CHAPTER-III

FIXED POINT THEOREMS IN L-SPACE

3.1 Kasahara [102] has introduced L-space, then Yeh has given few fixed points theorem in L-space. It seems that notion of metric is not necessary in Banach contraction theorem and some of its generalization.

For other definitions we refer to Chapter II. Our Object have to take (with many modification) the conditions of Pal and Maiti [103], [104] to prove fixed point theorems in L-space.

In fact, we prove:

Theorem 1: Let $(M \rightarrow)$ be a separated L-space with d-completeness for a non negative extended real valued function d on MXM with $d(x, x) = 0$ for each x in M. Let T be continuous self mapping satisfying the conditions:

(3.1.1) $d(x,Tx) + d(y, T^2x) \leq \alpha\, d(x,y)\ , \quad 0 \leq \alpha < 2$

(3.1.2) $d(x, Tx) + d(y, T^2 x)$

$\leq \beta\{d(x, T^2x) + d(Ty, T^2x) + d(x, y)\}\ , \ \tfrac{1}{2} \leq \beta < 2/3$

(3.1.3) $d(x, Tx) + d(Tx, T^2x) + d(y, T^2x)$

$\leq \gamma\{d(x, T^2x) + d(Ty, T^2x)\},\ 1 \leq \gamma < 3/2$

[102] Kasahara, S. (49) [103] Pal,T.K.andMaiti,M.(61)
[104] Pal, T.K. and Maiti, M. (60)

(3.1.4) $d(Tx, Ty) \leq \delta \max \{d(x, y), d(x, Tx),$

$d(y, T^2x), \frac{1}{2}[d(x, T^2x) + d(Ty, T^2 x)]\}, 0 < \delta < 1,$

for all x, y in M with $d(x, y) < q$, $0 < q < \infty$ and

(3.1.5) there exists $b \in M$ such that $d(b, Tb) < q$.

Then T has a fixed point and the sequence $\{T^n b\}_{n \in N}$ covnerges to the fixed point of T.

Proof:

We define a sequence $\{x_n\}$ with $x_o = b$ and $x_{n+1} = T_{xn}$, $n = 1, 2, 3,$ -------

Let us define a sequence $\{c_n\}$ such that c_n = $cn = d(x_n, x_{n+1})$ where $x_n = T^n x$, $n = 0, 1, 2,$-------

Now suppose that (3.1.1) is true for the pair $x_n, x_{n+1} \in M$. Then

$$d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \leq \alpha\, d(x_n, x_{n+1})$$

or $$d(x_{n+1}, x_{n+2}) \leq (\alpha - 1)\, d(x_n, x_{n+1})$$

implying

(3.1.6) $$c_{n+1} \leq (\alpha - 1)c_n$$

Similarly if (3.1.2) is true,

$$d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$$
$$\leq \beta\{d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+2}) + d(x_n, x_{n+1})\}$$

$$\leqslant \beta \{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$$

$$+ d(x_{n+2}, x_{n+2}) + d(x_n, x_{n+1})\}$$

$$< 2\beta\, d(x_n, x_{n+1}) \quad + \beta\, d(x_{n+1}, x_{n+2})$$

or $(1-\beta)\ d\ (x_{n+1}, x_{n+2}) \leqslant (2\beta-1)\ d(x_n, x_{n+1})$

or $\quad d\ (x_{n+1}, x_{n+2}) \leqslant \dfrac{2\beta-1}{1-\beta}\ d\ (x_n, x_{n+1})$

(3.1.7) $\quad c_{n+1} \leqslant \dfrac{2\beta-1}{1-\beta}\ c_n$

also if (3.1.3) is true then

$$d\ (x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+1}, x_{n+2})$$

$$\leqslant \gamma \{\ d\ (x_n, x_{n+2}) \ + d\ (x_{n+2}, x_{n+2})\ \}$$

(3.1.8)

$$c_{n+1} \leqslant \frac{\gamma - 1}{2 - \gamma}\ c_n$$

Similarly if (3.1.4) is true, we get

(3.1.9) $\quad c_{n+1} \leqslant \delta\, c_n$

Collecting (3.16), (3.1.7), (3.1.8) and (3.1.9)

we have

(3.1.10) $\quad c_{n+1} \leqslant \lambda\, c_n$

for all n, where

$$\lambda = \max \{ \alpha -1, \frac{2\beta -1}{1-\beta} \quad \frac{\gamma -1}{2-\gamma}, \delta \} < 1$$

