This chapter is a collection of definitions and basic theorems which will be needed in the ensuing chapters.

All basic rings considered in this thesis will be associative with unit element and modules will be unitary.

(1): Sums and products of modules:

Definition 1: - Let \((M_i)_I\) be a given family of left \(R\)-modules. The cartesion product

\[ \prod_{i \in I} M_i = \{ (x_i)_{i \in I} : x_i \in M_i \} \]

can be made into module by defining the operations component-wise.

\[ (x_i)_{i \in I} + (y_i)_{i \in I} = (x_i + y_i)_{i \in I} \]

\[ a \cdot (x_i)_{i \in I} = (ax_i)_{i \in I} \quad (a \in R) \]

This module is called the product of the family of modules \(M_i\). It contains a sub-module.

\[ \emptyset \subset M_i = \{ (x_i)_{i \in I} : x_i = 0 \text{ for almost all } i \in I \} \]

which is called the external direct sum of the modules \(M_i\). There are canonical homomorphisms \(P_j : \prod_{i \in I} M_i \rightarrow M_i\) given by
Morita theorem 5 [40]:- An R-module $M$ is faithful if and only if $R^R$ can be embedded in $M^I$ for some set $I$ (If $R$ has minimum condition on left ideals, $I$ can be taken to be finite).

Definition 17:- Let $M'$ be any submodule of a left $R$-module $M$. $M'$ is said to be a pure submodule of $M$ if for all $r \in R$ and for all $m \in M$, $rm \in M'$ implies there exists $m' \in M'$ such that $rm = rm'$. $M'$ is said to be ideal pure in $M$ if for every right ideal $I$ in $R$, $M' \cap IM = IM'$.

Theorem 6 [37]:- Suppose $M_2 \subseteq M_1 \subseteq M$ are left $A$-module. Then

1. $M_2$ pure in $M_1$ and $M_1$ pure in $M$ implies $M_2$ pure in $M$.

2. $M_2$ pure in $M$ implies $M_2$ pure in $M_1$.

3. $M_1$ pure in $M$ implies $M_1/M_2$ pure in $M/M_2$.

4. If $M_2$ is pure in $M$ then $M_1/M_2$ pure in $M/M_2$ implies $M_1$ pure in $M$.

5. If $M_2$ is pure in $M$ then under the one-one correspondence between the sub-modules of $M$ containing $M_2$ and the submodules of $M/M_2$ pure submodules correspond to pure submodules.
\[ p_j((x_i)_I) = x_j \text{ called projections and} \]

\[ j_{i_0}: M_{i_0} \rightarrow \prod_{i} M_i \text{ given by } j_{i_0}(x) = (u_i)_I \text{ called} \]

injections where

\[ u_i = x \text{ if } i = i_0 \]

\[ = 0 \text{ if } i \neq i_0 \]

when each \( M_i = M \) we write \( M^I \) for \( \prod_{i} M_i \) and \( M(I) \) for \( \otimes M_i \).

For a positive integer \( n \), \( M^n \) will denote the product of \( n \) copies of \( M \).

**Definition 2:** Let \( M \) be a module and \((M_i)_I\) a family of submodules of \( M \), then \( \sum_{i} M_i \) which consists of all possible finite sums of elements from the various modules \( M_i \), is the smallest submodule of \( M \) containing \( \bigcup_{i} M_i \). We call \( \sum_{i} M_i \) the sum of submodules \( M_i \).

**Definition 3:** A module \( M \) is said to be a direct sum of a family \((M_i)_I\) of submodules of \( M \) if

(i) \[ M = \sum_{i} M_i \]

(ii) \[ M_i \cap \sum_{i \neq i_0} M_i = 0 \text{ for all } i \in I \]
i.e. \((M_i)_I\) is an independent set of submodules.
Theorem 1: Let $(M_i)_I$ be a family of modules indexed by $I$. Let $N$ be a module and $(f_i)_I$ be homomorphisms $f_i : N \rightarrow M_i$. Then there exists a unique homomorphism $f : N \rightarrow \prod_{i \in I} M_i$ such that for each $i \in I$ the following diagram commutes:

\[
\begin{array}{c}
N \\
\downarrow f \\
\prod_{i \in I} M_i \\
\downarrow f_i \\
M_i
\end{array}
\]

