CHAPTER-II

ON NON UNIQUE FIXED POINTS IN L-SPACE

(2.1) Kasahara[96] has introduced L-space, then Yeh [97] has given few fixed point theorems in L-space. It seems that the notion of metric is not necessary in the Banach contraction theorem [98] and some of its generalization.

Denote by N the set of all non-negative integers. A pair $(x, \rightarrow)$ of a set X and a subset $\rightarrow$ of the set $X^N \times X$ is called an L- space if the following conditions are satisfied:

(2.1.1) If $x_n = x$ in X for all n in N, then $(\{x_n\}_{n \in N}, x) \in \rightarrow$

(2.1.2) If $(\{Xn\}_{n \in N}, x) \in \rightarrow$, then $(\{x_{n_i}\}_{i \in N},$

for every subsequence $\{X_{n_i}\}_{i \in N}$ of $\{x_n\}_{n \in N}$,

In what follows, we shall write $\{x_n\}_{n \in N} \rightarrow x$ or $x_n \rightarrow x$ instead of $(\{x_n\}_{n \in N}, x)$ and read $\{x_n\}_{n \in N}$ convergers to x.

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[96] Kasahara, S. (49)

[97] Yeh, C.C. (118)

[98] Banach, S. (2)

Let $(X, \rightarrow)$ be an L-space. It is said to be separated.

If each sequence in X converges to at most one point of X. A mapping f of X into an L-space $(X', \rightarrow')$ is said to be continuous if $x_n \rightarrow x$ implies $f(x_{n_i}) \rightarrow' f(x)$ for some subsequnece $\{x_{n_i}\}_{i \in N}$ of $\{x_n\}_{n \in N}$. By the product space of $(X, \rightarrow)$ and $(X', \rightarrow')$ we mean the L-space $(X \times X', \rightarrow")$ where $\rightarrow"$ is defined as $(x_n, y_n) \rightarrow" (x, y)$ if and only if $x_n \rightarrow x$ and $y_n \rightarrow y$. Let d be a non-negative extended real valued function on X x X such that

$0 < d(x, y) < \infty$ for all $x, y \in X$. The L-space $(X, \rightarrow)$ is said to be d - complete if each sequence $\{x_n\}_{n \in N}$ in X with $\sum_{n=o}^{\infty} d(x_{n+1}, x_n) < \infty$ converges to at most one point of X.

We know [Kasahara, same]

Lemma K. Let $(X, \rightarrow)$ be an L- space which is d- complete for a non negative extended real valued function.

d on X x X. If $(X, \rightarrow)$ is separated, Then $d(x, y) = d(y, x) = 0$ implies $x = y$ for every x, y in X.

Ciric [99] has obtained the following result regarding the existence of non unique fixed points for orbitally continuous self mapping on a orbitally complete metric space.

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[99] Ciric, L.B. (12)

He proved the following:

Theorem C. Let (M, d) be orbitally complete metric space and T be a orbitally continuous self mapping of X, satisfying

(2.1.3) $\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\}$

$- \min\{d(x, Ty), d(y, Tx)\} \leq q\, d(x, y)$

for all x, y in M and some q (0, 1). Then for each x in M the sequence $\{T^n x\}$ converges to a fixed point of T.

Taskovic [100] proved some fixed point theorem when the mapping T of a metric space (X, d) satisfies.

(2.1.4) $a_1\, d(Tx, Ty) \pm a_2\, d(x, Tx) + a_3\, d(y, Ty)$

$q \min\{d(x, Ty), d(y, Tx)\} \leq q\, d(x, y)$

for each x, y in X where a_i $(i = 1,2,3)$ and q are real numbers, $a_1 + a_2 + a_3 > P$ and $P - a_2 \geq 0$

2.2 In what follows we shall first prove:

Theorem 1. Let $(M, \rightarrow)$ be a separated L- space which is d- complete for a non negative real valued function d on $M \times M$ with $d(x, x) = 0$ for each x in M and T be continuous self-mapping satisfying,

