CHAPTER-IV

SOME FIXED POINT THEOREMS FOR EXPANSION MAPPINGS

4.1 During the last fifteen years or so many results on fixed or common fixed point theorems were studied for certain expansion mappings Fisher [106] proved:

Theorem F: Let T be a selfmapping of a metric space X such that, for all $x, y \in X$

(4.1.1) $d(Tx, Ty) \geq \frac{1}{2} [d(x, Tx) + d(y, Ty)]$

Then T is the identity mapping.

Wang, Li, Cao and Iseki [107] obtain the following results:

Theorem W1: If there is a constant $\alpha > 1$, such that

(4.1.2) $d(fx, fy) \geq \alpha\, d(x, y)$

for any x, y in X and f is onto, then f has a unique fixed point.

Theorem W2: If there exists non-negative real numbers a, b, c with $a + b + c > 1$ and $a < 1$ such that

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[106] Fisher, B. (28)

[107] Wang, S.Z; Li, B.Y; Gao, Z.M. and Iseki, K. (115)

(4.1.3)

$$d\,(fx,\ fy) \geq a\, d\,(x,\ fx) + b\, d\,(y,\ fy) + c\, d\,(x,y)$$

for every x, y in X with $x \neq y$ and f is onto, then f has a fixed point.

<u>Theorem W3</u>: If there exists a real number $a > 1$ such that

(4.1.4)

$$d\,(fx,\ fy) > a \min \{d\,(x,fx),\ d(y,fy),\ d\,(x,y)\}$$

for each x, y in X and f is onto and continuous, then T has a fixed point.

In 1985 Rhoades [108] generalize the above results for pair of maps for common fixed point. In fact he proved:

<u>Theorem R1</u>: Let f, g be surjective selfmap of a complete metric space (X, d). Suppose there exists a constant $q > 1$ such that

(4.1.5) $\quad d\,(fx,\ gy) \geq a\, d\,(x,\ y)$

for each x, y in X. Then f and g have a unique common fixed point .

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[108] Rhoades, B.E. (82)

<u>Theorem R2</u>: Let f, g be surjective continuous self maps of a complete metric space X. If there exists a real number $a > 1$ such that

(4.1.6)

$$d(fx, gy) \geq a \min \{d(x,fx), d(x, gy), d(x, y)\}$$

for each x, y in X, then f or g has a fixed point or f and g have a common fixed point.

4.2 In this section, we establish some generalized results on fixed point theorem taking expansion mappings. In fact we prove:

<u>Theorem 1:</u> Let f, g be surjective selfmaps of a complete metric space (X, d) where

(4.2.1)

$$d(fx, gy) \geq k \min \{d(x,y), [d(x,fx)\, d(y,fy)]^{\frac{1}{2}} + [d(x,fy)\, d(y,fx)]^{\frac{1}{2}}\}$$

for all x, y in X, $x \neq y$. The f and g have common fixed point in X.

<u>Proof</u>: Let x_o in X. Since f is surjective there exists a point x_1 in $f^{-1}x_o$. Since g is surjective there exists a point x_2 in $g^{-1}x_1$. Continuing this we obtain a sequence $\{x_n\}$ with x_{2n+1} in $f^{-1}x_{2n}$, x_{2n+2} in $g^{-1}x_{2n+1}$.

Suppose $x_{2n+1} = x_{2n}$ for some n.

If $x_{n+1} \neq x_{2n+2}$, then from (4.2.1)

$$d\ (x_{2n+1}\ ,\ x_{2n}) = d\ (gx_{2n+2}, fx_{2n+1})$$

$$= d\ (fx_{2n+1}, gx_{2n+2})$$

$$\geqslant k \min\ \{ d\ (x_{2n+1}, x_{2n+2}),$$

$$[d(x_{2n+1}, x_{2n})\ d(x_{2n+2}, x_{2n+1})]^{\frac{1}{2}}$$

$$+ [d(x_{2n+1}, x_{2n+1})\ d(x_{2n+2}, x_{2n})]^{\frac{1}{2}} \}$$

Thus

$$0 \geqslant 0$$

Therefore $d\ (x_{2n+1}, x_{2n+2}) = 0$. The condition

$x_{2n+1} = x_{2n} = fx_{2n+1}$ implies that x_{2n+1} is a fixed point of f.