Definition 4: The unique homomorphism $f : N \rightarrow \prod_{i \in I} M_i$ is called the direct product of $(f_i)_I$ and is often denoted by $f = \prod_{i \in I} f_i$. It is characterised by

\[p_i \left( \prod_{i \in I} f_i \right) = f_i.
\]

Theorem 2: Let $(M_i)_I$ be an indexed set of modules. Let $N$ be a module and $(f_i)_I$ be an indexed class of homomorphisms $f_i : M_i \rightarrow N$ for $i \in I$. Then there exists a unique homomorphism $f : \bigoplus_{i \in I} M_i \rightarrow N$ such that the following diagram commutes:

\[
\begin{array}{c}
M_i \\
\downarrow f_i \\
N \\
\downarrow f \\
\bigoplus_{i \in I} M_i
\end{array}
\]
The unique homomorphism is denoted by \( f = \bigoplus_{i} f_i \).

**Definition 5:** Let \( N \) be a submodule of \( M \), if the inclusion mapping \( N \rightarrow M \) is a homomorphism of \( R \)-modules. If, in addition, \( N \neq M \) (so that \( N \) is strictly contained in \( M \)) then \( N \) is called a **proper submodule** of \( M \).

**Theorem 3:** Let \( M \) be an \( R \)-module and \( \{N_i\}_{i \in I} \) an indexed family of submodules of \( M \). Then their intersection \( \bigcap_{i \in I} N_i \) is also a submodule of \( M \).

**Definition 6:** If \( K \) is a submodule of an \( R \)-module \( M \), then \( M/K \) (endowed with the \( R \)-module structure described above) is called the **factor module** of \( M \) with respect to \( K \).

**Theorem 4:** Let \( K \) be a fixed submodule of an \( R \)-module \( M \) and \( A \) a variable sub-module satisfying \( K \subseteq A \subseteq M \). Then \( A/K \) is a submodule of \( M/K \).

Furthermore, if \( B \) is a given submodule of \( M/K \), then there is one and only one sub-module \( A \) (\( K \subseteq A \subseteq M \)) such that \( B = A/K \).

**Definition 7:** Given a pair of homomorphisms \( f \) and \( g \), the sequence
\[ M' \xrightarrow{f} M \xrightarrow{g} M'' \]

is said to be exact at \( M \) in case \( \text{Im} f = \text{Ker} g \).

A sequence (finite or infinite) of homomorphisms

\[ \cdots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} M_{n+1} \xrightarrow{f_{n+2}} \cdots \]

is exact in case it is exact at each \( M_n \); that is, in case for each successive pair \( f_n, f_{n+1} \),

\[ \text{Im} f_n = \ker f_{n+1}. \]

Thus the sequence

\[ 0 \xrightarrow{} N \xrightarrow{f} M \]

is exact at \( N \) if and only if \( 'f' \) is one to one. Likewise the sequence

\[ M \xrightarrow{g} N \xrightarrow{} 0 \]

is exact at \( N \) if and only if \( 'g' \) is onto.

**Definition 8:** If \( f: M \xrightarrow{} N \) and \( f': N \xrightarrow{} M \) are homomorphisms with \( ff' = I_N \), we say that \( f \) is a split epimorphism and \( f' \) is a split monomorphism.
Definition 9: An R-module M is called cyclic if \( M' = R \mathbf{x} \) for some \( x \in M \).

Definition 10: The annihilator of an R-module M is defined as

\[
\text{Ann} (M) = \{ r \in R \mid rM = 0 \}
\]

M is said to be faithful in case \( \text{Ann} (M) = 0 \).

Definition 11: A left ideal L is called nilpotent if there is a positive integer n such that \( L^n = \{0\} \), where \( L^n = L^{n-1}(L) \). Equivalently, L is nilpotent if for some n the product of any n elements of L is zero.

Definition 12: A submodule N of M is called a direct summand of M if \( M = N \oplus N' \) for some submodule \( N' \) of M. If \( K \subset N \subset M \) are submodules, show the following.

(i) If N is a direct summand of M, N/K is a direct summand of M/K.

(ii) If K is a direct summand of N and N is a direct summand of M, then K is a direct summand of M.

