Chapter 3

Higher dimensional analogue of Erdös-Ginzburg-Ziv Theorem

3.1 Introduction

Additive number theory, factorization theory and graph theory provide a good source for combinatorial problems in finite abelian groups (See [56, 57, 62, 27, 12], for instance). Among them, zero sum problems have been of growing interest. The cornerstone of almost all recent combinatorial research on zero-sum problems is a theorem of Erdös-Ginzburg-Ziv [32] and a question of H. Davenport on an invariant of finite abelian groups.

In general, our notations and terminology will be the same as the one in factorization theory (cf. survey articles by Chapman, Halter-Koch and Geroldinger in [12] and the paper of Gao and Geroldinger [38]). Here for a finite sequence $S = \{g_1, g_2, \cdots, g_l\} = g_1 g_2 \cdots g_l$ of elements of $G$, repetitions are allowed and the order is disregarded. If $S = \{a_1, a_2, \cdots, a_k\}$, $T = \{b_1, b_2, \cdots, b_k\}$ are two sequences of elements of group $G$ then $ST$ will denote the sequence $\{a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_k\}$. In an analogous way one defines the sequence $\prod_{i=1}^k A_i$ for given sequences $A_1, A_2, \cdots, A_k$. A sequence $T$ is
called a subsequence of $S$ if there exists a sequence $T'$ such that $TT' = S$; clearly the sequence $T'$ is uniquely determined by $S$ and $T$ and we denote it by $ST^{-1}$.

Let $G$ be a finite abelian group. We denote a cyclic group of order $n$ by $C_n$. As defined in Section 2.1, by $s(G)$ (or $\eta(G)$ respectively) we denote the smallest integer $\ell \in \mathbb{N}$ such that every sequence $S$ of length $|S| \geq \ell$ of elements of $G$ has a zero-sum subsequence $T$ of length $|T| = \exp(G)$ (or $1 \leq |T| \leq \exp(G)$ respectively). The investigation of these invariants has a long tradition, and in recent years the investigation of these invariants and of the related inverse problems, i.e., the investigation of the structure of extremal sequences with, and without, the respective properties, received a good deal of attention. Among others, this is due to applications in the theory of non-unique factorizations. We refer to the monograph of Geroldinger and F. Halter-Koch [45], in particular to Chapter 5, for a detailed account of results on these invariants and their applications in the theory of non-unique factorizations, and to the recent survey article of Gao and Geroldinger [39] for an exposition of the state of the knowledge and numerous references.

Still, many questions are wide open. The precise value of $s(G)$ for cyclic groups is known by the classical Erdős-Ginzburg-Ziv Theorem [32], but $s(G)$ for groups of rank 2 has only recently been determined (see [71, 45]) and the precise value of $s(G)$ is unknown for most groups of rank greater than 2, as is the value of $\eta(G)$. We will describe the bounds and precise value in certain cases.

**Lemma 3.1.1** (Chi, Ding, Gao et. al. [21]). Let $G$ be a finite abelian group and let $H \subset G$ be a subgroup such that $\exp(G) = \exp(H)\exp(G/H)$. Then

$$s(G) \leq \exp(G/H)s(H) + s(G/H) - \exp(G/H).$$

**Theorem 3.1.2.** Let $m, n, r \in \mathbb{N}$ with $m|n$.

(i) $\eta(C_m \oplus C_n) = 2m + n - 2$ and $s(C_m \oplus C_n) = 2m + 2n - 3$ (See [71], [45]).
(ii) $\eta(C_r^n) \geq (2^r - 1)(n - 1) + 1$ and $s(C_r^n) \geq 2^r(n - 1) + 1$. If $n$ is a power of 2, then equality holds (See [49], [30]).

(iii) If $n$ is odd, then $\eta(C_3^n) \geq 8n - 7$ and $s(C_3^n) \geq 9n - 8$. If $n$ is a power of 3, then equality holds (See [31]).