Also $x_{2n+2} = x_{2n+1} = gx_{2n+2}$ implies x_{2n+2} is a fixed point of g. Similarly $x_{2n+2} = x_{2n+1}$

impliex $x_{2n+2} = gx_{2n+2}$ and $fx_{2n+1} = x_{2n+1}$ leads to x_{2n+1} is a commom fixed point of f and g.

Assuming $x_n \neq x_{n+1}$ for each n. From (4.2.1), we write

$$d(x_{2n}, x_{2n+1}) = d(f_{x2n+1}, gx_{2n+2})$$

$$\geq k \min \{ d(x_{2n+1}, x_{2n+2}),$$

$$[d(x_{2n+1}, x_{2n}) d(x_{2n+2}, x_{2n+1})]^{\frac{1}{2}}$$

$$+ [d(x_{2n+1}, x_{2n+1}) d(x_{2n+2}, x_{2n})]^{\frac{1}{2}}\}$$

This implies

(4.2.2)

$$d(x_{2n+1}, x_{2n+2}) \leq (1/k) d(x_{2n}, x_{2n+1})$$

Similarly

(4.2.3)

$$d(x_{2n}, x_{2n+1}) \leq (1/k) d(x_{2n-1}, x_{2n})$$

By (4.2.2) and (4.2.3), it follows that the sequence $\{x_n\}$ is a Cauchy sequence. Thus there exists some point z in X to which x_n converges. Since $x_{2n} = fx_{2n+1}$, $x_{2n+1} = gx_{2n+2}$ and f and g are continuous, we obtain that

$$z = fz = gz$$

This completes the proof.

This theorem generalizes the theorem of Rhoades.

Remark: Khan [109] has proved:

Theorem K. Let T be a self mapping of a complete metric space (X, d) and satisfy

$$d(Tx, Ty) \leq \alpha \{d(x, Tx)\, d(y, Ty)\}^{\frac{1}{2}}$$

for all $x, y \in X$ and $0 \leq \alpha < 1$, then T has a unique fixed point. Kannan [110] mapping is like Arithematic mean type and Khan is like G.M. Since Arithematic mean is greater than G.M; this result is not intresting for contraction mapping. However it will be better for expansive mappings.

(4.3) A Contraction mapping is a mapping T on a metric space (X, d) into itself satisfying

$$d(Tx, Ty) \leq \alpha\, d(x, y)$$

for all x, y in X where $0 \leq \alpha < 1$.

It is well known that a contraction mapping. On a complete metric space has a unique fixed point Kannan[111].

The condition under which two mappings on a metric space have a common fixed point and proved:

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[109] Khan, M.S. (53)

[110] Kannan, R. (48)

[111] Kannan, R. (48)

<u>Theorem K</u>. If T_1 and T_2 are mappings on a complete metric space (X, d) into itself and if there is a constant α such that $0 < \alpha < \frac{1}{2}$ and

(4.3.1)

$$d(T_1 x, T_2 y) \leq \alpha [d(x, T_1 x) + d(y, T_2 y)]$$

for all x, y in X and T_1 and T_2 have a unique common fixed point.

Fisher [112] defined a mapping T on a metric space (X, d) to be a Kannan mapping if it satisfies (4.3.1) with $T_1 = T_2 = T$ and obtain a relation between Kannan mapping on a metric space into itself then T^n is a Kannan mapping for large n and on the other hand if T is a Kannan mapping on (X, d) satisfying-

(4.3.2)

$$d(x, Tx) + d(y, Ty) \leq \beta d(x, y)$$

for all x, y in X and fixed $\beta > 0$, T^n is a contraction mapping for large n.

