4

Entanglement of Formation for Gaussian states

4.1 Introduction

Entanglement is an essential resource for many quantum information processing tasks, and hence it is important to be able to quantify this resource. In Section 1.6, we outlined a set of demands, that any good measure of entanglement should satisfy. In the case of bipartite pure states, the demands lead to a simple and unique measure for this resource: it is the von Neumann entropy of either subsystem [62, 338, 339]. For mixed states however, many different entanglement measures have been explored [340], and there is no measure which justifies itself to be unique. Of these measures, the entanglement of formation (EOF) [37] is the most natural extension of the pure state measure of entanglement, to the case of mixed states. To recall the definition of EOF in Eq. (1.87), the EOF for a bipartite state $\hat{\rho}^{(ab)}$ is defined as an infimum:

$$\text{EOF} (\hat{\rho}^{(ab)}) \equiv \inf \left\{ \sum_j p_j E(\psi_j) \mid \hat{\rho}^{(ab)} = \sum_j p_j \ket{\psi_j}\bra{\psi_j} \right\}.$$  \hspace{1cm} (4.1)

The infimum is to be taken over all possible ensemble realizations of the given mixed state $\hat{\rho}^{(ab)}$ as convex sum of pure states, and $E(\psi_j) \equiv S(\text{tr}_B[\ket{\psi_j}\bra{\psi_j}])$, where $S(\cdot)$ is the von Neumann entropy. The regularised version of EOF is the entanglement cost [37, 63]. EOF has been computed in closed form for arbitrary two-qubit states [67], and for highly symmetric states like the isotropic states [69] and the Werner states [71].

The role of Gaussian states in quantum information theory has already been outlined in Section 1.9. Their use in teleportation [83, 84] and quantum cryptography [97] has been demonstrated. Questions related to their separability [57, 58, 241, 242] and distillability [217] have been resolved. More recently, analytic expression for their EOF has been obtained in the symmetric case [70]. This notable achievement seems to be the first computation of EOF for states of infinite rank. These authors exploit a certain extremal-
ity that the two-mode-squeezed vacuum enjoys in respect of the Einstein-Podolsky-Rosen (EPR) correlation [341] on the one hand and entanglement on the other.

An interesting Gaussian-state-specific generalisation of EOF, the Gaussian entangle-
ment of formation, has also been explored [342]. But the EOF of asymmetric Gaussian state has remained an open problem [343] in spite of considerable effort [344]. Naturally, the problem of EOF for general (asymmetric) Gaussian states should be solved before the important issue of additivity of entanglement for Gaussian states could be addressed [342].

In this work we compute, under a conjecture, the EOF for arbitrary two-mode Gaussian states. Our analysis rests on two principal ingredients. The first one is a four-parameter canonical form we develop for the covariance matrix; one of these parameters, the squeeze parameter, proves to be a measure of EOF. The second one is a family of generalised EPR correlations for noncommuting pairs of nonlocal variables; this family is indexed by a continuous parameter \( \theta \). And the conjecture is in respect of an extremal property of this generalised EPR correlation.

4.2 Canonical Form for Covariance Matrix

Given a two-mode Gaussian state, with the mode on Alice’s side described by canonical quadrature variables \( \hat{x}_a, \hat{p}_a \) and that on Bob’s side by \( \hat{x}_b, \hat{p}_b \), we can assume without loss of generality that the first moments of all four variables vanish [57, 70]. Such a zero-mean Gaussian state is fully described by the covariance matrix [57, 70]

\[
V_G = \frac{1}{2} \begin{bmatrix}
\alpha \beta n & 0 & \beta k_x & 0 \\
0 & \alpha^{-1} \beta^{-1} n & 0 & -\beta^{-1} k_p \\
\beta k_x & 0 & \alpha^{-1} \beta m & 0 \\
0 & -\beta^{-1} k_p & 0 & \alpha \beta^{-1} m
\end{bmatrix},
\]

(4.2)

where the phase space variables are assumed to be arranged in the order \((x_a, p_a, x_b, p_b) \equiv \xi\), and we have retained through the parameters \(\alpha, \beta > 0\) the freedom of independent local unitary (i.e., symplectic) scalings on the Alice’s and Bob’s sides. This freedom will be used shortly.