In order to prove the main theorem of this paper we will need following upper bound obtained by Alon and Dubiner [10]:

When $G = C_r^n$ then $s(G)$ is bounded above by a linear function of $n$ and they showed that,

$$s(C_r^n) \leq c(r)n,$$  \hspace{1cm} (1)

where $c(r)$ is a constant which depends on $r$. It is known that $c(1)$ can be taken as 2 (due to Erdős, Ginzburg and Ziv [32]), and $c(2)$ can be taken as 4 (due to C. Reiher [71]). In general, in our current state of knowledge $c(r)$ can be taken satisfying,

$$c(r) \leq 256(r \log_2 r + 5)c(r - 1) + r + 1, \text{ for } r \geq 3.$$  

\textbf{Remark 3.1.1.} From above expression $c(3)$ turns out to be approximately 9994. So

$$s(C_3^n) \leq 9994n.$$  

\textbf{Conjecture 3.1.1} (Gao, Hou, Schmid, Thangadurai [41]). \textit{Let $n \in \mathbb{N}$. Then}

$$s(C_3^n) = \begin{cases} 8n - 7, & \text{if } n \text{ is even} \\ 9n - 8, & \text{if } n \text{ is odd}. \end{cases}$$
Remark 3.1.2. If one assumes Conjecture 3.1.1 then from the fact that \( s(G) \geq \eta(G) + \exp(G) - 1 \) and Theorem 3.1.2\( (iii) \) it follows that,

\[
\eta(C_n^3) = 8n - 7, \text{ if } n \text{ is an odd integer.}
\]

Remark 3.1.3. If one assumes Conjecture 3.1.1 then from the fact that \( s(G) \geq \eta(G) + \exp(G) - 1 \) and \( \eta(Z_n^3) \geq 7n - 6 \), for \( n \) an even integer, it follows that

\[
\eta(C_n^3) = 7n - 6, \text{ if } n \text{ is an even integer.}
\]

Remark 3.1.4. It can be easily proved that for \( n, m \) both positive integers if \( s(C_r^n) = a_r(n - 1) + 1 \) and \( s(C_r^m) = a_r(m - 1) + 1 \), then \( s(C_r^{nm}) \leq a_r(nm - 1) + 1 \). According to Dr. Wolfgang Schmid, there doesn’t seem to be a general process available to get the lower bound on \( s(C_r^{nm}) \) in this situation. For example note, \( s(C_3^5) = 45(3 - 1) + 1 \) is known, but as per him nobody knows \( s(C_5^5) \) and there are reasons to believe that they are not related in the form one might expect; for \( s(C_3^5) = 45(3 - 1) + 1 \) is known but the best lower bound for \( s(C_9^5) \) is \( 42(9 - 1) + 1 \) and the upper bound is \( 45(9 - 1) + 1 \). One can see the paper by Y. Edel [29, Theorem 1] for the information on this. Also it has been proved that, \( s(C_3^3) = 9(3) - 8 = 19 \) (see [49, Satz 4]); also cf. [30, Corollary 4.5]. So in view of Theorem 3.1.2\( (iii) \), \( s(C_3^{3k}) = 9(3^k) - 8 \), for \( k \in \mathbb{N} \).

We prove the following Theorem:

**Theorem 3.1.3.** Assume that \( m \geq 3 \) is a fixed positive integer such that \( \eta(C_r^m) = a_r(m - 1) + 1 \), for some constant \( a_r \) depending on \( r \). Further, assume that

\[
n \geq \frac{m^r(c(r)m - a_r(m - 1) + m - 3)(m - 1) - (m + 1) + (m + 1)(a_r + 1)}{m(m + 1)(a_r + 1)}
\]

is a fixed positive integer such that \( s(C_r^n) = (a_r + 1)(n - 1) + 1 \). In the above lower bound on \( n \), \( c(r) \) is the Alon-Dubiner constant. Then, \( s(C_r^{nm}) = (a_r + 1)(nm - 1) + 1 \).
3.2 Proof of the Main Theorem

Proof of Theorem 3.1.3. Let $S$ be a sequence in $C_{nm}^{r}$. $|S| = (a_r + 1)(nm - 1) + 1$. Let $S_m$ be the sequence of all elements of $S$ modulo $m$. Then, we see that there exists an element $x \in S_m$ which is repeated maximum number of times. We can assume $x$ to be the zero element of $S_m$, if necessary by translating the elements of $S$. Note that in $S_m$ at least $\left\lceil \frac{(a_r + 1)(nm - 1) + 1}{m} \right\rceil$ zeros are available. (For a real number $x$ by $\lfloor x \rfloor$ we mean, the least integer $\geq x$).