It is more simple matter to show that

$$c_n \to \text{as } n \to \infty$$

Since

$d(x_o, Tx_o) = d(b, Tb) < q$ we have in all four

cases $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$

Hence by d-completeness of M, the sequence $\{T^n b\}_{n \in N}$ converges to some $u \in M$, in each of the four cases. Then the continuity of T implies that there exists a subsequence $\{T^{ni} b\}_{i \in N}$ of $\{T^n b\}_{n \in N}$ such that

$T(T^{ni}b) \to Tu$, but for $T(T^{ni} b)_{i \in N}$ also being a subsequence of $\{T^n b\}_{n \in N}$, we have

$T(T^{ni}b)_{i \in N} \to u$ and hence

$Tu = u$, for each of the four cases.

This complete the proof of the theorem.

3.2 Now we shall prove a common fixed point theorem. In fact, we prove.

Theorem-2:

Let $(M \rightarrow)$ be a separated L-space with d-completeness for a non-negative extended real valued function d on MXM with $d(x, x) = 0$ for each x in M.

Let T_1, T_2 be continuous self mapping satisfying

(3.2.1)

$$d(x,T_1x) + d(y, T_2^2x) \leq \alpha\, d(x, y),\ 0 \leq \alpha < 2$$

(3.2.2)

$$d(x, T_1x) + d(y, T_1^2 x) \leq \beta \{ d(x, T_1^2x) + d(T_2y, T_2^2 x) + d(x, y)\},\ \tfrac{1}{2} < \beta < 2/3$$

(3.2.3)

$$d(x, T_1x)+d(T_1x,T_1^2 x) + d(y, T_2^2 x) \leq \gamma \{d(x, T_1^2 x) + d(T_2y, T_2^2x)\},\ 1 \leq \gamma < 3/2$$

(3.2.4)

$$d(T_1x, T_2y) \leq \delta \max\{d(x,y),d(x,T_1x), d(y,T_2^2x), \tfrac{1}{2}[d(x, T_1^2 x)+d(T_2y,T_2^2 x)]\},\ 0 < \delta < 1$$

for all x, y in M with $d(x, y) < q$, $0 < q < \infty$ and

(3.2.5) there exists $b \in M$ such that $d(b, Tb) < q$.

The T_1, T_2 have a common fixed point and the sequence $\{T_1^n, b\}_{n \in N}$ and $\{T_2^n\ b\}_{n \in N}$ converges to the common fixed point.

Proof:

Define a sequence $\{c_n\}$ such that

$c_n = d(x_{2n}, x_{2n+1})$, where

$x_1 = T_1 x_o, x_2 = T_2 x_1$

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$x_{2n-1} = T_1 (x_{2n-2})$

$x_{2n} = T_2 (x_{2n-1})$

Suppose (3.2.1) is true for all pairs $x_{2n}, x_{2n+1} \in M$.

Then $d(x_{2n}, T_1 x_{2n}) + d(x_{2n+1}, T_2^2 x_{2n})$

$\leqslant \alpha\, d(x_{2n}, x_{2n+1})$

Or $d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})$

$\leqslant \alpha\, d(x_{2n}, x_{2n+1})$

or $d(x_{2n+1}, x_{2n+2}) \leqslant (\alpha - 1)\, d(x_{2n}, x_{2n+1})$

implying that

$$c_{n+1} \leqslant (\alpha - 1)\, c_n$$

Similarly if (3.2.2), (3.2.3) and (3.2.4) are true then we obtain correspondingly

$$c_{n+1} \leqslant \frac{2\beta - 1}{1 - \beta} c_n$$

$$c_{n+1} \leq \frac{\gamma - 1}{2-\gamma} c_n$$

$$c_{n+1} \leq \delta c_n$$

Therefore we have

$$c_{n+1} \leq \lambda c_n$$

for all n, where

$$\lambda = \max \{ (\alpha - 1), \frac{2\beta - 1}{1-\beta}, \frac{\gamma - 1}{2-\gamma}, \delta \} < 1$$

Since $\lambda < 1$, therefore

$$\sum_{n=o}^{\infty} d(x_{2n}, x_{2n+1}) < \infty .$$

Hence by d-completeness of M, the sequence $\{T^{2n}b\}_{n \in N}$ converges to some $u \in M$, in each cases.

Then by usual argument $x_{2n} \to u$.

