{M} = \text{rank } M \). Further, the diagonal entries of the upper triangular \( \tilde{M} \) are all nonzero, being the nonzero entries of \( B_N(\kappa) \).

Now, the rank of an upper triangular matrix is not less than that of its diagonal part. Thus, rank \( \tilde{M} \) is not less than rank \( B_N(\kappa) = \infty \), thus completing the proof.
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6.5 Amplifier channel $C_2(\kappa), \kappa \geq 1$

The two-mode metaplectic unitary operator describing a single-mode quantum-limited amplifier channel corresponds to the following symplectic transformation on the mode operators [158]:

$$S = \begin{pmatrix}
\cosh \nu & 0 & \sinh \nu & 0 \\
0 & \cosh \nu & 0 & -\sinh \nu \\
\sinh \nu & 0 & \cosh \nu & 0 \\
0 & -\sinh \nu & 0 & \cosh \nu
\end{pmatrix}.$$  \hfill (6.57)

As in the earlier two cases of $D(\kappa)$ and $C_1(\kappa)$, the position and momentum variables do not mix under the action of $C_2(\kappa)$. The position variables transform as

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \rightarrow \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} = M \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \cosh \nu & -\sinh \nu \\ -\sinh \nu & \cosh \nu \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$  \hfill (6.58)

and the momentum variables transform according to $M^{-1}$. Thus the parameter $\kappa$ in $C_2(\kappa)$ is related to the two-mode squeeze parameter $\nu$ through $\kappa = \cosh \nu$. The function $F(z_1, z_2, \eta_1, \eta_2)$ in (6.14) is readily computed to be

$$F(z_1, z_2, \eta_1, \eta_2) = \kappa^{-1} \exp \left\{ \kappa^{-1}(\eta_1 z_1 + \eta_2 z_2) + (\sqrt{1 - \kappa^2})(\eta_1 \eta_2 - z_1 z_2) \right\}.$$  \hfill (6.59)

As in the earlier cases of $D(\kappa)$ and $C_1(\kappa)$, the differentiation on $F(z_1, z_2, \eta_1, \eta_2)$ can be performed to obtain the matrix elements of the unitary operator corresponding to the symplectic $S$ in (6.57). We obtain, after some algebra patterned after the earlier two cases,

$$C_{n_1 n_2}^{m_1 m_2} = \frac{\kappa^{-1}}{\sqrt{n_1!n_2!m_1!m_2!}} \sum_{r=0}^{n_2} \sum_{j=0}^{m_1} n_2^{r} m_1^{j} C_{r}^{m_1} C_{j}^{m_2} (-1)^{r} (\sqrt{1 - \kappa^2})^{r+m_1-j} \times \kappa^{-1} \delta_{n_1-r+j} \delta_{n_2+m_1+r-j}.$$  \hfill (6.60)

The Kraus operators are obtained from $C_{n_1 n_2}^{m_1 m_2}$ by setting $n_2 = 0$, and $m_2 = \ell$. Setting $n_2 = 0 \Rightarrow r = 0$, and we have

$$A_{\ell}(\kappa) = \kappa^{-1} \sum_{m=0}^{\infty} \sqrt{m + \ell} C_{\ell}^{m} (\sqrt{1 - \kappa^2})^{\ell} (\kappa^{-1})^{m}|m + \ell\rangle \langle m|,$$  \hfill (6.61)

as the Kraus operators of the quantum-limited amplifier channel $C_2(\kappa), \kappa > 1$ [367].
6.5.1 Duality between the attenuator family \(C_1(\cdot)\) and the amplifier family \(C_2(\cdot)\)