Let $S_m^*$ be the sequence of all non-zero elements of $S_m$. From $S_m^*$ take out all possible $k \geq 0$ disjoint non-empty subsequences $R_1, R_2, \cdots, R_k$ with $|R_i| = m$ such that $\sum_{a \in R_i} a \in C_r^n$, $\forall i = 1, 2, \cdots, k$. Hence, $W := S_m^* \left( \prod_{i=1}^{k} R_i \right)^{-1}$ contains no $m$-element subsequence whose sum is zero in $C_r^n_m$. Hence by a theorem of Alon and Dubiner [10], we have $|W| \leq c(r)m - 1$.

If $|W| \leq a_r(m-1)$ then we shall show that there exist a subsequence of $S$ of length $nm$ which sum up to zero in $C_{nm}^{r}$. Let us count all the disjoint $m$-element subsequences of $S_m$ whose sum is zero in $C_{nm}^{r}$. First remove all possible disjoint $m$-element zero sum subsequences from $S_m (S_m^*)^{-1}$ and say the remaining sequence be $A$ (It may happen that there is no such subsequence. In this case take $A = S_m (S_m^*)^{-1}$). Let the number of all possible disjoint $m$-element subsequences from $S_m (S_m^*)^{-1}$ be $t$. Clearly, $0 \leq |A| \leq m - 1$. Then,

$$tm + km = (a_r + 1)(nm - 1) + 1 - |W| - |A|.$$  

Hence,

$$t + k \geq \frac{1}{m}((a_r + 1)(nm - 1) + 1 - a_r(m - 1) - m + 1) = (a_r + 1)(n - 1) + \frac{1}{m}.$$  

Since $(t+k)$ is an integer, $(t+k) \geq (a_r + 1)(n - 1) + 1$. Hence we have pairwise disjoint $m$-element subsequences, $I_1, I_2, \cdots, I_{(a_r + 1)(n - 1) + 1}$ of $S$ such that $\sum_{b \in I_j} b = 0 \in C_{m}^{r}$ for
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every \( j = 1, 2, \cdots, (a_r+1)(n-1)+1 \). Write \( c_j = \frac{1}{m} \sum_{b \in I_j} b \), for every \( j = 1, 2, \cdots, (a_r+1)(n-1)+1 \). Since \( s(C_n^r) = (a_r+1)(n-1)+1 \) and we have \( (a_r+1)(n-1)+1 \) number of integer lattice points \( c_1, c_2, \cdots, c_{(a_r+1)(n-1)+1} \), there exist \( n \) element subsequence \( c_{i_1}, c_{i_2}, \cdots, c_{i_n} \) such that its sum is zero in \( C_n^r \). Thus we get,

\[
\sum_{j=1}^{n} c_{i_j} = 0 \in C_n^r \implies \sum_{j=1}^{n} \sum_{b \in I_j} b = 0 \in C_{nm}^r.
\]

Hence we are done in this case.

Now we assume that \( |W| > a_r(m - 1) \) and we show that

1. There are disjoint subsequences \( B_1, B_2, \cdots, B_\ell \) of \( W \) of length \( < m \) with zero sum in \( C_m^r \) and with \( |B_i| + |B_j| > m \) for \( i \neq j \),

2. There are at least \( \frac{(c(r)m - a_r(m-1) + m - 3)(m-1)}{m+1} \) zeroes in \( S_m(S_m^*)^{-1} \), and

3. \( |W \left( \prod_{i=1}^{\ell} B_i \right)^{-1}| \leq a_r(m - 1) \).

Since \( |W| > a_r(m - 1) \) we can find a natural number \( t, 2 \leq t \leq m - 1 \) such that one finds such a \( t \)-element subsequence of \( W \) sums to zero in \( C_m^r \). Let \( B_1 \) be a maximal subsequence of \( W \) such that \( |B_1| = t_1 \) with \( 2 \leq t_1 \leq m - 1 \) and its sum is zero in \( C_m^r \).

Then we can take a subsequence \( A_1 \) of \( S_m \) which contains \( m - t_1 \) zeros and together with \( B_1 \) we get an \( m \)-element subsequence whose sum is zero in \( C_m^r \).