Now we shall show that u is a common fixed point of T_1 and T_2, i.e.

$$u = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} T_2 (x_{2n-1})$$

$$= T_2 \lim_{n\to\infty} x_{2n-1} = T_2 u$$

Similarly

$$u = T_1 u$$

Therefore u is a common fixed point of T_1 and T_2

Remark:

(1) The condition (3.1.4) of theorem 1, may be split into four conditions.

(3.2.6) $d(Tx, Ty) \leq q\, d(x, y)$, $0 \leq q < 1$

(3.2.7) $d(Tx, Ty) \leq r\, d(x, Tx)$, $0 \leq r < 1$

(3.2.8) $d(Tx, Ty) \leq s\, d(y, T^2x)$, $0 \leq S < 1$

(3.2.9) $d(Tx, Ty) \leq t\, [d(x, T^2x) + d(Ty, T^2x)]$ $0 \leq t < \frac{1}{2}$

(2) If only condition (3.1.4) is satisifed by T in theorem 1, then u is a unique fixed point of T.

In other case, however, if T further satisfies at least one of the condition.

(3.2.10) $d(x_o, Tx) < d(x, x_o) + d(x, Tx)$

(3.2.11) $d(x, x_o) < d(x_o, Tx) + d(x, Tx)$

for all $x \in X$ and $x \neq x_o$, then unicity of the fixed point is guaranted.

3.3 Now we shall prove

Theorem 3: Let $(M, \to)$ be a separated L- Space for a non-negative extended real valued function d on MxM with $d(x,x)=0$ for each x in M. Let T and T^2 are continuous at x_o, and T satisfies at least one of the following conditions.

(3.3.1) $d(x, Tx) + d(y, T^2x) \leq 2\, d(x, y)$

(3.3.2) $d(x, Tx) + d(y, T^2x)$

$$\leq 2/3\, \{d(x, T^2 x) + d(y, Tx) + d(x, y)\}$$

(3.3.3) $d(x, Tx) + d(y, T^2x) + d(Tx, Ty)$

$$\leq 3/2\, \{ d(x, T^2x) + d(y, Tx) \}$$

(3.3.4) $d(Tx, Ty) < \text{Max}\, \{d(x,y), d(x, Tx),$

$$d(y, T^2x), \tfrac{1}{2} [d(x, T^2x) + d(Ty,T^2x)]\}$$

for all x, y in M with $d(x, y) < v$, $0 < v < \infty$.

Then x_o is a fixed point of T.

Proof:

Setting c_n as in theorem 1, it possible to show that $c_{n+1} < c_n$ for all case (3.3.1) to (3.3.4).

Therefore the sequence $\{c_n\}$ is monotonic decreasing and bounded also. Then $c_n \to p$ as $n \to \infty$,

where $p = \inf \{c_n\}$.

Since x_o is a cluster point of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$, such that $x_{n_i} \to x_o$ as $i \to \infty$.

Also

$$x_{n_{i+1}} = Tx_{n_i} \to Tx_o$$

$$x_{n_{i+2}} = T^2 x_{n_i} \to T^2 x_o \quad \text{as} \quad i \to \infty$$

because T and T^2 are continuous at x_o, then we obtain:-

$$p = \lim_{i\to\infty} d\,(x_{n_i} , x_{n_{i+1}})$$

$$= \lim_{i\to\infty} d\,(x_{n_i} , Tx_{n_i})$$

$$= d\,(x_o , Tx_o)$$

Again

$$p = \lim_{i\to\infty} d\,(x_{n_{i+1}} , x_{n_{i+2}})$$

$$= \lim_{i\to\infty} d\,(x_{n_{i+1}} , T^2 x_{n_i})$$

$$= d\,(Tx_o, T^2 x_o)$$

Suppose that $x_o \neq Tx_o$ if (3.3.1) holds for the pair x_o, Tx_o; then

$$d(x_o, Tx_o) + d(Tx_o, T^2 x_o) < 2\, d\,(x_o, Tx_o)$$

or $d(Tx_o, T^2x_o) < d(x_o, Tx_o)$

which is impossible.

Hence

$$Tx_o = x_o$$

Similar results can be derived for (3.3.2) or (3.3.3) or (3.3.4) in place of (3.3.1)

Then x_o is a fixed point of T.

This complete the proof of the theorem.