The purpose of this article is to extend these results to much wider class of mapping and apply it to expansion mapping

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[112] Fisher, B. (28)

<u>Definition</u>: A pair of mappings (f, g) on X into itself is said to have generalized

Kannan property if there exists constants

$$(4.3.3) \qquad k_2 + k_4 + k_5 + k_6 > 1$$

$$(4.3.4) \qquad 1 - k_1 + k_4 > 0$$

and

$$(4.3.5) \qquad d(fx, gy) \geq k_1 d(x, fx) + k_2 d(y, gy) + k_3 d(x, gy) + k_4 d(y, fx) + k_5 d(x, y) + k_6 \frac{d(y, gy)\,[1+\{d(x,y)\, d(y,gy)\}^{\frac{1}{2}}]}{1 + d(x, y)}$$

for all x, y in X.

Now we shall prove theorem 2.

<u>Theorem 2:</u> If (f, g) is a generalized Kannan mapping and surjective mapping of a complete metric space into itself, for all x, y in X, $x \neq y$. Then f and g have a common fixed point.

<u>Proof:</u> Let x_o in X. Since f is surjective there exists a point x_1 in $f^{-1}x_o$. Since g is surjective there exists a point x_2 in $g^{-1}x_1$. Continuing in this manner we obtain a sequence $\{x_n\}$ with x_{2n+1} in $f^{-1}x_{2n}$, x_{2n+2} in $g^{-1}x_{2n+1}$, Suppose $x_{2n+1} = x_n$ for some n.

If $x_{2n+1} = x_{2n+2}$ then from hypothesis,

$$d(x_{2n+1}, x_{2n}) = d(gx_{2n+2}, fx_{2n+1})$$

$$= d(fx_{2n+1}, gx_{2n+2})$$

$$\geq k_1\, d(x_{2n+1}, x_{2n}) + k_2 d(x_{2n+2}, x_{2n+1})$$

$$+ k_3\, d(x_{2n+1}, x_{2n+1}) + k_4 d(x_{2n+2}, x_{2n})$$

$$+ k_5\, d(x_{2n+1}, x_{2n+2})$$

$$+ k_6 \frac{d(x_{2n+2}, x_{2n+1})\,[1+\{d(x_{2n+1}, x_{2n+2})\, d(x_{2n+1}, x_{2n+2})\}^{\frac{1}{2}}]}{1 + d(x_{2n+1}, x_{2n+2})}$$

Thus

$$0 = (k_2 + k_4 + k_5 + k_6)\, d(x_{2n+2}, x_{2n+1}),$$

where k_2, k_4, k_5, k_6 are evaluated at

$$(x, y) = (x_{2n+1}, x_{2n+2}),$$

By (4.3.3), $d(x_{2n+2}, x_{2n+1}) = 0$,

The condition $x_{2n+1} = x_{2n} = fx_{2n+1}$ implies that x_{2n+1} is a fixed point of f.

Also $x_{2n+2} = x_{2n+1} = gx_{2n+2}$ implies x_{2n+2} is a fixed point of g.

Similarly

$x_{2n+2} = x_{2n+1}$ implies $x_{2n+2} = gx_{2n+2}$ and

$fx_{2n+1} = x_{2n+1}$ leads to x_{2n+1} being a common fixed point of f and g.