Note that \(V_G\) is left with no correlation between the ‘spatial’ variables \(\hat{x}_a, \hat{x}_b\) and the ‘momentum’ variables \(\hat{p}_a, \hat{p}_b\). Thus it is sometimes convenient to view \(V_G\) as the direct
sum of $2 \times 2$ matrices:
\[
X_G = X_G \oplus P_G, \\
V_G = X_G \oplus P_G, \\
P_G = \frac{\beta^{-1}}{2} \begin{bmatrix} \alpha^{-1}n & -k_p \\ -k_p & \alpha m \end{bmatrix}.
\] (4.3)

Let $|\Psi_r\rangle$ denote the standard two-mode-squeezed vacuum state with squeeze parameter $r$. It takes the Schmidt form in the standard Fock basis:
\[
|\Psi_r\rangle = \sum_{n=0}^{\infty} c_n |n\rangle_A \otimes |n\rangle_B \equiv \sum_{n=0}^{\infty} c_n |n, n\rangle, \\
c_n = \tanh^n r / \cosh r.
\] (4.4)

Denoting by $E_r$ the entanglement of $|\Psi_r\rangle$, we have
\[
E_r = \cosh^2 r \log_2 (\cosh 2r) - \sinh^2 r \log_2 (\sinh^2 r).
\] (4.5)

The covariance matrix of $|\Psi_r\rangle$ has the form
\[
V_{\Psi_r} = X_{\Psi_r} \oplus P_{\Psi_r}, \\
X_{\Psi_r} = \frac{1}{2} \begin{bmatrix} C & S \\ S & C \end{bmatrix}, \\
P_{\Psi_r} = \frac{1}{2} \begin{bmatrix} C & -S \\ -S & C \end{bmatrix}, \\
C \equiv \cosh 2r, \; S \equiv \sinh 2r.
\] (4.6)

**Proposition 4.1** Given a two-mode covariance matrix $V_G$, the local scale parameters $\alpha, \beta$ can be so chosen that $V_G$ gets recast in the form
\[
V_0 = \frac{1}{2} \begin{bmatrix} C + u c^2 & 0 & S + u cs & 0 \\ 0 & C + v c^2 & 0 & -S - v cs \\ S + u cs & 0 & C + u s^2 & 0 \\ 0 & -S - v cs & 0 & C + v s^2 \end{bmatrix}, \\
C \equiv \cosh 2r_0, \; S \equiv \sinh 2r_0; \; c \equiv \cos \theta_0, \; s \equiv \sin \theta_0.
\] (4.7)

Note: We will call $V_0$ the canonical form of a two-mode covariance matrix; our results below will justify this elevated status. We assume without loss of generality $n \geq m$ or, equivalently, $0 < \theta_0 \leq \pi/4$. For a given $V_G$ there will be two solutions for the above form. Canonical form will always refer to the one with the smaller squeeze parameter $r$, which
is ensured by the restriction
\[ \tan \theta_0 \geq \tanh r_0. \]  \hfill (4.8)

This condition proves central to our analysis. Its origin may be appreciated by inverse
two-mode-squeezing the Gaussian state \( V_0 \) until it becomes just separable, and noting
that there exists a range of further squeezing in which the mixed Gaussian state remains
separable before becoming inseparable again. The parameters \( u, v \geq 0 \).

**Theorem 4.1** The essence of the canonical form is that \( V_0 \) differs from the covariance
matrix of a two-mode-squeezed vacuum \( |\Psi_{r_0}\rangle \) by a positive matrix which is a direct sum
of two singular \( 2 \times 2 \) matrices which are, modulo signature of the off-diagonal elements,
multiples of one another.