The Kraus operators \(A_\ell(\kappa), \kappa > 1\) of the amplifier channel \(C_2(\kappa)\) have an interesting dual relationship to the Kraus operators \(B_\ell(\kappa^{-1}), \kappa > 1\) of the attenuator channel \(C_1(\kappa^{-1})\). While \(\sum_{\ell=0}^\infty A_\ell^\dagger(\kappa)A_\ell(\kappa) = \mathbb{I}, \kappa > 1\) and \(\sum_{\ell=0}^\infty B_\ell^\dagger(\kappa')B_\ell(\kappa') = \mathbb{I}, \kappa' < 1\), consistent with the trace-preserving property of \(C_2(\kappa)\) and \(C_1(\kappa')\), we have

\[
\sum_{\ell=0}^\infty A_\ell(\kappa)A_\ell^\dagger(\kappa') = \kappa^{-2}\mathbb{I},
\]

\[
\sum_{\ell=0}^\infty B_\ell(\kappa')B_\ell^\dagger(\kappa^{'}) = (\kappa')^{-2}\mathbb{I}.
\]

Thus the (trace-preserving) families \(C_1\) and \(C_2\) are not unital. But they are ‘almost unital’, for the failure to be unital is by just a scalar factor. This shows that the family \(\{\kappa A_\ell(\kappa)^\dagger, \kappa > 1\}\) and the family \(\{\kappa'^{-1} B_\ell(\kappa')^\dagger, \kappa' < 1\}\) too describe trace-preserving CP maps, and we may ask what these ‘new’ channels stand for.

The meaning of these channels may be easily seen by considering the adjoints \(A_\ell(\kappa)^\dagger, \kappa > 1\) of the Kraus operators of the amplifier channel:

\[
A_\ell(\kappa)^\dagger = \kappa^{-1} \sum_{m=0}^\infty \sqrt{m+\ell} C_\ell \left( \sqrt{1 - \kappa^{-2}} \right)^\ell \kappa^{-m}|m\rangle\langle m + \ell| = \kappa^{-1} B_\ell(\kappa^{-1})
\]

Thus \(\{\kappa A_\ell(\kappa)^\dagger\}, \kappa > 1\) are the Kraus operators of the beamsplitter channel \(C_1(\kappa')\) with \(\kappa' = \kappa^{-1} < 1\). Similarly it can be seen that \(\{\kappa' B_\ell(\kappa')^\dagger\}, \kappa' < 1\) represents the amplifier channel \(C_2(\kappa)\) with \(\kappa = (\kappa')^{-1} > 1\). Thus we have

**Theorem 6.7** The amplifier family \(C_2(\kappa)\) and the attenuator family \(C_1(\kappa^{-1}), \kappa > 1\) are mutually dual: their Kraus operators are connected through the adjoint operation.

6.5.2 Action of the Kraus operators

Under the action of the amplifier channel \(C_2(\kappa)\) the Weyl-ordered characteristic function transforms as follows, and this may be identified with the very definition of the channel:

\[
\chi_W(\xi) \rightarrow \chi_W^\prime(\xi) = \chi_W(\kappa \xi) \exp[-(\kappa^2 - 1)|\xi|^2/2].
\]
Chapter 6. Operator-sum representation for Bosonic Gaussian channels

Given a Weyl-ordered characteristic function $\chi_W(\xi)$, the corresponding antinormal ordered characteristic function corresponding to the $Q$-distribution is [169]

$$\chi_A(\xi) = \chi_W(\xi) \exp[-|\xi|^2/2].$$

(6.65)

Therefore the channel action Eq. (6.64), written in terms of $\chi_A(\xi)$, reads

$$\chi_A(\xi) \rightarrow \chi_A(\xi) = \chi_A(\kappa \xi).$$

(6.66)

That is, $\chi_A(\xi)$ simply scales under the action of the amplifier channel, a fact that should be profitably compared with the scaling behaviour (6.50) for the attenuator channel. Since $\chi_A(\xi)$ and the $Q$-function form a Fourier transform pair, the action of the amplifier channel is fully described as a scaling transformation of the $Q$-function: $Q(\alpha) \rightarrow Q'(\alpha) = \kappa^{-2} Q(\kappa^{-1} \alpha)$, $\kappa > 1$ [368].