(iii) If K is a direct summand of M, then K is a direct summand of N. If further N/K is a direct summand of M/K, then N is a direct summand of M.
Definition 13: A module $M$ is said to satisfy ascending chain condition (Abbreviated to ACC) if there is no infinite strictly ascending sequence $A_1 \subset A_2 \subset A_3 \cdots$ of submodules of $M$. Similarly, if $M$ does not contain any infinite strictly descending sequences of submodules then it is said to satisfy descending chain condition (DCC).

Definition 14: A ring $R$ with element is called a left noetherian ring if any one of the following equivalent statements holds:

(i) $R$ satisfies the ascending chain condition for its left ideals.

(ii) Every non-empty set of left ideals of $R$ contains a maximal member.

(iii) Every left ideal in $R$ is finitely generated.

Definition 15: A ring $R$ is called an artinian ring if $R$ satisfies the descending chain condition for its left ideals.

Definition 16: A non-zero module $M$ is called a simple module if $M$ contains no nontrivial proper submodules. A module $M$ is cyclic if it is generated by a single element.
Definition 18: A module $L$ over $R$ is finitely related or finitely presented if there is an exact sequence

$$
0 \longrightarrow K \longrightarrow R^n \longrightarrow L \longrightarrow 0
$$

(1)

where $n$ is an integer $> 0$, and $K$ is finitely generated. This is equivalent to the requirement that there exist integers $n$ and $m$ such that

$$
R^m \longrightarrow R^n \longrightarrow L \longrightarrow 0
$$

is exact. Also it is known that when a module $L$ is finitely presented then any exact sequence (1) implies that $K$ is finitely generated.

(2): Projective and injective modules:

Definition 19: A module $U$ will be called projective if given any morphism $f: U \longrightarrow M'$ and any epimorphism $g: M \longrightarrow M'$, there is a homomorphism $h: U \longrightarrow M$ with $g.h = f$. In the language of diagrams. This means that every diagram

```
                  U
                 /    |
                /     |
M  ---->  M'  ---->  0
```

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in which the row is exact, can be imbedded in a commutative diagram

\[
\begin{array}{c}
U \\
\text{h'} \downarrow \ \\
\text{f} \\
\text{M} \xrightarrow{g} \text{M'} \longrightarrow O
\end{array}
\]

**Theorem 7 [41]:** A direct sum of modules is projective if and only if each summand is projective.

**Definition 20 [14]:** A module \( V \) will be called injective if given any \( M' \) be a sub-module of \( M \) and a homomorphism \( M' \longrightarrow V \), there is an extension \( M \longrightarrow V \).

In the language of diagrams this means that every diagram

\[
\begin{array}{c}
0 \longrightarrow M' \longrightarrow M \\
\downarrow \ \\
V
\end{array}
\]

in which the row is exact, can be imbedded in a commutative diagram
Definition 21: A ring $R$ will be called left hereditary if every left ideal of $R$ is a projective module.

Definition 22: The ring $R$ will be called left semi-hereditary if each finitely generated left ideal of $R$ is a projective module.

Theorem 8 [14]: For each ring $R$, the following conditions are equivalent.

(i) $R$ is left semi-hereditary.

(ii) Each finitely generated submodule of a projective left $R$-module is projective.

Definition 23 [50]: An injective hull (or injective envelope) for a module $A$ is any injective module which is an essential extension of $A$. 
Theorem 9 [50]: [Bear, Eckmann-Schopf] Let $A$ be a module.

(i) Any injective module containing $A$ contains an injective hull for $A$. In particular, there exist injective hull for $A$. Let $E$ be any injective hull for $A$, and let $j : A \to E$ be the inclusion map.

(ii) Given any essential monomorphism $f : A \to B$, there exists a monomorphism $g : B \to E$ such that $g.f = j$.

(iii) Given any monomorphism $f : A \to E'$ with $E'$ injective, there exists a monomorphism $g : E \to E'$ such that $g.j = f$.

Definition 24 [50]: Given a module $A$, we use the notation $E(A)$ for an injective hull of $A$. As this is only unique up to isomorphism, general assertions about $E(A)$ must be valid for all injective hull of $A$. The equation $B = E(A)$ should only be used as an abbreviation for "$B$ is an injective hull for $A$".

(3): Quasi-projective and Quasi-injective modules :

Definition 25 [1]: A right $R$-module $M$ is said to be quasi-projective (resp. quasi-injective) if $M$ is $M$-projective
(resp. $M$ is $M$-injective).