Assume $x_n \neq x_{n+1}$ for each n. From the hypothesis

$$d\,(x_{2n}, x_{2n+1}) = d\,(fx_{2n+1}, gx_{2n+2})$$

$$\geq k_1\, d\,(x_{2n+1}, x_{2n})$$

$$+ k_2\, d\,(x_{2n+2}, x_{2n+1})$$

$$+ k_3\, d\,(x_{2n+1}, x_{2n+1})$$

$$+ k_4\, d\,(x_{2n+2}, k_{2n})$$

$$+ k_5\, d\,(x_{2n+1}\,, x_{2n+2})$$

$$+ k_6 \frac{d(x_{2n+2}, x_{2n+1})\,[1+\{d(x_{2n+1}, x_{2n+2})\; d(x_{2n+2},x_{2n+1})\}^{\frac{1}{2}}]}{1+ d\,(x_{2n+1}, x_{2n+2})}$$

or

(4.3.6) $(1 - k_1 + k_4)\; d\,(x_{2n}, x_{2n+1}) \geq$

$$(k_2+k_4k_5+k_6)\; d(x_{2n+1}, x_{2n+2})$$

where k_1, k_2, k_4, k_5, k_6 are evaluated at

(x_{2n+1}, x_{2n+2})

Similarly we can show that

$$(4.3.7)\quad (1 - k_1' + k_4')\ d(x_{2n+1}, x_{2n+2}) \geq (k_2' + k_4' + k_5' + k_6')\ d(x_{2n+2}, x_{2n+3})$$

where p', q', r' and s' are evaluated at (x_{2n+3}, x_{2n+2}),

In equalities (4.3.6) and (4.3.7) along with (4.3.3) and (4.3.4) imply that $\{x_n\}$ is Cauchy, hence convergent to some x in X.

Without loss of generality we may assume that $x_n = x$ for infinitely many n since, otherwise, f and g have a common fixed point. If there exists an infinite number of integers n such that $x_{2n} \neq x$, define $y \in g^{-1} x$. Then from hypothesis

$$\begin{aligned} d(x_{2n}, x) &= d(fx_{2n+1}, gy) \\ &\geq k_1\ d(x_{2n+1}, x_{2n}) \\ &\quad + k_2\ d(y, gy) + k_3\ d(x_{2n+1}, gy) \\ &\quad + k_4\ d(y, x_{2n}) + k_5\ d(x_{2n+1}, y) \end{aligned}$$

$$+ k_6 \frac{d(y, gy)\ [1 + \{d(x_{2n+1}, y)\ d(y, gy)\}^{\frac{1}{2}}]}{1 + d(x_{2n+1}, y)}$$

where the constants are evaluated at (x_{2n+1}, y). The above inequality implex

$$d(x_{2n}, x) \geqslant (k_1+k_2+k_3+k_4+k_5)$$

$$\min \{d(x, y), d(y, x_{2n}), d(x_{2n+1}, y)\}$$

$$\geqslant \inf_{x,y \in X} (k_1 + k_2 + k_3 + k_4 + k_5)$$

$$\min \{ d(x, y), d(y, x_{2n}), d(x_{2n+1}, y)\}$$

Taking the limits as $n \to \infty$

$$0 \geqslant \inf_{x,y \in X} (k_1 + k_2 + k_3 + k_4 + k_5) \, d(x, y)$$

which implies $x = y$

Similar results holds for $d(x_{2n+1}, x)$,

This completes the proof of the theorem.

Remark 1: If we take $f = g$ and $k_1 = k_2 = k_3 = k_4 = k_6 = 0$ we get a theorem of Wang and others [113].

Remark 2: If we take $k_1 = k_2 = k_3 = k_4 = k_6$, we get a result of Rhoades [114].

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[113] Wang, S.Z; Li, B.Y; Gao, Z.M. and Iseki, K. (115)

[114] Rhoades, B.E. (82)

4.4 Now we shall be concerned with three surjective mappings in place of two. In fact we prove:

Theorem 3. Let f, g, h be three surjective selfmaps of a complete metric space (X, d). Suupose that there exists a constant $\alpha > 1$ such that

(4.4.1) $[d(fhx, ghy)]^2 \geq \alpha \min \{d(hx,fhx)\, d(hy\, ghy),$
$d(hx, fhx)\, d(hx, hy),\, d(hy,ghy)\, d(fhx, ghy),$
$d(fhx, ghy)\, d\,(hx, hy)\}$

for each x, y in X.

(4.4.2) f and g are continuous mappings and h is orbitally continuous mapping.