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[100] Taskovic, M.R. (112)

(2.2.1) $\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\}$

$+ a \min\{d(x, Ty), d(y, Tx)\}$

$< a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, Ty)$

$+ a_4 d(y, Tx) + a_5 d(x, T^2x) + a_6 d(y, T^2x)$

$+ a_7 d(Tx, T^2x) + a_8 d(Ty, T^2x) + a_9 d(x,y)$

for all x, y in M and a and a_i are real numbers such that

(2.2.2) $0 < a_i < 1$ and $\sum_{1}^{9} a_i < 1.$

(2.2.3) $a_1 + a_2 + 2a_3 + 2a_5 + a_6 + a_7 + a_9 < 1$

(2.2.4) there exists b M such that $d(b, Tb) < q$,

$d(x, y) < q$, $0 < q <$

Then T has a fixed point and the sequence $\{T^n b\}_{n \in N}$ converges to a fixed point.

Proof. Let x in M be arbitrary, then we define a sequence $\{x_n\}$ by

(2.2.5) $x_o = x, x_1 = Tx_o, x_2 = Tx_1 \ldots, x_n = Tx_{n-1}$

If for some n in N, $x_n = x_{n+1}$, then $\{x_n\}$ is a Cauchy sequence and the limits of $\{x_n\}$ is a fixed point of T. Suppose that $x_n \neq x_{n+1}$, for each $n \neq 0, 1, 2, $-----

Then for $x = x_{n-1}$ and $y = x_n$, by (2.2.1) we have

$\min\{d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$

$+ a \min\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}$

$$\leq a_1 d(x_{n-1}, x_n) + a_2 d(x_n, x_{n+1}) \pm$$

$$+ a_3 d(x_{n-1}, x_{n+1}) + a_4 d(x_n, x_n)$$

$$+ a_5 d(x_{n-1}, x_{n+1}) \quad a_6 d(x_n, x_{n+1})$$

$$+ a_7 d(x_n, x_{n+1}) + a_8 d(x_{n+1}, x_{n+1})$$

$$+ a_9 d(x_{n-1}, x_n)$$

i.e. $\text{Min} \{ d(x_n, x_{n+1}), \; d(x_{n-1}, x_n)\}$

$$\leq (a_1 + a_3 + a_5 + a_9) \; d(x_{n-1}, x_n)$$

$$+ (a_2 + a_3 + a_5 + a_6 + a_7) \; d(x_n, x_{n+1})$$

or $$d(x_n, x_{n+1}) \leq \frac{a_1 + a_3 + a_5 + a_9}{1-a_2-a_3-a_5-a_6-a_7} \; d(x_{n-1}, x_n)$$

Thus we have by induction

(2.2.6)

$$d(x_n, x_{n+1}) \leq \left(\frac{a_1 + a_3 + a_5 + a_9}{1-a_2-a_3-a_5-a_6-a_7} \right)^n d(b, Tb)$$

for every n in N and so we have $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$

Hence the d- completeness of M implies that the sequence $\{T^n b\}_{n \in N}$ converges to some u in M. So by the continuity of T .

There is a subsequence $\{T^{n_i} b\}_{i \in N}$ of $\{T^n b\}_{n \in N}$ such that $T(T^{n_i} b) \rightarrow Tu$.

But then since $\{T(T^{n_i} b)\}_{i \in N}$ is a subsequence of $(T^n b)_{n \in N}$ we have

$$T\ (T^{ni}b) \rightarrow u$$

Therefore $Tu = u$. This completes our proof. Now we shall prove:

Theorem 2. Let $(M \rightarrow)$ be a L- space which is d- complete for a continuous non-negative extended real function d on M X M such that.

(2.2.7) $d\ (x, y) = 0$ implies $x = y$,

(2.2.8) $d\ (x, x) = 0$ for each x in M.

If T is continuous self mapping of M satisfying (2.2.1), (2.2.2) and (2.2.3), (2.2.4) then T has a fixed point.