It is instructive to see in some detail how our Kraus operators $A_\ell(\kappa)$ bring out this behaviour. Given a state

$$\rho = \sum_{n,m=0}^{\infty} |n\rangle \langle n| \rho \langle m| m\rangle = \sum_{n,m=0}^{\infty} \rho_{nm} |n\rangle \langle m|,$$

(6.67)

its corresponding $Q$ function is [169]

$$Q_\rho(\alpha) = \langle \alpha| \rho |\alpha\rangle = \exp[-|\alpha|^2] \sum_{n,m=0}^{\infty} \frac{(\alpha^*)^n (\alpha)^m}{\sqrt{n!} \sqrt{m!}} \rho_{nm}.$$

(6.68)

To see the action of the linear map $C_2(\kappa)$ on an arbitrary $\rho$, it is sufficient to exhibit its action on the operators $|n\rangle \langle m|$, for all $n, m \geq 0$. We have

$$|n\rangle \langle m| \rightarrow \sum_{\ell=0}^{\infty} A_\ell(\kappa) |n\rangle \langle m| A_\ell^\dagger(\kappa)$$

$$= \kappa^{-2} (\kappa)^{-(n+m)} \sum_{\ell=0}^{\infty} \frac{(1 - \kappa^{-2})^\ell}{\ell!} \sqrt{(n+\ell)!} \sqrt{(m+\ell)!} |n+\ell\rangle \langle m+\ell|.$$  

(6.69)

Thus, under the action of the channel $C_2(\kappa)$, $\rho$ goes to

$$\rho' = \kappa^{-2} \sum_{n,m=0}^{\infty} \rho_{nm} \frac{(1 - \kappa^{-2})^\ell}{\ell!} \sqrt{(n+\ell)!} \sqrt{(m+\ell)!} |n+\ell\rangle \langle m+\ell|.$$  

(6.70)
The $Q$ function of the resultant or output state $\rho'$ is
\[
\langle \alpha | \rho' | \alpha \rangle = \kappa^{-2} \exp[-|\alpha|^2] \sum_{n,m=0}^{\infty} \rho_{nm} \frac{\kappa^{-(n+m)}}{\sqrt{n!m!}} (\alpha^*)^n (\kappa^{-1}\alpha)^m \left( \sum_{\ell=0}^{\infty} \frac{(1-\kappa^{-2})^\ell}{\ell!} |\alpha|^2 \right).
\]
\[
= \kappa^{-2} \exp[-|\kappa^{-1}\alpha|^2] \sum_{n,m=0}^{\infty} \frac{(\kappa^{-1}\alpha^*)^n (\kappa^{-1}\alpha)^m}{\sqrt{n!m!}} \rho_{nm}
\]
\[
= \kappa^{-2} Q(\kappa^{-1}\alpha).
\]
(6.71)

We thus conclude

**Theorem 6.8** The scaling $Q_\rho(\alpha) \rightarrow Q_{\rho'}(\alpha) = \kappa^{-2}Q_\rho(\kappa^{-1}\alpha)$, $0 < \kappa^{-1} < 1$, is a completely positive map whose Kraus decomposition is given by $\{A_i(\kappa)\}$.

This result may be compared with Theorem 6 for the $C_1(\cdot)$ family of channels.