(4.4.3) h commute with f and g.

Then f, g, h have unique common fixed point.

Proof: Let hx_o be in X. Since f is surjective, there exists a point hx_1 in $f^{-1}hx_o$.

Since g is surjective there exists a point hx_2 in $g^{-1}hx_1$.

Continuing in this manner we obtain a sequence $\{hx_n\}$ with hx_{2n+1} in $f^{-1}x_{2n}$, and hx_{2n+2} in $g^{-1}x_{2n+1}$.

If $hx_n = hx_{n+1}$, for any x implies that hx_n is fixed point of f.

Also $hx_{n+1} = hx_n$, hx_n is also fixed point of g.

Similarly $hx_{n+1} = hx_n$ leads to hx_{n+1} is a fixed point of f and g.

Now assume $hx_n \neq hx_{n+1}$ for each n, From the hypothesis we have

$$[d(hx_{2n}, hx_{2n+1})]^2 = [d(fhx_{2n+1}, ghx_{2n+2})]^2$$

$$\geq \alpha \min\{d(hx_{2n+1}, hx_{2n})\, d(hx_{2n+2}, hx_{2n+1}),$$

$$d(hx_{2n+1}, hx_{2n})\, d\,(hx_{2n+1}, hx_{2n+2}),$$

$$d(hx_{2n+2}, hx_{2n+1})\, d(hx_{2n}, hx_{2n+1})$$

$$d\,(hx_{2n+1}, hx_{2n+1})\, d\,(hx_{2n+1}, hx_{2n+2})\}$$

or

$$[d(hx_{2n}, hx_{2n+1})]^2 \geq d(hx_{2n+1}, hx_{2n})\, d(hx_{2n+2}, hx_{2n+1})$$

i.e. $d(hx_{2n}, hx_{2n+1}) \geq \alpha\ d\,(hx_{2n+1}, hx_{2n+2})$

$$[d(hx_{2n+1}, hx_{2n+2})]^2 = [d(ghx_{2n+2}, fhx_{2n+3})]^2$$

$$= [d\,(fhx_{2n+3}, ghx_{2n+2})]^2$$

$$\geq \alpha \min\{d(hx_{2n+3}, hx_{2n+2})\, d(hx_{2n+2}, hx_{2n+1}),$$

$$d\,(hx_{2n+3}, hx_{2n+2})\, d(hx_{2n+3}, hx_{2n+2}),$$

$$d(hx_{2n+2}, hx_{2n+1})\ d(hx_{2n+1}, hx_{2n+2}),$$

$$d(hx_{2n+1}, hx_{2n+2})\ d(hx_{2n+3}, hx_{2n+2})\}$$

or $[d(hx_{2n+1}, hx_{2n+2})]^2 \geq \alpha \min\{d(hx_{2n+1}, hx_{2n+2})$

$$d(hx_{2n+2}, hx_{2n+3}), [d(hx_{2n+2}, hx_{2n+3})]^2,$$

$$[d(hx_{2n+1}, hx_{2n+2})]^2\}$$

Therefore

$$d(hx_{2n+1}, hx_{2n+2}) \geq \alpha^{\frac{1}{2}} d(hx_{2n+2}, hx_{2n+3}),$$

Since

$$[d(hx_{2n+1}, hx_{2n+2})]^2 \geq \alpha [d(hx_{2n+1}, hx_{2n+2})]^2$$

is not possible and if

$$[d(hx_{2n+1}, hx_{2n+2})]^2 \geq \alpha d(hx_{2n+1}, hx_{2n+2})\ d(hx_{2n+2}, hx_{2n+3})$$