3.4 Now we shall prove a theorem for fixed and periodic points for sets of operators.

In fact we prove:

<u>Theorem 4:</u> Let $(M, \rightarrow)$ be a separated L-space with d-completeness for a non-negative extended real valued function d on MXM with $d(x, x) = 0$ for each $x \in M$, Let Ti and Tj be two sets of operators mapping the L-space into itself and let there exists positive integers ρ and q such that for each pair x, y in M at least one of the following is true:

(3.4.1) $d(x, T_i^p x) + d(y, T_j^q y)$

$\leq \alpha\, d(x, y)$, $1 \leq \alpha < 2$

(3.4.2) $d(x, T_i^p x) + d(y, T_j^q y)$

$\leq \beta \{ d(x, T_j^q y) + d(y, T_i^p x)$

$+ d(x, y)\}$, $\frac{1}{2} \leq \beta < 2/3$

(3.4.3) $d(x, T_i^p x) + d(y, T_j^q y) + d(T_i^p x, T_j^q y)$

$\leq \gamma \{ d(x, T_j^q y) + d(y, T_i^p x)$, $1 < \gamma < 3/2$

(3.4.4) $d(T_i^p x, T_j^q y) \leq \delta \max \{ d(x,y), d(x, T^2 x),$

$d(y, T_j^q y), \frac{1}{2} [d(x, T_j^q y) + d(y, T_i^p x) \}$

$0 \leq \delta < 1$

for all x, y in M with $d(x, y) < r$, $0 < r < \infty$ and

(3.4.5) there exists $b \in M$ such that $d(b, Tb) < r$

Then Ti, Tj have common periodic point.

<u>Proof</u>: Let a sequence $\{x_n\}$ in X be defined as follows:

$x_{2n+1} = T_{2n+1}^p \; x_{2n}$

$x_{2n+2} = T_{2n+2}^q \; x_{2n+1}$, n = 0, 1, 2, ------

Then Ti and Tj have a common periodic point u for

i = 1, 3, ----, 2 n+1 , ----

j = 2, 4, --- 2 n+2, ----

Let $c_{2n} = d(x_{2n}, x_{2n+1})$.

Suppose x_{2n} , x_{2n-1} , Satisfy (3.4.1), then

$d(x_{2n}, x_{2n+1}) \; d(x_{2n+1}, x_{2n+2}) \leq \alpha \; d(x_{2n}, x_{2n+1})$

Therefore,

(3.4.6) $$c_{2n+1} \leq (\alpha -1)\, c_{2n}$$

Similarly, if x_{2n}, x_{2n+1} satisfy (3.4.2), (3.4.3) or (3.4.4) we get respectively

(3.4.7) $$c_{2n+1} \leq \frac{2\beta -1}{1-\beta}\, c_{2n}$$

(3.4.8) $$c_{2n+1} \leq \frac{\gamma - 1}{2-\gamma}\, c_{2n}$$

(3.4.9) $$c_{2n+1} \leq \delta\, c_{2n}$$

Combining (3.4.6), (3.4.7), (3.4.8), (3.4.9) we have

$$c_{2n+1} \leq \lambda\, c_{2n}$$

where

$$\lambda = \max \left\{ \alpha -1, \frac{2\beta - 1}{1-\beta}, \frac{\gamma -1}{2-\gamma}, \delta \right\} < 1$$

Repeating the above argument with x_{2n-1}, x_{2n} we obtain

$$c_{2n} \leq \lambda\, c_{2n-1}$$

Thus

$$\sum_{n=o}^{\infty} d(x_n, x_{n+1}) < \infty$$

and by usual argument $\{x_n\}$ converges to a point u in X.

We wish to show that

$$T_i^p u = T_j^q u = u$$

Suppose (3.4.1) holds. Then we have

$$d(u, T_i^p x) + d(x_{2n+1}, T_{2n+1}^p x_{2n+1})$$

$$\leq \alpha\, d(x_{2n+1}, u)$$

Proceeding to the limit $n \to \infty$, we get

$$u = T_i^p u.$$

Similar conclusion follows from other inequalities i.e. (3.4.2), (3.4.3) and (3.4.4)

Thus

$$u = T_i^p u$$

Similarly one can show that

$$T_j^q u = u$$

Thus u is a common periodic point of T_i and T_j.

<u>Remark</u>: If in addition to (3.4.1) to (3.4.4), one has

(3.4.10) $d(u, T_i^p x) < d(x, u) + d(x, T_i^p x)$,

for $x \in X$ and

(3.4.11) $d(u, T_j^q x) < d(y, u) + d(y, T_j^q y)$,

for $y \in X$.