Theorem 10 [1]: Let $(M_i)_{1 \leq i \leq n}$ be a finite family of right $R$-modules. Then $\bigoplus_{i=1}^{n} M_i$ is quasi-injective (resp. quasi-projective) if and only if $M_i$ is $M_j$-injective (resp. $M_i$ is $M_j$-projective) for all $i, j$ with $1 \leq i \leq n$.

(4): $M$ - projective and $M$-injective modules :-

For left $R$-modules $M, N$, the set of module homomorphism

$f : M \rightarrow N$ is denoted by $\text{Hom}_R(M, N)$.

This is an abelian group with respect to the operation of addition $(f, g) \rightarrow f + g$ defined by

$$(f + g)(x) = f(x) + g(x) \quad (x \in M)$$

Notation: Given a ring $R$ the category of left $R$-modules is the system

$$R^1M = (R^M, \text{Hom}_R, O)$$

where $\mathcal{R}^M$ is the class of left $R$-modules, $\text{Hom}_R(M, N) \rightarrow \text{Hom}_R(M, N)$ and 'O' is the usual composition of the functions. In this category the objects are left $R$-modules and the morphisms are left $R$-module
homomorphisms.

Definition 26 i:- "Hom Functors" Let \( R \) and \( S \) be rings and let \( U = R U_S \) be a bimodule. Let \( f:R^M \rightarrow R^N \) be an \( R \)-homomorphism in \( R^1M \). Then for each \( h \in \text{Hom}_R(U,M) \), we have \( fh \in \text{Hom}_R(U,N) \),

\[
\text{Hom}_R(U,f): \text{Hom}_R(U,M) \rightarrow \text{Hom}_R(U,N) \text{ given by }
\]

\[h \mapsto fh\] can be shown to be left \( S \)-homomorphism.

For simplicity in notation, if there is no ambiguity with the module \( U \), \( \text{Hom}_R(U,f) \) will be denoted by \( f_* \).

Thus we do have a function

\[
\text{Hom}_R(U,-): R^1M \rightarrow S^1M \text{ defined by }
\]

\[\text{Hom}_R(U,-): M \rightarrow \text{Hom}_R(U,M)\]

\[\text{Hom}_R(U,-): f \rightarrow \text{Hom}_R(U,f)\]

on the other hand, for the \( R \)-homomorphism \( f:R^M \rightarrow R^N \) we can define a mapping

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\[ f^* = \text{Hom}_R(f,U): \text{Hom}_R(N,U) \longrightarrow \text{Hom}_R(M,U) \text{ via} \]

\[ \text{Hom}_R(f,U): h \longrightarrow hf. \]

\[ f^* \text{ is a right } S\text{-homomorphism. Here then we have a function} \]

\[ \text{Hom}_R(-,U): R^1M \longrightarrow S^1M \text{ defined by} \]

\[ \text{Hom}_R(-,U): M \longrightarrow \text{Hom}_R(M,U) \]

\[ \text{Hom}_R(-,U): f \longrightarrow \text{Hom}_R(f,U). \]

**Definition 27:** "Exact functor" Let \( R \) and \( S \) be rings and \( T: R^1M \longrightarrow S^1M \) an additive functor. \( T \) is an exact functor if it carries exact sequences in \( R^1M \) into exact sequences in \( S^1M \). \( T \) is left exact if it has weaker property that for an exact sequence

\[ 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \]

in \( R^1M \), it gives exactness for the sequence

\[ 0 \longrightarrow T (M') \longrightarrow T (M) \longrightarrow T (M'') \longrightarrow 0 \]

in \( S^1M \). Similarly one defines right exactness.
Theorem 11 [3]: The Hom functors are left exact. Thus in particular if \( R^U \) is a module, then for every exact sequence

\[
0 \longrightarrow K \overset{f}{\longrightarrow} M \overset{g}{\longrightarrow} N \longrightarrow 0
\]

in \( R^M \) the sequences

\[
0 \longrightarrow \text{Hom}_R (U,K) \overset{f^*}{\longrightarrow} \text{Hom}_R (U,M) \overset{g^*}{\longrightarrow} \text{Hom}_R (U,N)
\]

and

\[
0 \longrightarrow \text{Hom}_R (N,U) \overset{g^*}{\longrightarrow} \text{Hom}_R (M,U) \overset{f^*}{\longrightarrow} \text{Hom}_R (K,U)
\]

are exact.