**Proof:** The canonical form demands, as a necessary condition, that \( \alpha, \beta, \) and \( r \) be chosen
to meet
\[ \det(X_\alpha - X_{\Psi_0}) = 0, \quad \det(P_\alpha - P_{\Psi_0}) = 0. \]  \hfill (4.9)

These being two constraints on three parameters, one will expect to get a one-parameter
family of solutions to these constraints. For each such solution we may denote the vector
annihilated by the singular matrix \( X_\alpha - X_{\Psi_0} \) by \( (\sin \theta, -\cos \theta) \), and that annihilated by
\( P_\alpha - P_{\Psi_0} \) by \( (\sin \theta', \cos \theta') \). The canonical form corresponds to that solution for which
\( \theta' = \theta \); it is this degenerate value that equals \( \theta_0 \) of the canonical form.

That there exists such a degenerate value can be seen as follows. We may fix the scale
parameter \( \alpha \) through \( \alpha = \sqrt{m/n} \), and then solve Eqs. \( (4.9) \) for \( \beta \) and \( r \), the smaller \( r \)
being the relevant one. We will find \( \theta = \pi/4 \) and \( \theta' < \pi/4 \) in this case. On the other
hand if we take \( \alpha = \sqrt{n/m} \) and then solve Eqs. \( (4.9) \), we will find \( \theta' = \pi/4 \) and \( \theta < \pi/4 \).

It follows from continuity that there exists an intermediate value \( \alpha_0 \) for the parameter
\( \alpha \), in the range \( \sqrt{m/n} < \alpha < \sqrt{n/m} \), for which \( \theta' = \theta \) \((< \pi/4 \) since \( n > m \)). And this
yields the canonical form.

Viewed alternatively, the canonical form places the following two requirements on the
scale factors \( \alpha, \beta \):
\[ \frac{\det X_\alpha - 1/4}{\det P_\alpha - 1/4} = \frac{\tr(\sigma_3 X_\alpha)}{\tr(\sigma_3 P_\alpha)}, \]
\[ \det(X_\alpha - \sigma_3 P_\alpha \sigma_3) = 0, \]  \hfill (4.10)

where \( \sigma_3 \) is the diagonal Pauli matrix. These are simultaneous equations in \( \alpha, \beta, \) and
solving these equations yields, in terms of \( n, m, k_x, k_p \), the values of \( \alpha, \beta \) corresponding to the canonical form.

Two special cases may be noted. If \( m = n \) we have \( \alpha = 1 \) (since \( \sqrt{n/m} = \sqrt{m/n} \)), and hence \( \beta = (n - k_p) / (n - k_x) \), so that the canonical squeeze parameter \( r_0 \) is given by \( e^{-2r_0} = (n - k_x)(n - k_p) / (n - k_x)(n - k_p) \), reproducing the results of Ref. [70]. The parameter \( \theta_0 \) always equals \( \pi/4 \) in this (symmetric) case. On the other hand, if \( k_x = k_p = k \), the canonical form corresponds to \( \alpha = \beta = 1 \), and one obtains \( r_0 \) by simply solving

\[
\det \begin{bmatrix} n - \cosh 2r_0 & k - \sinh 2r_0 \\ k - \sinh 2r_0 & m - \cosh 2r_0 \end{bmatrix} = 0 ,
\]

which yields this closed-form expression for \( r_0 \):

\[
\cosh(2\eta - 2r_0) = \frac{nm - k^2 + 1}{\sqrt{(n + m)^2 - 4k^2}} ,
\]

\[
e^{\pm 2\eta} = \frac{(n + m) \pm 2k}{\sqrt{(n + m)^2 - 4k^2}} .
\]

### 4.3 Generalised EPR Correlation

To proceed further, we need to generalise the familiar EPR correlation in Eq. (1.120) [70]. Given any bipartite state \(|\psi\rangle\), define

\[
\hat{x}_\theta = \sin \theta \hat{x}_a - \cos \theta \hat{x}_b ,
\]

\[
\hat{p}_\theta = \sin \theta \hat{p}_a + \cos \theta \hat{p}_b ,
\]

\[
\Lambda_\theta(\psi) = \langle \psi | (\hat{x}_\theta)^2 | \psi \rangle + \langle \psi | (\hat{p}_\theta)^2 | \psi \rangle .
\]