Proof: As in the proof of theorem 1, we can prove that (2.2.6) holds and the sequence $\{T^n\ b\}_{n \in N}$ converges to some u in M and that $T\ (T^{n(k)}b) \rightarrow Tu$ for some subsequence $\{T^{n(k)}b\}$ of the sequence $\{T^n\ b\}$.

Therefore the continuity of T implies

$$d\ (T\ (T^{n(ki)}\ b)) \rightarrow d\ (Tu, u)$$

for some subsequence $\{T^{n(ki)}\ b\}_{i \in N}$ of $\{T^{n(k)}b\}_{k \in N}$

Hence by theorem 1,

$$d\ (T\ (T^{n(ki)}\ b), T^{n(ki)}b) \rightarrow 0$$

Hence $d\ (u, Tu) = 0$ and Thus $Tu = u$.

2.3 Further we shall prove.

Theorem 3:

Let (M, →) be a separated L- space which is d-complete for a non-negative real valued function d on MXM with d (x, x) = 0 for each x in M and T be continuous selfmapping satisfying:

(2.3.1)

$$\min \{(d(Tx, Ty))^2, d(x, y) d(Tx, Ty), (d(y, Ty))^2\}$$

$$+ a \min \{ d(x, Tx) d(y, Ty), d(x, Ty) d(y, Tx) \}$$

$$< a_1 d(x, Tx) d(y, Ty) + a_2 d(x,Ty) d(y, Tx)$$

$$+ a_3 d(y, T^2x) d(x, y) + a_4 d(x, T^2 x)$$

$$d(Ty\ T^2x) + a_5 d(Tx, T^2x) d(x, y)$$

for all x, y in M and a and a_i are real numbers such that

(2.3.2) $0 < \sum_{1}^{5} a_i < 1$

and

(2.2.4) holds, then T has a fixed point and the sequence $\{T^n, b\}_{n \in N}$ converges to a fixed point.

Proof:

Let x in M be arbitrary, we define a sequence

$x_o = x$, $x_1 = Tx_o$, $x_2 = Tx_1$ ---- $x_n = Tx_{n-1}$

Now if for some n in N , $x_n = x_{n+1}$. Then $\{x_n\}$ is a cauchy sequence and the limit of $\{x_n\}$ is a fixed point of T. Suppose that $x_n \neq x_{n+1}$ for each n = 0, 1,2,-------

Now from (2.3.1) , for $x = x_{n-1}$ and $y = x_n$ we have

$$\min \{d(x_n, x_{n+1}))^2, d(x_{n-1}, x_n) d(x_n, x_{n+1}), (d(x_n, x_{n+1}))^2\}$$

$$+ a \min \{d(x_{n+1}, x_n) d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}) d(x_n, x_n)\}$$

$$\leqslant a_1 d(x_{n-1}, x_n) d(x_n, x_{n+1})$$

$$+ a_2 d(x_{n-1} x_{n+1}) d(x_n, x_n)$$

$$+ a_3 d(x_n, x_{n+1}) d(x_{n-1}, xn)$$

$$+ a_4 d(x_{n-1}, x_{n+1}) d(x_{n+1}, x_{n+1})$$

$$+ a_5 d(x_n, x_{n+1}) d(x_{n-1}, x_n)$$

i.e. $\min \{(d(x_n, x_{n+1})^2, d(x_{n-1}, x_n) d(x_n, x_{n+1})\}$

$$\leqslant (a_1 + a_3 + a_5) d(x_{n-1} x_n) d(x_n, x_{n+1})$$

Since $d(x_{n-1} x_n) d(x_n, x_{n+1})$

$$\leqslant (a_1 + a_3 + a_5) d(x_{n-1}, x_n) d(x_n, x_{n+1})$$

is impossible as $a_1 + a_3 + a_5 < 1$

Thus

$$d(x_n, x_{n+1}) \leqslant (a_1 + a_3 + a_5) d(x_{n-1}, x_n)$$

by induction

$$d(x_n, x_{n+1}) \leqslant (a_1 + a_3 + a_5)^n d(b, Tb)$$

for every n in N and so we have $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$. The rest of the proof follows from theorem 1.