If \( |W(B_1)^{-1}| \geq a_r(m - 1) + 1 \), we can find \( B_2 \) which is the maximal subsequence of \( W(B_1)^{-1} \) with \( |B_2| = t_2 \) with \( 2 \leq t_2 \leq m - 1 \), whose sum is zero in \( C_m^r \). Note that \( t_1 \geq t_2 \) and \( |B_1B_2| > m \). If not, we would have chosen \( B_1B_2 \) in the first step and it would have contradicted the maximality. Once we have chosen \( B_2 \), take \( A_2 \), a subsequence of \( S_m \) of all zeros disjoint from \( A_1 \) and having cardinality \( m - t_2 \). Then, \( A_2B_2 \) produces an \( m \)-element subsequence of \( S_m \) whose sum is zero in \( C_m^r \).
Continue this process, until we arrive at $|W(\prod_{i=1}^{\ell} B_i)^{-1}| \leq a_r(m - 1)$, where $\ell$ is a positive integer. We will calculate upper bound of $\ell$ now. By definition of $\ell$, we have

$$a_r(m - 1) + 1 + \sum_{i=1}^{\ell-1} |B_i| \leq |W| \leq a_r(m - 1) + \sum_{i=1}^{\ell} |B_i|$$  \hspace{1cm} (2)

**Case 1.** $\ell = 2k$, $k \in \mathbb{N}$.

Since $|B_i| \geq 2$, $\forall i \in \{1, 2, \ldots, \ell\}$ and $|B_i| + |B_j| \geq m + 1$, $\forall i, j \in \{1, 2, \ldots, \ell\}$, $i \neq j$, we have

$$(k - 1)(m + 1) + 2 \leq \sum_{i=1}^{\ell-1} |B_i| \leq |W| - a_r(m - 1) - 1$$

Hence,

$$(k - 1)(m + 1) + 2 \leq c(r)m - a_r(m - 1) - 2$$

$$\Rightarrow (k - 1) \leq \frac{c(r)m - a_r(m - 1) - 4}{m + 1}$$

$$\Rightarrow k \leq \frac{c(r)m - a_r(m - 1) - 4 + m + 1}{m + 1}$$

$$= \frac{c(r)m - a_r(m - 1) + m - 3}{m + 1}$$

Hence at most $X = \frac{(c(r)m - a_r(m - 1) + m - 3)(m - 1)}{m + 1}$ zeros are required in this case.

**Case 2.** $\ell = 2k + 1$, $k \in \mathbb{N} \cup \{0\}$.

**Sub-case (I) :** $k = 0$. Clearly, number of zeros required is at most $m - 2$.

**Sub-case (II) :** $k \in \mathbb{N}$. Then from (2), we have

$$k(m + 1) \leq \sum_{i=1}^{\ell-1} |B_i| \leq |W| - a_r(m - 1) - 1$$

$$\Rightarrow k \leq \frac{c(r)m - a_r(m - 1) - 2}{m + 1}$$

For this sub-case $\ell \geq 3$. Observe that there exists $B_i$ such that $|B_i| \geq \frac{m+1}{2}$. Hence the number of zeros required in this sub-case is at most,

$$Y = \frac{(c(r)m - a_r(m - 1) - 2)(m - 1)}{m + 1} + \frac{m - 1}{2}$$
Since $m \geq 3$, $X = \max\{X, Y, m - 2\}$. Hence the number of zeros needed in any case is at most, $X = \frac{(c(r)m - a_r(m - 1) + m - 3)(m - 1)}{m + 1}$.

Hence in order to make sure that there are at least $X$ zeros in $S_m$, we need the following condition,

$$|S_m(S_m^*)^{-1}| \geq \left\lceil \frac{(a_r + 1)(nm - 1) + 1}{m^r} \right\rceil \geq X.$$ 

This holds because

$$n \geq \frac{m^r(c(r)m - a_r(m - 1) + m - 3)(m - 1) - (m + 1) + (m + 1)(a_r + 1)}{m(m + 1)(a_r + 1)}$$

by hypothesis.

If $|S_m(S_m^*)^{-1} \left( \prod_{i=1}^{\ell} A_i \right)^{-1}| \geq m$, remove all possible disjoint $m$-element subsequences and say the remaining sequence be $A$. Let us say these subsequences are $t$ in number.