The amplifier channel has the following property in respect of nonclassicality of the output states:

**Corollary 6.2** The amplifier channel cannot generate nonclassicality.

**Proof:** By Eq. (6.64), the normal ordered characteristic function transforms as follows
\[
C_2(\kappa) : \chi_N(\xi) \rightarrow \chi_N'(\xi) = \chi_N(\kappa\xi) \exp[-(\kappa^2 - 2)|\xi|^2/2].
\]
(6.72)

This may be rewritten in the suggestive form
\[
\chi_N(\xi) \rightarrow \chi_N'(\xi) = \chi_N(\kappa\xi) \exp[-(\kappa^2 - 1)|\xi|^2].
\]
(6.73)

Fourier transforming, we see that the diagonal weight $\phi(\alpha)$ of the output state is the convolution of the (scaled) input diagonal weight with a Gaussian (corresponding to the last factor), and hence it is pointwise nonnegative whenever the input diagonal weight $\phi(\alpha)$ is pointwise nonnegative.

**Remark:** We are not claiming that the amplifier channel cannot destroy nonclassicality [compare the structure of Corollary 2 with that of Corollary 1 following Theorem 6]. Indeed, it is easy to show that nonclassicality of every Gaussian state will be destroyed by any $C_2(\kappa)$ with $\kappa \geq \sqrt{2}$ [184, 367-369]. It is also easy to show that there are states whose nonclassicality will survive $C_2(\kappa)$ even for arbitrarily large $\kappa$ [184, 367, 368]. To see this, note first of all, that any state $\rho$ whose $Q$-function $Q(\alpha) = \langle \alpha | \rho | \alpha \rangle$ vanishes for some $\alpha$ is necessarily nonclassical. The assertion simply follows from the fact that under the scaling $Q(\alpha) \rightarrow \kappa^{-2}Q(\kappa^{-1}\alpha)$ a zero $\alpha_0$ of $Q(\alpha)$ goes to a zero at $\kappa\alpha_0$. 

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Remark on entanglement breaking: It is well known that the quantum-limited amplifier channel is not entanglement breaking [361]. It may be pointed out in passing that this fact follows also from the structure of our Kraus operators \( \{ A_\ell (\kappa) \} \). Since these operators coincide with the transpose of the beamsplitter channel Kraus operators \( \{ B_\ell (\kappa^{-1}) \} \), apart from a \( \ell \)-independent multiplicative factor, there exists no finite rank operator in the support of the set of operators \( \{ A_\ell (\kappa) \} \). In particular, there are no rank one operators in the support of \( \{ A_\ell (\kappa) \} \). Hence, \( C_2(\kappa) \) is not an entanglement breaking channel.

6.6 The Singular case \( A_2 \)

We now consider briefly \( A_2 \), the last of the quantum limited Bosonic Gaussian channels. The two-mode metaplectic unitary operator representing \( A_2 \) produces a symplectic transformation on the quadrature variables which does not mix the position variables with the momentum variables [158]:

\[
\begin{pmatrix}
q_1 \\
q_2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
q_1' \\
q_2'
\end{pmatrix}
= M
\begin{pmatrix}
q_1 \\
q_2
\end{pmatrix},
\]

\[
\begin{pmatrix}
p_1 \\
p_2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
p_1' \\
p_2'
\end{pmatrix}
= (M^{-1})^T
\begin{pmatrix}
p_1 \\
p_2
\end{pmatrix},
\]

\[
M = \begin{pmatrix}
0 & 1 \\
1 & -1
\end{pmatrix}.
\]  \hspace{1cm} (6.74)

Therefore, our general scheme applies to this case as well. Unlike in the earlier cases of \( D(\kappa) \), \( C_1(\kappa) \), and \( C_2(\kappa) \), in the present case it turns out to be more convenient to evaluate the matrix elements of \( U^{(ab)} \) in a mixed basis:

\[
C_{n_1,n_2}^{m_1,q} = \langle m_1 | (q |U^{(ab)}| n_1) | n_2 \rangle.
\]  \hspace{1cm} (6.75)