In defining \( \Lambda_\theta(\psi) \) we have assumed \( \langle \psi | \hat{x}_\theta | \psi \rangle = 0 = \langle \psi | \hat{p}_\theta | \psi \rangle \); if this is not the case then \( \hat{x}_\theta \) and \( \hat{p}_\theta \) in \( \Lambda_\theta(\psi) \) should be replaced by \( \hat{x}_\theta - \langle \psi | \hat{x}_\theta | \psi \rangle \) and \( \hat{p}_\theta - \langle \psi | \hat{p}_\theta | \psi \rangle \) respectively. Clearly, the usual EPR correlation in Eq. (1.120) [70] corresponds to \( \theta = \pi/4 \). While \( \hat{x}_{\pi/4} , \hat{p}_{\pi/4} \) commute, the generalised EPR (nonlocal) variables \( \hat{x}_\theta, \hat{p}_\theta \) do not commute, and hence the name generalised EPR correlation for \( \Lambda_\theta(\Psi_r) \); indeed, we have

\[
[\hat{x}_\theta , \hat{p}_\theta ] = -i \cos 2\theta .
\]

For the two-mode-squeezed vacuum \(|\Psi_r\rangle\) the generalised EPR correlation reads

\[
\Lambda_\theta(\Psi_r) = \cosh 2r - \sin 2\theta \sinh 2r .
\]
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Let us combine the quadrature variables of the oscillators of Alice and Bob into boson operators $\hat{a} = (\hat{x}_a + i\hat{p}_a)/\sqrt{2}$ and $\hat{b} = (\hat{x}_b + i\hat{p}_b)/\sqrt{2}$. Then, $\Lambda_\theta(\psi)$ has this expression quadratic in the boson variables:

$$\Lambda_\theta(\psi) = \langle \psi | \hat{A}_\theta | \psi \rangle,$$

$$\hat{A}_\theta = 1 + 2 \sin^2 \theta \hat{a}^\dagger \hat{a} + 2 \cos^2 \theta \hat{b}^\dagger \hat{b} - 2 \cos \theta \sin \theta (\hat{a} \hat{b} + \hat{a}^\dagger \hat{b}^\dagger).$$

(4.16)

We may call $\hat{A}_\theta$ the generalised EPR operator.

The entanglement of $\Psi_r$ monotonically increases with increasing value of the squeezing parameter $r$. In order that $\Lambda_\theta(\Psi_r)$ be useful as an entanglement measure of $\Psi_r$, it should, for fixed value of $\theta$, decrease with increasing $r$. The restriction $\tan \theta \geq \tanh r$, encountered earlier in Eq. (4.8) from a different perspective, simply ensures this. Through the monotonic relationship (3) between $r$ and $E_r$, we will view this constraint as a restriction on the allowed range of values of $\theta$, for a fixed value of entanglement.

Given a squeezed state $|\Psi_r\rangle$, let us denote by $|\Psi'_r\rangle$ the state obtained from $|\Psi_r\rangle$ by independent local canonical transformations [57] $S_a, S_b \in Sp(2, R)$ acting respectively on the oscillators of Alice and Bob.

**Proposition 4.2** We have $\Lambda_\theta(\Psi'_r) \geq \Lambda_\theta(\Psi_r)$, $\forall \theta$ in the range $1 \geq \tan \theta \geq \tanh r$ and for all $S_a, S_b \in Sp(2, R)$.