Definition 28: Let \( R^U \) be a module. If \( R^M \) is a module, then \( U \) is \( M \)-projective (or \( U \) is relative to \( M \)) in case for each epimorphism \( g : R^M \longrightarrow R^N \) and each homomorphism \( h : R^U \longrightarrow R^N \) there is an \( R \)-homomorphism \( \tilde{h} : U \longrightarrow M \) such that the diagram

\[
\begin{array}{ccc}
U & \overset{\tilde{h}}{\longrightarrow} & M \\
\downarrow h & & \downarrow g \\
M & \overset{g}{\longrightarrow} & N \longrightarrow 0
\end{array}
\]

commutes. On the other hand \( U \) is \( M \)-injective (or \( U \) is
injective relative to $M$) in case for each monomorphism $f : K \rightarrow M$ and each homomorphism $h : R^K \rightarrow R^U$ there is an $R$-homomorphism $\tilde{h} : M \rightarrow U$ such that the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & K \\
& \downarrow & \downarrow \\
& K & \longrightarrow & M \\
& \downarrow & \downarrow \tilde{h} & \downarrow h \\
& U & \longrightarrow & U
\end{array}
\]

commutes.

Remark :- Regarding $R$ as a left module over itself in the usual way it turns out that $R$-injective modules are the same as the injective over $R$. How every $R$-projective modules are not the same as projective modules over $R$.

For example, $\mathbb{Z}_Q$ is $\mathbb{Z}_Z$ - projective but not projective over $\mathbb{Z}$.

Definition 29 :- A module $R^U$ is projective in case it is projective relative to every module $R^M$. A module $R^V$ is injective in case it is injective relative to every module $R^M$.  

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Theorem 12 [3]: Let \( M \) be a left \( R \)-module and let \( (U_i)_i \) be an indexed set of left \( R \)-modules. Then

(i) \( \bigoplus_i U_i \) is \( M \)-projective if and only if each \( U_i \) is \( M \)-projective.

(ii) \( \prod_i U_i \) is \( M \)-injective if and only if each \( U_i \) is \( M \)-injective.

Theorem 13: Let \( (U_i)_i \) be an indexed set of left \( R \)-modules then

(i) \( \bigoplus_i U_i \) is projective if and only if each \( U_i \) is projective.

(ii) \( \prod_i U_i \) is injective if and only if each \( U_i \) is injective.

Theorem 14 [1]: Let \( U \) be a right \( R \)-module. Then

(i) If \( 0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0 \) is an exact sequence in mod-\( R \) and \( U \) is \( M \)-injective, then \( U \) is \( M' \)-injective and \( U \) is \( M'' \)-injective.

(ii) If \( (M_i)_{i \in I} \) is a family of right \( R \)-modules and \( U \) is \( M_i \)-injective for each \( i \in I \), then \( U \) is \( \bigoplus_{i \in I} M_i \)-injective.
(5) : "Tensor product of modules"

Definition [30]:- Given a right module $M_R$ and a left module $R^N$ over a ring $R$ and an abelian group $A$, a function $h : M \times N \rightarrow A$ is said to be $R$-balanced in case for all $m, m_1 \in M$, $n, n_1 \in N$ and $r \in R$

(i) $h(m_1 + m_2, n) = h(m_1, n) + h(m_2, n)$

(ii) $h(m, n_1 + n_2) = h(m, n_1) + h(m, n_2)$

(iii) $h(mr, n) = h(m, rn)$.

Definition 31 i.- A pair $(T, \Theta)$ consisting of an abelian group $T$ and $R$-balanced map $\Theta : M \times N \rightarrow T$ is a tensor product of $M_R$ and $R^N$ in case for every abelian group $A$ and every $R$-balanced map $h : M \times N \rightarrow A$ there is a $Z$-homomorphism $f : T \rightarrow A$ such that the diagram

$$
\begin{array}{c}
M \times N \xrightarrow{\Theta} T \\
| \downarrow h | \\
| \downarrow f | \\
A
\end{array}
$$

Commutes.
Notation :- The tensor product of \((M_R, R^N)\) is denoted by \(M \otimes_R N\) or simply \(M \otimes N\) if the context is clear.