2.4 Still further we shall prove:

Theorem 4. let (M →) be a separated L- space which is d-complete for a non-negative real valued function d on MXM with d (x, x) = 0 for each x in M and T be continuous selfmapping satisfying

(2.4.1) $$\min \{(d(Tx, Ty))^3, d(x, y)\, d(x, Tx)\, d(y, Ty), (d(y, Ty))^3\}$$
$$+ \min \{(d(x, Ty))^3, (d(y, Tx))^3\}$$
$$\leq \alpha\, d(x, y)\, d(x, Tx)\, d(y, Ty)$$
$$+ \beta\, d(x, y)\, d(x, Tx)\, d(y, T^2 x)$$
$$+ \gamma\, d(x, y)\, d(x, Tx)\, d(Tx, T^2 x)$$
$$+ \delta\, d(x, y)\, d(x, Tx)\, d(Ty, T^2 x)$$

for all x, y in M and $\alpha + \beta + \gamma < 1$ and (2.2.4) holds. Then T has a fixed point and the sequence $\{T^n b\}_{n \in N}$ converges to a fixed point.

Proof: Let x in M be arbitrary, then we define a sequence $\{x_n\}$ by

(2.4.2) $x_o = x,\ x_1 = Tx_o,\ x_2 = Tx_1,\ ---,\ x_n = Tx_{n-1}.$

If for some n, $x_n = x_{n+1}$, then $\{x_n\}$ is a cauchy sequence and the limit of $\{x_n\}$ is a fixed point of T. Suppose that $x_n \neq x_{n+1}$ for each

$n = 0, 1, 2$ -------

By using the inequality (2.4.1) for $x = x_{n-1}$ and $y = x_n$, we have

$$\min \{(d(x_n, x_{n+1}))^3, d(x_{n-1}, x_n)\ d(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}),\ (d(x_n, x_{n+1}))^3\}$$

$$+ \min \{(d(x_{n-1}, x_{n+1}))^3, (d(x_n, x_n))^3\}$$

$$\leq (\alpha + \beta + \gamma)\ d(x_{n-1}, x_n)\ d(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1})$$

i.e. $$\min\{(d(x_n, x_{n+1}))^3, (d(x_{n-1}, x_n))^2$$

$$d(x_n, x_{n+1})\}$$

$$\leq (\alpha + \beta + \gamma)\ \{(d(x_{n-1}, x_n))^2\ \ d(x_n, x_{n+1})\}$$

Since $\alpha + \beta + \gamma < 1$

$$(d(x_{n-1}, x_n))^2\ d(x_n, x_{n+1})$$

$$\leq (\alpha + \beta + \gamma)\ (d(x_{n-1}, xn))^2\ d(x_n, x_{n+1})$$

is impossible and we have

$$d(x_n, x_{n+1}) \leq (\alpha + \beta + \gamma)^{\frac{1}{2}}\ d(x_{n-1}, x_n)$$

Proceeding in this manner.

$$d\ (x_n,\ x_{n+1})\ =\ (\alpha + \beta + \gamma)^{n/2}\ d\ (b,\ Tb)$$

for every n in N and so we have

$$\sum_{n=0}^{\infty}\ d\ (x_n,\ x_{n+1})\ <\ \infty .$$

Hence the d- completeness of M implies that the sequence $\{T^n b\}_{n \in N}$ converges to some u in M. The rest of the proof follows from theorem 1.

2.5 In this section we state and prove theorem on the common fixed points of self mappings of a d-complete L-space.

<u>Theorem 5</u>

Let f and g be continuous mapping of a d-complete L- space X into itself and d (x, x) = 0 also d (x, y) = d (y, x) for every x, y X. If f and g satisfies-

(2.5.1) $$\min\ \{[d\ (fx,\ gy)]^3,\ d\ (x,\ y)\ [d(x,\ fx)\ d\ (y,\ gy)]\ [d(y,\ gy)]^3\}$$

$$-\ \min\ \{[d(x,\ gy)]^3\ ,\ [d\ (y,\ fx)]^3\}$$

$$\leq\ \alpha\ d(x,y)\ d\ (x,\ fx)\ \ d\ (y,\ gy)$$

for all x, $y \in X$ and $\alpha \in (0,\ 1)$. Then f and g have a comon fixed point.