Here \( | q \rangle \) labels the position basis of the ancilla mode. With this mixed choice, the Kraus operators are labelled by a continuous index \( \ell \), and are given by

\[
V_q = \langle q |U^{(ab)}| 0 \rangle = \sum_{m_1,n_1} C_{n_1,n_2}^{m_1,q} |m_1 \rangle \langle n_1|,
\]  \hspace{1cm} (6.76)
where

\[ C_{n_10}^{m_1 q} = \int dq_1 \langle m_1 | q_1 \rangle \langle q_1 | U^{(ab)} | n_1 \rangle | 0 \rangle = \int dq_1 \langle m_1 | q_1 \rangle \langle q, q_1 - q | n_1, 0 \rangle. \]  

(6.77)

Here we have used, as in the earlier cases, the action of the unitary operator in the position eigenstates of the two-mode system. Employing the position space wavefunctions of the Fock states, we have

\[ C_{n_10}^{m_1 q} = \frac{\pi^{-3/4}}{\sqrt{2^{n_1 + m_1} n_1 ! m_1 !}} H_{n_1}(q) e^{-\frac{q^2}{2}} \int dq_1 H_{m_1}(q_1) e^{-\frac{(q_1 - q)^2}{2} - \frac{q^2}{4}}. \]

(6.78)

The above integral is easily evaluated \[370\], and we have

\[ C_{n_10}^{m_1 q} = \frac{\pi^{-1/4}}{\sqrt{2^{n_1 + m_1} n_1 ! m_1 !}} q^{m_1} H_{n_1}(q) \exp[-3q^2/4] = \langle m_1 | q/\sqrt{2} \rangle \langle q | n_1 \rangle, \]

(6.79)

where \(|q/\sqrt{2}\rangle\) is the coherent state \(|\alpha\rangle\) for \(\alpha = q/\sqrt{2}\), and the purpose of the round bracket being to distinguish the same from the position eigenket \(|q/\sqrt{2}\rangle\). With this notation the Kraus operators are

\[ V_q = |q/\sqrt{2}\rangle \langle q| \]  

(6.80)

That the trace-preserving condition on the Kraus operators is satisfied emerges from the fact that the position kets are complete: \(\int dq V_q^\dagger V_q = \int dq |q\rangle \langle q| = \mathbb{1}\).

To connect these Kraus operators \(V_q\) to the action of the channel in the phase space picture, we examine the behaviour of an arbitrary pure state \(|\psi\rangle\) under passage through the channel. We have

\[ A_2 : \rho = |\psi\rangle \langle \psi| \rightarrow \rho' = \int dq |q/\sqrt{2}\rangle \langle q/\sqrt{2}| \langle \psi \rangle \langle \psi| \langle \psi |q/\sqrt{2}\rangle \]

\[ = \int dq |\psi(q)|^2 |q/\sqrt{2}\rangle \langle q/\sqrt{2}| \]

\[ = \int dq \int dp |\psi(q)|^2 \delta(p) \langle [q + ip]/\sqrt{2}| [q + ip]/\sqrt{2}\rangle. \]  

(6.81)

The last expression is already in the ‘diagonal’ form in the coherent states basis, with \(|\psi(q)|^2 \delta(p), \alpha = (q + ip)/\sqrt{2}\) forming the diagonal weight function \(\phi(\alpha)\). It follows by
convexity that for an arbitrary input state $\rho$ the output of the channel is given by

$$\rho' = \pi^{-1} \int d^2\alpha \phi(\alpha) \langle \alpha | \rho | \alpha \rangle, \quad \phi(\alpha) = \langle q | \rho | q \rangle \delta(p). \quad (6.82)$$

It is seen that this transformation is the same as $\chi_W(\xi) \to \chi_W \left( \frac{1+|\alpha|^2}{2} \xi \right) \exp[-|\xi|^2/2]$, the expected behaviour of the characteristic function under passage through $\mathcal{A}_2$ [371].