Theorem 15 :- For each right \(R\)-module \(M_R\), \(M \otimes_R R \cong M\).

Definition 32 :- "Tensor functor"

Let \(U = S^U_R\) be a bimodule. Then follows that there is an additive covariant functor

\[
(U \otimes_R -) : R^1M \longrightarrow \mathbb{Z}^1M \text{ defined by }
\]

\[
(U \otimes_R -) : \begin{array}{c}
M \longrightarrow U \otimes_R M \\
\end{array}
\]

\[
(U \otimes_R -) : f \longrightarrow 1 \otimes_R f
\]

It is well known that if \(f : M \longrightarrow M'\) is an \(R\)-homomorphism then \(U \otimes_S f : S^U \otimes_M M \longrightarrow S^U \otimes_M M'\) is an \(S\)-homomorphism. This fact permits us to the functor as an additive functor from

\[
R^1M \longrightarrow S^1M \text{ and write it }
\]

\[
(S^U \otimes_R -) : R^1M \longrightarrow S^1M
\]

similarly there is an additive covariant functor.
\[ (-\mathcal{S}U_R) : \mathcal{S}M \to \mathcal{R}M \text{ defined by} \]
\[ (-\mathcal{S}U_R) : N \to N \mathcal{S} U_R \]
\[ (-\mathcal{S}U_R : g \to g \mathcal{S} 1_U \]

Notation: Let \( L \) be an injective cogenerator in the category \( Z^{\mathcal{M}} \) of a abelian groups.

Let
\[ (\quad)^* = \text{Hom}_{\mathcal{Z}} (\quad, L) \]

**Theorem 16:** Let \( f : M' \to M \) and \( g : M \to M'' \) be \( R \)-homomorphisms in \( R^{\mathcal{M}} \) and let \( U \) be a right \( R \)-module. Then
\[ U \mathcal{S} M' \to U \mathcal{S} M \to U \mathcal{S} M'' \]
is exact if and only if
\[ \text{Hom}(g, U^*) \to \text{Hom}(f, U^*) \]
\[ \text{Hom}_R(M'', U^*) \to \text{Hom}_R(M, U^*) \to \text{Hom}_R(M', U^*) \]
is exact.

**Theorem 17:** The tensor functor is right exact. In particular

\[ 0 \to M' \to M \to M'' \to 0 \]
is exact in \( R^{\mathcal{M}} \), then for every bimodule \( S U_R \)
\[
\begin{align*}
\text{UOM'} \xrightarrow{\text{U} \circ f} \text{UOM} \xrightarrow{\text{U} \circ g} \text{UOM''} \xrightarrow{} 0
\end{align*}
\]
is exact.

**Definition 33:** A R-module U is said to be flat if for every exact sequence of R-modules

\[
\begin{align*}
0 \xrightarrow{} M' \xrightarrow{f} M \xrightarrow{g} M'' \xrightarrow{} 0
\end{align*}
\]
the induced sequence

\[
\begin{align*}
0 \xrightarrow{} \text{UOM'} \xrightarrow{f_*} \text{UOM} \xrightarrow{g_*} \text{UOM''} \xrightarrow{} 0
\end{align*}
\]
is exact.

\[\text{[51]}\]

**Theorem 18:** For any R-module U, the following conditions are equivalent:

(i) For any exact sequence of R-modules

\[
\begin{align*}
0 \xrightarrow{} M' \xrightarrow{f} M \xrightarrow{g} M'' \xrightarrow{} 0
\end{align*}
\]
the sequence

\[
\begin{align*}
0 \xrightarrow{} \text{UOM'} \xrightarrow{f_*} \text{UOM} \xrightarrow{g_*} \text{UOM''} \xrightarrow{} 0
\end{align*}
\]
is exact.
(ii) If $M' \xrightarrow{f} M$ is injective, then $f_* : U \otimes M' \longrightarrow U \otimes M$ is injective.

(iii) If $M' \xrightarrow{f} M$ is injective with $M$ and $M'$ finitely generated, then $f_* : U \otimes M' \longrightarrow U \otimes M$ is injective.

[In view of the above theorem an $R$-module $U$ is flat if and only if for every injective map $f:M' \longrightarrow M$ the map $f_* : U \otimes M' \longrightarrow U \otimes M$ is injective. Moreover we way restrict both $M$ and $M'$ to be finitely generated].