then

$$d(hx_{2n+1}, hx_{2n+2}) > \alpha d(hx_{2n+1}, hx_{2n+2})$$

$$> \alpha^{\frac{1}{2}} d(hx_{2n+1}, hx_{2n+2}),$$

$$\alpha > 1.$$

Thus we obtain

$$d\,(hx_{2n}\;,\;hx_{2n+1}\;)\geqslant\;d(hx_{2n+1},\;hx_{2n+2})$$

$$\geqslant\alpha,\alpha^{\frac{1}{2}}d(hx_{2n+2},\;hx_{2n+3})$$

or

$$d\,(hx_{2n},\;hx_{2n+1})\geqslant\;\beta\,d(hx_{2n+2},\;hx_{2n+3})$$

where $\beta\;=\;\alpha\;.\;\alpha^{\frac{1}{2}}\;>1.$

In general,

$$d\,(hx_{2n},\;hx_{2n+1})\geqslant c^{k/2}\;d(hx_{2n+k},\;hx_{2n+k+1})$$

$$\geqslant c^{k}\;d(hx_{2n+2k},\;hx_{2n+2k+1})$$

for $c > 1$ $\{hx_n\}$ is cauchy, converges to a point x in X. The assumption $x_n \neq x_{n+1}$, for each n implies that $x_n \neq x$, for almost all n. Since f and g are continuous mappings and $\{fhx_n\}$ and $\{ghx_n\}$ are subsequences of the sequence $\{hx_n\}$, hence they will be also convergent to the same point $x \in X$.

Hence $f(hx_{2n+1}) \to fx$, $g\;(hx_{2n+2}) \to gx$,

Since h commutes with f an g, we have

$$f\;(hx_{2n+1})\quad=\quad h\;(fx_{2n+1})$$

$$g\;(hx_{2n+2})\quad=\quad h\;(gx_{2n+2})$$

Letting $n \to$, we have

$$fx = hx$$

$$gx = hx$$

i.e. $fx = gx = hx$

Since h is orbitally continuous mapping of a complete metric space (X, d) we have

$$x = \lim_{n\to\infty} h^n x = \lim_{n\to\infty} hh^n x = hx$$

Hence $fx = gx = hx = x$,

i.e. x is a common fixed point of f, g, h. The uniqueness of fixed point can be proved easily by the hypothesis of the theorem.

4.5. We know that harmonic mean is less than geometric mean we shall prove better result by proving.

<u>Theorem-4</u>: If there exists $\alpha + \beta > 1$ and $\alpha, \beta > 0$ such that

$$(4.5.1)\quad d(Tx, Ty) > \frac{2\alpha\, d(x, y)\, d(y, Ty)}{d(x,y) + d(y, Ty)} + \beta\, d(x,y)$$

for each $x, y \in X$ and with $x \neq y$ and T is onto in complete metric space X, then T has a fixed point where

$$\alpha + \beta > 1.$$

<u>Proof</u>: Let x_o be in X. Since T is onto, there is an element x_1, satisfying x_1 in $T^{-1}x_o$.

Thus

$$x_n \quad T^{-1}(x_{n-1}) \quad n = 2, 3, 4, \text{-----}$$

If $x_{m-1} = x_m$ for some m, then x_m is a fixed point of T.

Now we can suppose that $x_{n-1} \neq x_n$, for every n;

by hypothesis, we write,

$$d(x_{n-1}, x_n) = d(Tx_n, Tx_{n+1})$$

$$> \frac{2\alpha\, d(x_{n-1}, x_n)\; d(x_n, x_{n+1})}{d(x_{n-1}\, x_n) + d(x_n, x_{n+1})}$$

$$+ \beta\, d(x_{n-1}, x_n)$$

or

$$(1-\beta)\; [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \geqslant 2\alpha\, d(x_n, x_{n+1})$$

or

$$(1-\beta)\, d(x_{n-1}, x_n) \geqslant (2\alpha - 1 + \beta)\, d(x_n, x_{n+1})$$

or

$$d(x_n, x_{n+1}) \leqslant \frac{1-\beta}{2\alpha - 1 + \beta}\; d(x_{n-1}, x_n)$$

Since $\alpha + \beta > 1$, $\{x_n\}$ is a cauchy sequence and converges to some $x \in X$.