**Proof:** Clearly,

$$\Lambda_\theta(\Psi'_r) = \frac{1}{2} \left\{ \cosh 2r \left[ \sin^2 \theta \text{tr}(S_a S_a^T) + \cos^2 \theta \text{tr}(S_b S_b^T) \right] \right. - \sin 2\theta \sinh 2r \left. \text{tr}(\sigma_3 S_a \sigma_3 S_b^T) \right\}.$$  

(4.17)

If $e^{\pm \gamma_a}$ are the singular values of $S_a$, and $e^{\pm \gamma_b}$ those of $S_b$, then

$$\text{tr}(S_a S_a^T) = 2 \cosh 2\gamma_a,$$

$$\text{tr}(S_b S_b^T) = 2 \cosh 2\gamma_b,$$

and

$$\text{tr}(\sigma_3 S_a \sigma_3 S_b^T) \leq 2 \cosh(\gamma_a + \gamma_b).$$  

(4.18)

Thus the difference $\Delta(\gamma_a, \gamma_b) \equiv \Lambda_\theta(\Psi'_r) - \Lambda_\theta(\Psi_r)$ obeys

$$\Delta(\gamma_a, \gamma_b) \geq \cosh 2r \left[ \sin^2 \theta (\cosh 2\gamma_a - 1) + \cos^2 \theta (\cosh 2\gamma_b - 1) \right]$$

$$- \sin 2\theta \sinh 2r \left[ \cosh(\gamma_a + \gamma_b) - 1 \right].$$  

(4.19)
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It is easily seen that $\Delta(\gamma_a, \gamma_b)$ is extremal at $\gamma_a = \gamma_b = 0$ corresponding to the standard squeezed state $|\Psi_r\rangle$. To show that this extremum is indeed minimum, we note that the determinant of the Hessian matrix of the right hand side, evaluated at $\gamma_a = 0 = \gamma_b$, is proportional to $\sin 2\theta \cosh 2r - \sinh 2r$, and hence is positive if and only if $\tan \theta \geq \tanh r$.

Once again we see a role for the requirement $\tan \theta \geq \tanh r$. Let the equivalence $V_G \sim V_0$ denote the fact that the corresponding Gaussian states are connected by a local canonical transformation. The fact that $M \equiv V_0 - V_{\Psi_{r_0}} \geq 0$ implies $\Lambda_{\theta_0}(\rho_{\Psi_0}) \geq \Lambda_{\theta_0}(\Psi_{r_0})$. In view of Proposition 6.2, this implies

$$\Lambda_{\theta_0}(\rho_{\Psi_0}) \geq \Lambda_{\theta_0}(\rho_{\Psi_0}) \geq \Lambda_{\theta_0}(\Psi_{r_0}) = \cosh 2r_0 - \sin 2\theta \sinh 2r_0,$$

for any Gaussian state $V_G$ connected to $V_0$ by local canonical transformation. This assigns an alternative meaning to the canonical parameter $r_0$:

**Proposition 4.3** Given a Gaussian state described by $V_G \sim V_0$, the canonical squeeze parameter $r_0$ is the smallest $r$ for which the matrix inequality $V_G - V_{\Psi_r} \geq 0$ is true.

It is well known that the two-mode-squeezed vacuum has several extremal properties of interest to entanglement [70, 253]. It seems that this state enjoys one more such distinction, this time in respect of our generalised EPR correlation.

**Conjecture 4.1** Among all bipartite states of fixed entanglement numerically equaling $E_r$, and for every $\theta$ in the range $\tan r \leq \tan \theta$, the two-mode-squeezed vacuum $|\Psi_r\rangle$ yields the least value for the generalised EPR correlation $\Lambda_\theta(\cdot)$. In other words, no state $|\psi\rangle$ with entanglement $E(|\psi\rangle) \leq E_r$ can yield a generalised EPR correlation $\Lambda_\theta(\psi) < \Lambda_\theta(\Psi_r)$, for any $\theta$ in the range $\tan \theta \geq \tanh r$.

The special case $\theta = \pi/4$ is the basis of the important work of Ref. [70]. Hence the present assertion can be viewed as a generalisation of their Proposition 1. The original EPR correlation $\Lambda_{\pi/4}(\cdot)$ continuously decreases to zero with increasing entanglement. But this is not true of the generalised EPR correlation $\Lambda_\theta(\cdot)$.