The above results can be alternatively understood through the action of the channel in the Fock basis. Under passage through the channel,

$$|n\rangle\langle m| \to \int dq V_q |n\rangle\langle m| V_q^\dagger$$

$$= \int dq \frac{\pi^{-1/2}}{\sqrt{2^{n+m}n!m!}} H_n(q) H_m(q) e^{-q^2/2} (q/\sqrt{2}) (q/\sqrt{2}), \quad (6.83)$$

for all $n, m$. The outcome for an arbitrary input state $\rho$ follows by linearity, and we have

**Theorem 6.9** The channel $\mathcal{A}_2$ is both nonclassicality breaking and entanglement breaking.

**Proof:** We note from Eq. (6.80) that the Kraus operators are already in rank one form, thereby showing that the channel is entanglement breaking. And from Eq. (6.82) we see that the output of the channel, for every input state $\rho$, supports a diagonal representation with nonnegative weight $\langle q | \rho | q \rangle \delta(p) \geq 0$, for all $\alpha = (q + ip)/\sqrt{2}$, showing that the output is classical for all input states.

### 6.7 Single Quadrature classical noise channel $\mathcal{B}_1(a)$, $a \geq 0$

The channel $\mathcal{B}_1(a)$, whose action is to simply inject Gaussian noise of magnitude $a$ into one quadrature of the oscillator, and is not quantum limited. It can be realized in the form

$$\mathcal{B}_1(a) : \rho \to \rho' = \frac{1}{\sqrt{\pi}a} \int dq \exp[-q^2/a] D(q/\sqrt{2}) \rho D(q/\sqrt{2})^\dagger, \quad (6.84)$$

where $D(\alpha)$’s are the unitary displacement operators. $\mathcal{B}_1(a)$ is thus a case of the so-called random unitary channels [364], a convex sum of unitary channels. The continuum

$$Z_q \equiv (\pi a)^{-1/4} \exp[-q^2/2a] D(q/\sqrt{2}) \quad (6.85)$$
are the Kraus operators of this realization. The quantum-limited end of $B_1(a)$ is obviously
the identity channel, corresponding to $a \to 0$ \(\lim_{a \to 0} \sqrt{\pi a} \exp[-q^2/a] = \delta(q)\), and
$Z_{q=0} = \text{identity}$. One may assume $a = 1$ without loss of generality. The reason we
present a brief treatment of this channel here is just to demonstrate that this case too
subjects itself to our general scheme.

The two-mode metaplectic unitary operator representing $B_1$ produces a symplectic
transformation on the quadrature variables which, as in the earlier cases of $D(\kappa)$, $C_1(\kappa)$,
$C_2(\kappa)$, and $A_2$, does not mix the position variables with the momentum variables [158]:

\[
\begin{pmatrix}
q_1 \\
n_2
\end{pmatrix}

\to

\begin{pmatrix}
q_1' \\
n_2
\end{pmatrix}

= M

\begin{pmatrix}
q_1 \\
n_2
\end{pmatrix},
\]

\[M = \begin{pmatrix}1 & -1 \\0 & 1\end{pmatrix}.\]  \hspace{1cm} (6.86)

And $p_1, p_2$ transform according to $(M^{-1})^T$.

As in the immediate previous case $A_2$, the matrix elements of $U_{(ab)}$ are

\[C_{n_1m_2}^{m_1q} = \langle m_1|q U_{(ab)}|n_1\rangle |n_2\rangle,\]  \hspace{1cm} (6.87)

where $|q\rangle$’s are the position eigenvectors. In view of this the Kraus operators are labelled
by a continuous index ‘$q$’ and are given by

\[\langle q | U_{(ab)} | 0 \rangle = \sum_{m_1, n_1} C_{n_1m_2}^{m_1q} |m_1\rangle \langle n_1|,\]  \hspace{1cm} (6.88)

where

\[C_{n_1m_2}^{m_1q} = \int dq_1 \langle m_1|q_1\rangle \langle q_1 | U_{(ab)} | n_1 \rangle | 0 \rangle\]

\[= \int dq_1 \langle m_1|q_1\rangle \langle q_1 - q , q | n_1 , 0 \rangle.\]  \hspace{1cm} (6.89)

Here we made the two-mode metaplectic unitary operator act on the position basis.