<u>Proof</u>: Let $x_o \in X$ be arbitrary, we define a sequence of elements $\{x_n\}$ in X as follows

$$x_{2n+1} = f(x_{2n})$$

$$x_{2n+2} = g(x_{2n+1}), \qquad n = 0, 1, 2 ----$$

Where all x_n are distinct, then from (2.5.1) we write.

$$\min \{[d(fx_{2n}, gx_{2n+1})]^3, d(x_{2n}, x_{2n+1}) [d(x_{2n}, fx_{2n})$$

$$d(x_{2n+1}, gx_{2n+1})], [d(x_{2n+1}, gx_{2n+1})]^3\}$$

$$-\min \{[d(x_{2n}, gx_{2n+1})]^3, [d(x_{2n+1}, fx_{2n})]^3\}$$

$$\leq \alpha\, d(x_{2n}, x_{2n+1})\, d(x_{2n}, fx_{2n})\, d(x_{2n+1}, gx_{2n+1})$$

or

$$\min \{[d(x_{2n+1}, x_{2n+2})]^3, d(x_{2n}, x_{2n+1}) [d(x_{2n}, x_{2n+1}).$$

$$d(x_{2n+1}, x_{2n+2})], [d(x_{2n+}, x_{2n+2})]^3\}$$

$$\leq \alpha\, d(x_{2n}, x_{2n+1})\, d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})$$

Since $\alpha < 1$,

$$[d(x_{2n}, x_{2n+1})]^2\, d(x_{2n+1}, x_{2n+2})$$

$$\leq \alpha\, d\, [d(x_{2n}, x_{2n+1})]^2\, d\, [x_{2n+1}, x_{2n+2}]$$

is impossible, thus we have

(2.5.2)

$$d(x_{2n+1}, x_{2n+2}) \leq \sqrt{\alpha}\, d(x_{2n}, x_{2n+1})$$

Similarly we have

(2.5.3) $d\ (x_{2n}, x_{2n+1}) \leq \sqrt{\alpha}\ d(x_{2n-1}, x_{2n})$

From (2.5.2) and (2.5.3) we have,

$$d\ (x_n, x_{n+1}) \leq \alpha^{n/2}\ d\ (x_o , x_1)$$

Hence $\Sigma\ d\ (x_n, x_{n+1}) < \infty$

By the d-completeness of X, $\{x_n\}$ converges to some point of X i.e. lim x_n = u, Since X is an L-space, a subsequence $\{x_{2n}\}$ also converges to u.

By the continuity of f, we have lim f (x_{2n}) = f(u).

Again $\{f(x_{2n})\}$ is a subsequnece of $\{x_n\}$,

Hence lim $f(x_{2n})$ = u

By the uniqueness of limit, we have f(u) = u,

Similarly g(u) = u

Therefore u is a fixed point of f and g.

Now we shall prove fixed point theorem in L-space involving four points of the space. In fact we prove.

<u>Theorem 6:</u>

Let f be continuous mapping of a d-complete L-space X into itself and d (x, x) = 0 also d(x,y) = d(y,x) for every x, y $\in$ X. If f satisfies

(2.5.4)

$$a_1\, d\,(fu_1,\, fu_2)\quad d(fu_3,\, fu_4)$$

$$+\, a_2\, d\,(u_1,\, u_2)\quad d\,(fu_1,\, fu_2)$$

$$+\, a_3\,[\, d\,(u_1,\, fu_3)]^2$$

$$+\quad \alpha\ \min\,\{d\,(u_1,\, fu_3)\quad d\,(u_2,\, fu_4),$$
$$d\,(u_1,\, fu_4)\quad d\,(u_2,\, fu_3)$$

$$\leq\quad \beta\quad d\,(u_1,\, fu_3)\quad d\,(u_2,\, fu_4)$$

for all $u_1, u_2, u_3, u_4 \in X$, a_i $(i=1,2,3)$; α and β are real numbers such that

$$a_1 + a_2 + a_3 > \beta\ ,\quad \beta - a_3 \geq 0,\quad \beta \in (0,\ 1)$$

Then f has a fixed point.