Clearly, $0 \leq |A| \leq m - 1$. Then,

$$tm + |A| = (a_r + 1)(nm - 1) + 1 - |S_m^*| - \sum_{i=1}^{\ell} |A_i|.$$ 

$$\Rightarrow tm + \ell m + |A| = (a_r + 1)(nm - 1) + 1 - |S_m^*| + \sum_{i=1}^{\ell} |B_i|.$$ 

$$= (a_r + 1)(nm - 1) + 1 - |W| - km + \sum_{i=1}^{\ell} |B_i|.$$ 

$$\Rightarrow (t + \ell + k)m = (a_r + 1)(nm - 1) + 1 - |W| + \sum_{i=1}^{\ell} |B_i| - |A|.$$ 

Hence,

$$(t + \ell + k) \geq \frac{1}{m}((a_r + 1)(nm - 1) + 1 - a_r(m - 1) - m + 1) = (a_r + 1)(n - 1) + \frac{1}{m}.$$ 

Since $(t + \ell + k)$ is an integer, $(t + \ell + k) \geq (a_r + 1)(n - 1) + 1$. Hence as in the case $|W| \leq a_r(m - 1)$ we can extract a zero sum subsequence of $S$ of length $nm$. Hence the theorem is proved. \qed
Following are some observations in rank 3 case.

**Observations :**

1. Take $r = 3$, $a_r = 8$ and $n, m$ odd in the Theorem 3.1.3. Then by Remark 3.1.1
   the lower bound on $n$ becomes $\frac{(9987m + 5)(m-1)m^3 + 8(m+1)}{9m(m+1)}$.

2. Take $r = 3$, $a_r = 7$ and $n, m$ even in the Theorem 3.1.3. Then by Remark 3.1.1
   the lower bound on $n$ becomes $\frac{(9988m + 4)(m-1)m^3 + 7(m+1)}{8m(m+1)}$.

3. There is one more bound on $s(G)$ which was given by W. D. Gao et al. [44]
   (see also [45, Theorem 5.7.4]), which says that for a finite abelian group $G$ of
   exponent $m$, $s(G) \leq |G| + m - 1$. For those $m$ for which this bound is lesser than
   that comes from Alon and Dubiner, we can get a better bound on $n$ following
   same procedure as in the main theorem apart from replacing Alon Dubiner
   bound by this particular bound. We will get to see the difference between two
   bounds when we see some examples at the end of this chapter. Following are
   the bounds that one will get in this situation,

   (i) Assume $\eta(C_3^m) = 8m - 7$, $m$-odd and $m \geq 3$. If $s(C_3^n) = 9n - 8$, $n$-odd and
       if $n \geq \frac{m^3(m^3 - 6m + 4)(m-1) + 8(m+1)}{9m(m+1)}$, then $s(C_{nm}^3) = 9nm - 8$.

   (ii) Assume $\eta(C_3^m) = 7m - 6$, $m$ is an even integer and $m \geq 4$. For $n$ even
       integer if $s(C_3^n) = 8n - 7$, and if $n \geq \frac{m^3(m^3 - 5m + 3)(m-1) + 7(m+1)}{8m(m+1)}$ then
       $s(C_{nm}^3) = 8nm - 7$.

Now, we will see that using condition on $n$ and $m$ in above Observations and following
Theorem one can get conjectured bound for few more groups.

**Theorem 3.2.1** (Gao [41]). *(i)* Let $n = 3^a5^b$ for $a, b \in \mathbb{N} \cup \{0\}$. Then

$$s(C_3^n) = \eta(C_3^n) + n - 1 = 9n - 8.$$
(ii) Let $n = 2^a3$ for $a \in \mathbb{N}$. Then

$$s(C_n^3) = \eta(C_n^3) + n - 1 = 8n - 7.$$ 

**Examples:** (1) Let $n = 3^{12}$, $m \in \{3, 5, 7\}$ (using Observation 1). Then

$$s(C_{nm}^3) = \eta(C_{nm}^3) + nm - 1 = 9nm - 8.$$ 

By using Observation 3(i) we can see that these holds true for $m \in \{3, 5, 7, 9, 11, 13, 15, 17, 19, 21\}$. 

(2) Let $n = 2^{20}3$, $m \in \{4, 6, 8, 10, 12, 14\}$ (using Observation 2). Then

$$s(C_{nm}^3) = \eta(C_{nm}^3) + nm - 1 = 8nm - 7.$$ 

By using Observation 3(ii) and Remark 3.1.4 we can see that relation holds true for $m \in \{2n : n \in [1, 15]\}$. 

(3) Let $n = 2^{16}$, $m = 4$ (using Observation 2). Then

$$s(C_{nm}^3) = \eta(C_{nm}^3) + nm - 1 = 8nm - 7.$$ 

By using Observation 3(ii) and Remark 3.1.4 we can see that relation holds true for $m \in \{2, 4, 6, 8, 10, 12, 14\}$. 