Let us denote by $r_\theta$ the value of $r$ determined by a given value of $\theta$ through the equation $\tan \theta = \tanh r$, and let $\theta_r$ denote the value of $\theta$ so determined by $r$. Then, for a given numerical $E_r$, the relevant range for $\theta$ in Conjecture 1 is $\theta_r \leq \theta \leq \pi/4$.

**Proposition 4.4** The generalised EPR correlation $\Lambda_\theta(\cdot)$ obeys the basic inequality $\Lambda_\theta(\cdot) \geq \cos 2\theta$. The two-mode-squeezed vacuum saturates this inequality if and only if the squeeze parameter $r$ respects $\tanh r = \tan \theta$.
Proof: It is clear that the relations $\tan \theta = \tanh r$, $\sin 2\theta = \tanh 2r$, and $\cos 2\theta = (\cosh 2r)^{-1}$ are equivalent to one another, and so also are the inequalities $\tan \theta \geq \tanh r$, $\sin 2\theta \geq \tanh 2r$, and $\cos 2\theta \leq (\cosh 2r)^{-1}$. Now consider the transformation $(\hat{a}, \hat{b}) \rightarrow U(r)(\hat{a}, \hat{b})U(r)^\dagger$ where $U(r) = \exp\{r(\hat{a}\dagger\hat{b} - \hat{a}\hat{b})\}$ is the unitary two-mode-squeeze operation:

$$\hat{a} \rightarrow \hat{a} \cosh r - \hat{b}^\dagger \sinh r, \quad \hat{b} \rightarrow \hat{b} \cosh r - \hat{a}^\dagger \sinh r.$$  \hspace{1cm} (4.21)

This implies the following transformation for the anticommutator $\{\hat{b}, \hat{b}\} \equiv \hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b}$:

$$\{\hat{b}, \hat{b}\} \rightarrow (\hat{b}\hat{b}^\dagger - \hat{a}\hat{a}^\dagger) + \frac{1}{2}(\{\hat{a}, \hat{a}\dagger\}) \cosh 2r - (\hat{a}\hat{b} + \hat{a}^\dagger\hat{b}^\dagger) \sinh 2r = \cosh 2r \hat{\Lambda}_{\theta_r}, \quad \theta_r \equiv \arctan(\tanh r).$$ \hspace{1cm} (4.22)

Since $\{\hat{b}, \hat{b}\} \geq 1$, so is also its unitary transform $\cosh 2r \hat{\Lambda}_{\theta_r}$. That is, $\hat{\Lambda}_{\theta_r} \geq (\cosh 2r)^{-1} = \cos 2\theta_r$.

Thus, saturation of the inequality $\Lambda_{\theta_r}(\psi') \geq \cos 2\theta_r$ is equivalent to the condition $\langle \psi|\{\hat{b}, \hat{b}\}|\psi\rangle = 1$, where $|\psi'\rangle = U(r)|\psi\rangle$. A pure state which satisfies $\langle \psi|\{\hat{b}, \hat{b}\}|\psi\rangle = 1$, is of the form $|\psi\rangle = |\phi\rangle_a \otimes |0\rangle_b$, where $|\phi\rangle_a$ is any vector in Alice’s Hilbert space $\mathcal{H}_a$. It follows that states saturating the inequality $\Lambda_{\theta_r}(\hat{\rho}) \geq \cos 2\theta_r$ constitute the set $\{\hat{\rho}^{\otimes b} = U(r)|\hat{\rho}(a)\rangle \otimes |0\rangle_b \langle 0|U(r)^\dagger\}$, where $\hat{\rho}(a)$ is any (pure or mixed) state of Alice’s oscillator. Finally, Conjecture 6.1 claims that among all these states saturating this inequality the two-mode-squeezed vacuum $|\Psi_{r_0}\rangle$, corresponding to the choice $\hat{\rho}(a) = |0\rangle_a\langle 0|$, has the least entanglement.