To evaluate the Kraus operator, it is sufficient to evaluate the matrix elements

\[C_{m_10}^{m_1q} = \frac{\pi^{-3/4}}{\sqrt{2n_1 + m_1 + m_1 |m_1\rangle |m_1\rangle}} e^{-\frac{q^2}{2}} \int dq_1 H_{n_1}(q_1 - q) H_{m_1}(q_1) e^{-\frac{q^2}{2}} e^{-\frac{(q_1 - q)^2}{2}}.\]  \hspace{1cm} (6.90)
The above integral can be readily performed [372], and we obtain

\[
C_{m_1}^{n_1 \theta} = \pi^{-1/4} e^{-\frac{q^2}{4}} \left[ e^{-\frac{q^2}{4} \sqrt{\frac{m_1!}{n_1!}}} \left( \frac{-q}{\sqrt{2}} \right)^{n_1-m_1} L_{n_1-m_1}^{n_1-m_1} \left( \frac{q^2}{2} \right) \right]
\]

\[
\equiv \pi^{-1/4} e^{-\frac{q^2}{4}} \langle m_1 | D(q/\sqrt{2}) | n_1 \rangle = Z_q.
\]  

(6.91)

We have thus recovered (6.85), but staying entirely within our general scheme.

6.8 Summary

We have obtained operator-sum representations for all single-mode Bosonic Gaussian channels presented in their respective canonical forms. Evidently, the operator-sum representation of a channel not in the canonical form follows by adjoining of appropriate unitary Gaussian evolutions before and after the channel. The Kraus operators were obtained from the matrix elements of the two-mode metaplectic unitary operator which effects the channel action on a single mode. The two-mode symplectic transformation in each case did not mix the position and momentum variables and this fact proved valuable for our study. The Kraus operators for the quantum-limited channels except the singular case were found to have a simple and sparse structure in the Fock basis.

It was shown that the phase conjugation channels \( D(\kappa) \) and \( D(\kappa^{-1}) \) are dual to one another, and the attenuator and the amplifier families \( C_1(\kappa) \) and \( C_2(\kappa^{-1}) \), \( \kappa < 1 \) are mutually dual. The channels \( D(\kappa), C_1(\kappa), \) and \( C_2(\kappa) \) were found to be almost unital; in the sense that the unit operator was taken to a scalar times the unit operator.

In the case of the phase conjugation channel, the action in phase space was brought out explicitly through the action of the Kraus operators on the Fock basis. The attenuator channel resulted in the scaling of the diagonal weight function \( \phi(\alpha) \) and the amplifier channel resulted in the scaling of the Husimi \( Q \)-function as expected. Further, the output of the channel with respect to classicality/nonclassicality was studied. It was found that the phase conjugation channel \( D(\kappa) \) and the singular channel \( A_2 \) are classicality breaking while the attenuator channel \( C_1(\kappa) \) and the amplifier channel \( C_2(\kappa) \) do not generate nonclassicality.

The Kraus operators of the phase conjugation channel was brought to a rank one form, thereby explicitly bringing out the entanglement breaking nature of this channel. It was further shown that there is no finite rank operator in the support of the Kraus operators of either the amplifier or the attenuator channel, and this explicitly demonstrates that the quantum-limited attenuator and the amplifier families of channels are not entanglement breaking. The Kraus operators of the singular channel \( A_2 \) was also obtained in
the rank one form thereby manifestly showing that this channel is entanglement breaking.

**Note**: A more detailed analysis on the operator-sum representation of single-mode Bosonic Gaussian channels can be found at [373]. This includes an analysis on fixed points, an analysis on interrupted evolution, a proof of the extremality of all quantum limited single-mode Gaussian channels, and the operator-sum representation of composite channels.