Let $y \in T^{-1}x$. For infinitely many n, $x_n \neq x$ for each n,

$$d(x_n, x) = d(Tx_{n+1}, Ty)$$

$$\geq \frac{2\alpha\, d(x_{n+1}, x_n)\, d(x,y)}{d(x_n, x_{n+1}) + d(x,y)} + \beta\, d(x_{n+1}, x_n)$$

$\therefore$ $d(x_n, x)\ \{ d(x_n, x_{n+1}) + d(x, y) \}$

$$\geq 2\alpha\, d(x_{n+1}, x_n)\ d(x, y)$$

$$+ \beta\, d(x_{n+1}, x_n)\ \{d(x_n, x_{n+1}) + d(x, y)\}$$

Since $d(x_n, x) \to 0$ as $n \to \infty$ we have

$$d(x, y) = 0 \text{ as } x_n \neq x_{n+1}.$$

Therefore $x = y$ so T has a fixed point.

This proves the result.

(4.6) The concept of 2-metric spaces has been investigated by S. Gahler in a series of papers [115], [116] and [117].

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[115] Gahler, S. (30)

[116] Gahler, S. (31)

[117 Gahler, S. (32)

On the other hand, many authors have studied the aspects fixed point theory for several types of contractive mappings in the setting of the 2-metric spaces.

We need the following definitions:

Definition 1. A-2 metric space is a space X in which, for each triple points x, y and z in X, there exists a real valued function d (x, y, z) such that

(4.6.1) to eachpair of distinct points x, y in X, there eists a point z in X satisfying $d(x, y, z) \neq 0$,

(4.6.2) $d(x, y, z) = 0$ when at least two of the points are equal,

(4.6.3) $d(x, y, z) = d(y, z, x) = d(x, z, y)$

(4.6.4) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all w in X.

It is easily seen that d is non-negative.

Definition 2. A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be convergent to a point x in X if $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all a in X.

Definition 3. A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be a cauchy sequence if $\lim_{m,n\to\infty} d(x_m, x_n, a) = 0$ for all a in X.

Definition 4. A 2-metric space (X, d) is said to be complete if every cauchy sequence in X is convergent.

Definition 5. Let S be a mapping from a 2- metric space (X, d) into itself. Then S is said to be sequentially continuous at every point x in X if for every sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all a in X,

$$\lim_{n\to\infty} d(Sx_n, Sx, a) = 0$$

Throughout this paper, let F denote the family of mapping such that for each $\phi \in F$, $\phi : [o, \infty] \to [o, \infty]$ is upper semi-continuous from the right and non-decreasing in each co-ordinate variable with $\phi(t) < t$ for all $t > 0$.

It is well known that if $\phi(t) = t$ for every $t > 0$, then $\lim_{n\to\infty} \phi^n(t) = 0$ where $\phi^n(t)$ denotes the composition of $\phi(t)$ with itself n-times.

Our aim in this article is to prove:

Theorem 5. Let f and g be surjective sequentially continuous mappings from a complete 2-metric space (X,d) into itself, Suppose that there exists $\phi \in F$ such that

(4.6.5) $\phi\,[d(fx, gy, a)] \geq \min\{d(x, fx, a), d(y, gy, a), d(x, gy, a), d(y, fx, a), d(x, y)\}$

for all x, y and a in X where $\sum_{n=0}^{\infty} \phi^n(t) < \infty$

for all $t > 0$.

Then f or g has a fixed point or S and T have a common fixed point.

Proof: Let $x_0 \in X$ be given. Since f and g are surjective there exists a point x_1 in X such that $fx_1 = x_0$. Also there exists a point x_2 in X such that $Tx_2 = x_1$. Inductively we can define a sequence $\{x_n\}$ in X such that

$$fx_{2n+1} = x_{2n}$$

and

$$gx_{2n+2} = x_{2n+1}$$

If $x_n = x_{n+1}$, for any n, then f or g has a fix point.