4.4 Entanglement of Formation

With the canonical form and the generalised EPR correlations in hand, we are now fully equipped to compute the EOF of an arbitrary two-mode Gaussian state.

Proposition 4.5 Given an inseparable zero-mean two-mode Gaussian state $\rho_{V_0}$ with covariance matrix $V_0$ specified in the canonical form by $u$, $v$, $\theta_0$ and $r_0$ with $u$, $v \geq 0$ and $0 < \tanh r_0 \leq \tan \theta_0 \leq 1$, its EOF equals $E_{r_0}$, the entanglement of the squeezed vacuum $|\Psi_{r_0}\rangle$.

Proof: The fact that $M \equiv V_0 - V_{\Psi_{r_0}} \geq 0$ guaranties that $\rho_{V_0}$ can be realized as a convex sum of displaced versions $D(\xi)|\Psi_{r_0}\rangle$ of the squeezed vacuum states $|\Psi_{r_0}\rangle$, all of which
have the same entanglement $E_{r_0}$ as $|\Psi_{r_0}\rangle$:

$$\rho_{V_0} \sim \int d^2\xi D(\xi)|\Psi_{r_0}\rangle\langle\Psi_{r_0}|D(\xi)^\dagger \exp(-\frac{1}{2}\xi^TM^{-1}\xi).$$

(4.23)

Here $D(\xi)$ is the unitary phase space displacement operator. The rank of $M$ equals 2, and both $M^{-1}$ and the two-dimensional integral refer to the restriction of the phase space variable $\xi$ to the range of $M$.

Since a specific ensemble realization with average entanglement $E_{r_0}$ is exhibited, EOF($\rho_{V_0}$) $\leq E_{r_0}$. On the other hand, evaluation of the generalised EPR correlation $\Lambda_\theta(\rho_{V_0}) = \text{tr}(\Lambda_\theta\rho_{V_0})$, for the particular value of $\theta$ occurring in $V_0$ shows that $\Lambda_{\theta_0}(\rho_{V_0}) = \cosh 2r_0 - \sin 2\theta_0 \sinh 2r_0$. And by Conjecture 6.1, this implies EOF($\rho_{V_0}$) $\geq E_{r_0}$. We have thus proved EOF($\rho_{V_0}$) $= E_{r_0}$.

An attractive feature of the canonical form of the covariance matrix is that the two-mode-squeezing $U(r)$ acts on it in a covariant or form-preserving manner.

**Proposition 4.6** Under the two-mode-squeezing transformation $U(r)$ we have

$$V_0(r_0, \theta_0, u, v) \rightarrow V_0(r_0', \theta_0', u', v');$$
$$r_0' = r_0 + r, \quad \sin 2\theta_0' = \frac{\sinh 2r + \cosh 2r \sin 2\theta_0}{\cosh 2r + \sin 2\theta_0 \sinh 2r},$$
$$(u', v') = (u, v) \times (\cosh 2r + \sin 2\theta_0 \sinh 2r).$$

(4.24)

This is easily verified by direct computation. While the canonical squeeze parameter $r_0$ simply gets translated by $r$, the parameters $u$ and $v$ get scaled by a common factor. If we define $r_{\theta_0}$, $r_{\theta_0'}$ through $\tan \theta_0 \equiv \tan r_{\theta_0}$ and $\tan \theta_0' \equiv \tan r_{\theta_0}$, the transformation law for $\theta_0$ takes the form of translation: $r_{\theta_0'} = r_{\theta_0} + r$.

As a consequence of this covariance, the convex decomposition which minimizes the average entanglement goes covariantly to such a decomposition under two-mode-squeezing: the minimal decomposition commutes with squeezing. This implies, in particular, the following simple behaviour of EOF under squeezing: $E_{r_0} \rightarrow E_{r_0 + r}$.

Finally, the just separable Gaussian states on the separable-inseparable boundary, correspond to the canonical form with $r_0 = 0$ [57]. As was to be expected, the condition (4.8) places no restriction on $\theta_0$ in this case.