Then u is a unique common fixed point for T_i and T_j.

Proof:

Suppose (3.4.10) is satisfied and assume $v \neq u$ is also a periodic point of T_i of period p. Then

$$d(u, v) < d(u, v) + d(u, T_i^p v) = d(u, v),$$

a contradiction

Therefore u = v and u is unique. Now $u = T_i^p(u)$ implies

$$T_i^p(T_i u) = Ti\ (T_i^p u) = T_i u$$

But u is the unique periodic point of T_i

and so $T_i u = u$

Similarly from (3.4.1) $T_j u = u$

Then u is the unique fixed point of T_i and T_j.

This completes the proof of the theorem.

3.5 Now we shall prove a theorem through rational expression which generalizes the result of Ciric[105].

In fact we prove:

Theorem-5: Let (M $\rightarrow$) be a separated L- space with d-completeness for non-negative extended real valued function d on MXM with d (x, x) = 0 for each x in M. Let

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[105] Ciric, L.B. (12)

T be continuous and one of the following condition is satisfied.

(3.5.1)

$$d(x, Tx) + d(y, T^2x) \leq \alpha \left[d(x, y) + \frac{d(y,Tx)[1+d(x,Tx)+d(y,Tx)]}{1 + d(x, y)} \right]$$

$$1 \leq \alpha < 2.$$

(3.5.2)

$$d(x, Tx) + d(y, T^2 x)$$

$$\leq \beta \left[d(x,y) + \frac{d(x, T^2x)\,[1+d(x,Tx)+d(y,Tx)]}{1 + d(x, y)} \right]$$

$$\tfrac{1}{2} < \beta < 2/3$$

(3.5.3)

$$d(x, Tx) + d(Tx, T^2x) + d(y, T^2x)$$

$$\leq \gamma \left[d(x, y) + \frac{d(y,Ty)[1+d(x,Tx) + d(y,Tx)]}{1 + d(x, y)} \right]$$

$$1 \leq \gamma < 3/2$$

(3.5.4)

$$d\,(Tx, Ty) \leq \delta \max \{ d(x, y), d(x, Tx), d(y, T^2x), \tfrac{1}{2} \{ d\,(x, T^2x) + d\,(Ty, T^2x)\}$$

$$0 < \delta < 1$$

for all x, y in M with $d\,(x, y) < q$, $0 < q < \infty$ and

(3.3.5) there exists $b \in M$ such that $d\,(b, Tb) < q$.

Then T has a fixed point and the sequence $\{T^n b\}_{n \in N}$ converges to the fixed point of T.

Proof:

Define a sequence $\{c_n\}$ such that

$c_n = d\,(x_n, x_{n+1})$

where $x_n = T^n x\ (x = x_o)$, n = 0, 1, 2, ----

Now suppose that (3.5.1) is true for the pair x_n, x_{n+1}, then

$$d\,(x_n, x_{n+1}) + d\,(x_{n+1}, x_{n+2})$$

$$\leq \alpha\ [d\,(x_n, x_{n+1}) + \frac{d(x_{n+1}, x_{n+1})\ [1+d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+1})]}{1 + d\,(x_n, x_{n+1})}$$

$$= \alpha\ d\,(x_n, x_{n+1})$$

which implies

(3.5.6)

$$c_{n+1} \leq (\alpha - 1)\ c_n$$

Similarly if (3.5.2), (3.5.3) and (3.5.4) are true, then we obtain correspondingly

$$(3.5.7) \quad c_{n+1} \leq \left(\frac{2\beta - 1}{1-\beta}\right) c_n$$

$$(3.5.8) \quad c_{n+1} \leq \left(\frac{\gamma - 1}{2 - \gamma}\right) c_n$$

$$(3.5.9) \quad c_{n+1} \leq \delta\ c_n$$

From (3.5.6) to (3.5.9), we get

$$(3.5.10) \quad c_{n+1} < q\ c_n$$

for all n, where

$$q = \max \left\{ \alpha - 1, \frac{2\beta - 1}{1-\beta}, \frac{\gamma - 1}{2-\gamma}, \delta \right\} < 1$$

By the routine calculation we have

$$\sum_{n=o}^{\infty} d\,(x_n, x_{n+1}) < \infty .$$

Hence by d-completeness of M, the sequence $\{T^n b)_{n \in N}$ converges to some $u \in M$, in each of the above four cases.

The rest follows from theorem 1.

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