Now, suppose that $x_n \neq x_{n+1}$ for each n. From $\phi(t) < t$ for all $t > 0$, for all a in X, we have

$$\phi[d(x_{2n}, x_{2n+1}, a)] = \phi\,[d(fx_{2n+1}, gx_{2n+2}, a)]$$

$$\geq \min\{d(x_{2n+1}, fx_{2n+1}, a), d(x_{2n+2}, gx_{2n+2}, a), d(x_{2n+1}, gx_{2n+2}, a), d(x_{2n+2}, fx_{2n+1}, a), d(x_{2n+1}, x_{2n+2}, a)\}$$

$$= d(x_{2n+1}, x_{2n+2}, a)$$

Thus

$$d\ (x_{2n+1}, x_{2n+2}, a) \leq \emptyset\ [d(x_{2n}, x_{2n+1}, a]$$

for all a in X. Similarly

$$d\ (x_{2n}, x_{2n+1}, a) \leq \emptyset\ [d\ (x_{2n-1}, x_{2n}, a)]$$

for all a in X. For arbitrary n,

$$d\ (x_n, x_{n+1}, a) \leq \emptyset\ [d\ (x_{n-1}, x_n, a]$$

for all a in X.

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$$\leq \emptyset^n\ [d\ (x_o, x_1, a)]$$

for all a in X.

By the well known result cited above,

$$\lim_{n \to \infty} d(x_n, x_{n+1}, a) = 0 \text{ for all a in X.}$$

From $\emptyset(o) = 0$, we have for every non negative interg m,

$$d\ (x_o, x_1, x_m) \leq d\ (x_o, x_1, x_{m-1}) + d(x_m, x_{m-1}, x_o) + d\ (x_m, x_{m-1}, x_1)$$

$$\leq d[x_o, x_1, x_{m-1}) + \emptyset^{m-1} \{d(x_1, x_o, x_o) + d(x_1, x_o, x_1)\}$$

$$= d(x_o, x_1, x_{m-1})$$

$$\leq d(x_o, x_1, x_{m-2})$$

- - - - -

$$< d(x_o, x_1, x_1) = 0$$

Therefore we obtain $d(x_n, x_{n+1}, x_m) = 0$.

For arbitrary non-negative integer i, j and k

$(0 \leq i \leq j < k)$

$$d(x_i, x_j, x_k) \leq d(x_i, x_j, x_{i+1}) + d(x_i, x_{i+1}, x_k)$$

$$+ d(x_{i+1}, x_j, x_k)$$

$$= d(x_{i+1}, x_j, x_k)$$

- - - - - -

$$\leq d(x_{j-1}, x_j, x_k) = 0$$

For any $m < n$, for all a in X, we write

$$d(x_m, x_n, a) \leq d(x_m, x_n, x_{m+1}) + d(x_m, x_{m+1}, a)$$

$$+ d(x_{m+1}, x_n, a)$$

$$= d(x_m, x_{m+1}, a) + d(x_{m+1}, x_n, a)$$

$$\leq d(x_m, x_{m+1}, a) + d(x_{m+1}, x_{m+2}, a)$$

$$+ - - - + d(x_{n-1}, x_n, a)$$

$$= \phi^m[d(x_o, x_1, a)] + - - - + \phi^{n-1}[d(x_o, x_1, a)].$$

From $\sum_{n=o}^{\infty} \phi^n(t) < \infty$ for all $t > 0$, it follows that $\{x_n\}$ is a cauchy sequence and it converges to some point x in X.

Consequently $\{x_{2n}\}$, $\{x_{2n+1}\}$ and $\{x_{2n+2}\}$ converges to x.

By the sequential continuity of f and g,

$$fx_{2n+1} = x_{2n} \rightarrow fx$$

and $gx_{2n+2} = x_{2n+1} \rightarrow gx$ as $n \rightarrow \infty$,

Thus f and g have a common fixed point.

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