Definition 34 :- A module $U_R$ is flat relative to a module $R^M$ (or that $U$ is $M$-flat) in case the functor $(U \otimes_R -)$ preserves the exactness of all short exact sequence with middle term $M$. Then of course, $U$ is $M$-flat if and only if for every submodule $K \subseteq M$ the sequence

$$0 \longrightarrow U \otimes K \longrightarrow U \otimes g \longrightarrow U \otimes M$$

is exact.

Theorem 19 [3]: - Let $M$ be a left $R$-module. A right $R$-module $U$ is $M$-flat if and only if $U^* = \text{Hom}_R (U, L)$ is $M$-injective where $L$ is an injective cogenerator in $\mathcal{M}$.

In particular $U$ is flat if and only if $U^*$ is injective.
Definition 35 [3] - Given modules \( R^M \) and \( U_R \), let \( T(M) \) denote the class of right \( R \)-modules that are \( M \)-flat and let \( T^{-1}(U) \) denote the class of left \( R \)-modules \( N \) such that \( U \) is \( N \)-flat. It is well known that

\[ T(M) \text{ is closed under direct sums and direct summands.} \]

\[ T^{-1}(U) \text{ is closed under submodules, factor modules and direct sums.} \]

Definition 36 - Let \( U_R \) and \( R^M \) be modules. Define the annihilator in \( M \) of \( U \) to be

\[ \text{Ann}_M \left( U \right) = \{ m \in M | u \otimes m = 0 \text{ in } U \otimes M \text{ for all } u \in U \} \]

\( U \) is \( M \)-faithful in case \( \text{Ann}_M \left( U \right) = 0 \)

Definition 37 - Let \( R \) be a commutative ring and left \( U \) and \( X \) be \( R \)-modules then \( \text{Hom}_R \left( U, X \right) \) is called \( X \)-dual of \( U \).

Let \( A \) be an \( R \)-algebra, \( M \) and \( N \) are \( R \)-modules. Then there exists a natural \( A \)-homomorphism

\[ \phi_M : A \otimes \text{Hom}_R \left( M, N \right) \xrightarrow{R} \text{Hom}_A \left( A \otimes M, A \otimes N \right) \]

given by

\[ \phi_M \left( a \otimes f \right) \{ (b \otimes x) \} = ab \otimes f(x), \text{ } ab \in A, \text{ } x \in M, \text{ } f \in \text{Hom}_R \left( M, N \right) \} \]
Theorem 20 [51]:- Let \( A \) be a flat \( R \)-algebra. \( M \) and \( N \) are \( R \)-modules. Let \( \phi_M \) be the natural map defined above. Then

(i) \( \phi_M \) is injective if \( M \) is finitely generated.

(ii) \( \phi_M \) is an isomorphism if \( M \) is finitely presented.

Definition 38 :- (Direct limit functor).

A partially ordered set \((I, \preceq)\) is called directed if for every couple \( i, j \in I \) there exist \( K \in I \) such that \( i \preceq K \) and \( j \preceq K \). Assume that \( I \) is directed. A direct system of \( R \)-modules over \( I \) consists of

(i) a module \( M_i \) for each \( i \in I \)

(ii) a homomorphism \( \alpha_{ij} : M_i \longrightarrow M_j \) for each \( i \preceq j \) satisfying

\[
\alpha_{ii} = i \text{ and } i \preceq j \preceq K \text{ implies } \alpha_{jk} \alpha_{ij} = \alpha_{ik}
\]

Given such a direct system \((M_i, \alpha_{ij})\), we define its direct limit \( \lim_{\longrightarrow} M_i \), usually abbreviated as \( \lim_{\longrightarrow} M_i \), as follows: form disjoint union \( \bigcup_{i \in I} M_i \) of the sets \( M_i \), and define a binary relation \( \sim \) on \( \bigcup_{i \in I} M_i \), by putting \( x \sim y \) if \( x \in M_i, \ y \in M_j \) and there
exists $K \in I$ such that $i \leq K$, $j \leq K$ and $\phi_{ik} (x) = \phi_{jk} (y) \in M_k$.

Then the relation is an equivalence relation.

We now set
$$\text{limit } M_i = U M_i | \sim .$$

**Theorem 21**: Direct limit is an exact functor.