<u>Proof</u>: Let $x, y \in X$. Define $u_1 = fy$, $u_2 = fx$, $u_3 = x$, $u_4 = y$ then from (2.5.4) we obtain

$$a_1 d(f^2y,\, f^2x)\quad d(fx,\, fy) + a_2\, d(fy,\, fx)\, d(f^2y,\, f^2x)$$

$$+a_3\,[d\,(fy,\, fx)]^2$$

$$+\ \alpha\ \min\,\{d(fy,\, fx)\ d\,(fx,\, fy),\ d\,(fy,\, fy)\ d\,(fx,\, fx)\}$$

$$\leq \beta\, d\,(fy,\, fx)\quad d\,(fx,\, fy)$$

i.e.

(2.5.5) $$a_1\, d\,(f^2x,\, f^2y)\ d\,(fx,\, fy) + a_2\, d\,(fy,\, fx)\ d(f^2y, f^2x)$$

$$+a_3[d(fx,fy)]^2 \leq \beta\, d\,(fy,\, fx)\quad d\,(fx,\, fy)$$

Let $x \in X$ be arbitrary, we define a sequence $\{x_n\}$ by (2.5.6) $x_o = x_1$ $x^1 = fx_o$, $x_2 = fx_1$, $-- x_n = fx_{n-1}$

If for some n, $x_n = x_{n+1}$, then $\{x_n\}$ is a chauchy sequence and the limit of $\{x_n\}$ is a fixed point of f. Suppose $x_n \neq x_{n+1}$ for each $n_O = 0, 1, 2,$ by (2.5.5) for $x = x_{n-1}$, $y = x_{n-2}$, we have

$$a_1 \, d\,(x_n, x_{n+1})\, d\,(x_n, x_{n-1}) + a_2 \, d\,(x_{n-1}, x_n)\, d(x_n, x_{n+1})$$

$$+ a_3 \;[d(x_n, x_{n-1})]^2$$

$$\leq \beta \quad d\,(x_{n-1}\, x_n) \quad d\,(x_{n-1}\, x_n)$$

i.e. $$d\,(x_n, x_{n+1}) \leq \frac{\beta - a_3}{a_1 + a_2}\, d\,(x_{n-1}, x_n)$$

Proceeding in this manner, we get

$$d\,(x_n, x_{n+1}) \leq \left(\frac{\beta - a_3}{a_1 + a_2}\right)^n d\,(x, fx)$$

Hence $\Sigma\, d\,(x_n, x_{n+1}) < \infty$. By d-completeness of X, $\{x_n\}$ converges to some point of X, i.e. $\lim x_n = u$. Since X is an L-space, a subsequence $\{x_{2n}\}$ also covnerges to u. By the continuity of f, we have

$$f\,(x_{2n}) = f(u),$$

Again $\{f(x_{2n})\}$ is a subsequence of $\{x_n\}$

Hence $\lim f(x_{2n}) = u$. By the uniqueness of limit we have $f(u) = u$.

This complete the proof.

Remarks:

(1) Theorem 1 generalizes the result of Taskovic (which is for metric space) to L-space and we have used nine mappings in place of five mappings used by Taskovic. Theorem 3 also generalize the result of Pachpatte (metric-space) to L-space and also using nine mapping in place of five used by Pachpette[101]. Theorem 4 is more general result proved up till now, where product of three functions is used for the first time inL-space. In theorem 5 we have used common fixed point for mapping of type used in theorem. In theorem 6, we extend the result of Pachpatte for four points in L-space.

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[101] Pachpatte, B.G. (69)