CHAPTER-I

INTRODUCTION

1.1 A topological space X is said to have a fixed point property, if every continuous function $f: X \to X$ has a point in the sense that $f(x) = x$ for some $x \in X$. Under such condition it may happen that f admits one point or more points. L, E, J. Brower in 1912 gave the first result on fixed points. This fundamental result was later extended by Schauder[1] to compact covex sets in Banach space, then he[2] relaxed the stringent condition of compactness also in 1930.

Mean while, a Polish mathematician Banach[3] proved in 1922:

If f is a mapping of a complete metric space (x, d) into itself satisfying,

(1.1.1) $d(fx, fy) \leq \alpha\, d(x, y)$

for all x, y in X and $0 \leq \alpha < 1$, then f has a unique fixed point.

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[1] Schauder, J. (86)

[2] Schauder, J. (87)

[3] Banach, S. (2)

A mapping f satisfying (1.1.1) is known as contraction mapping. It is to be remarked that any such contraction is obviously continuous but the converse is not necessarily a contraction. Fixed point theorems have extensive applications in proving the existence and uniqueness of the solution of differential equaltions, integral equations, random differential equations, space mechanics and statistics etc, see Smart [4], Kolmogorov and Fomin [5] and Swaminathan [6]. In 1965 Chu and Diaz [7] has shown that even if f is not contraction, it is possible that f^n, the nth iterate of f, is a contraction, n being positive integer. Sehgal[8] generalized above result, which was generalized by Holmes [9] in 1969.

For the further generalization along this type

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[4] Smart, D.R. (110)

[5] Kolomgorov, A.N. and Fomin,S.J. (58)

[6] Swaminathan, S. (111)

[7] Chu, S.C. and Diaz, J.B. (9)

[8] Sehgal, V.M. (97)

[9[Holmes, R.D. (36)

of work we refer Khazanchi [10], Iseki [11], Guseman [12], Ciric [13], Sharma [14] and Rhoades [15].

Went, in 1969, Meir and Keeler [16] first defined a mapping known as weakly uniformly strict contraction, Let $f:X \to X$, if for given $\epsilon > 0$, there exists a $\delta = \delta(t) > 0$ such that

(1.1.2) $\epsilon < d(x,y) < \epsilon + \delta \Rightarrow d(fx, fy) < \epsilon$

then f is called a weakly uniformly strict contraction mappings.

Further Maiti and Pal [17] obtain further generalization of Meir and Keeler [18] and of Boyd and Wong[19]. Further Maiti, Pal and Achari[20] obtained more general results along these lines.

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[10]	Khazanchi, L	(56)
[11]	Iseki, K.	(38)
[12]	Guseman, L.F.	(34)
[13]	Ciric, L.B.	(11)
[14]	Sharma, A.K.	(99)
[15]	Rhoades, B.E.	(81)
[16]	Meir, A. and Keeler, A.	(65)
[17]	Maiti, M. and Pal, T.K.	(60)
[18]	Meir, A. and Keelr, A.	(65)
[19]	Boyd, D.W. and Wong J.S.W.	(4)
[20]	Maiti, M. and Achari, J. & Pal T.K.(62)	

We also refer Chung [21], Matkowski- Wegrzyk [22], Park [23], Park- Bac [24], Park-moon [25], Park-Rhoades [26], Rao-Rao [27], Shisheng [28], Singh [29], Singh- virendra [30].

In 1942, Karl Menger [31] generalized the metric axioms by associating a distributive function with each pair of points of an abstract set X. (A distributive function is a mapping $F:R \to R^+$ which is non-decreasing, left continuous, with if F = 0 and sup F = 1). Thus for any pair of points u, v of x, we associate a distributive function denoted by Fu, v (or F(u, v)) and for any positive number x, Fu, v(x) or F(u, v:x) as the probability that the distance between u and v is less than x. This new theory of probabilistic metric space

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[21]	Chung, K.J.	(10)
[22]	Matkowski, J. and Wegrzyk, R.	(64)
[23]	Park, S.	(70)
[24]	Park, S. and Bac, J.S.	(72)
[25]	Park, S. and Moon, <.P.	(73)
[26]	park, S. and Rhoades, B.E.	(74)
[27]	Rao, I.H.N. and Rao, K.P.R.	(77)
[28]	Shisheng, Z	(102)
[29]	Singh, S.L.	(103)
[30]	Singh, S.L. and Virendra	(106)
[31]	Menger, K.	(66)

started to develope after the paper of Schweizer and Sklar [32]. For basic work in this direction we refer [33], [34], [35], [36], [37-41], [42] and [43].

In chapter VI namely "Fixed points of Meir-Keeler type contractive mappings on Menger space" we use three mappings f, g, h and use mild continuity condition and prove two theorems.

(The details are discussed in the respective chapter)

1.2 In a paper Gahler [44] investigated the notion of 2-metric, a real valued function for a point triple and on a set X, whose abstract properties, were suggested

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[32] Schweizer, B. and Sklar (90)

[33] Constantin, G. and Istrateseu, I (15)

[34] Schweizer, B. (88)

[35] Schweizer, B. (89)

[36] Schweizer, B. (96)

[37] Schweizer, B. and Sklar, A. (91)

[38] Schweizer, B. and Sklar, A. (92)

[39] Schweizer, B. and Sklar, A. (93)

[40] Schweizer, B and Sklar, A. (94)

[41] Schweizer, B. and Sklar, A & Thorp, E. (95)

[42] Sherwood, H. (100)

[43] Sherwood, H. (101)

[44] Gahler, S. (30)

by the area function for a triangle determined by a triple in Euclidean space. The theory of 2- metric space has been exclusively studies and developed by Gahler [45, 46] White [47], Iseki [48]. Definitions of 2-metric space, cauchy sequence, convergence and completeness is given in chapter V. For the first time Iseki and Sharma [49], and afterward Rao and Rao [50], Khan [51, 52], Singh [53] have studied contractive mappings on 2- metric space.

We also refer many other papers in chapter V. Jungck's fixed point theorem [54] was proved in 1976 and improved by conserva [55], Das-Naik [56], Fisher [57],

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[45] Gahler, S. (31)

[46] Gahler, S. (32)

[47] White, A.G. (116)

[48] Iseki, K. (39)

[49] Iseki, K; Sharma, P.L. and Sharma,B.K.(44)

[50] Rao, I.H.N. and Rao, K.P.R. (78)

[51] Khan, M.S. (50)

[52] Khan, M.S. (51)

[53] Singh, S.L. (104)

[54] Jungck, G. (46)

[55] Conserva, V. (14)

[56] Das, K. M. and Naik, K.V. (18)

[57] Fisher, B. (26)

Jungck [58], Park [59], Rhoades [60], Sessa [61], Tiwari and Singh [62] is generally called as Jungck contraction principle. On the other hand, with a view to generalize the Banach contraction principle, Matkowski [63] proved a fixed point theorme for a system of n-transformation on a product of n-metric spaces. This result has been extended and generalized by Czerwick [64, 65], Kominck [66], Reddy and Subrahmanyam [67], Singh and Gairola [68] and Singh and Kulshrestha [69]. The other details are given chapter V where we present common fixed point theorem for two systems of multivalued transformations in 2- metric space which generalizes the well known result of Nadler [70].

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[58]	Jungck, G.	(47)
[59]	Park, S.	(71)
[60]	Rhoades, B.E, Sessa, S; Khan, M.S. and Swaleh, M.	(85)
[61]	Sessa, S.	(98)
[62]	Tiwari, B.M.L. and Sing, S.L.	(114)
[62]	Metkowski, J.	(63)
[64]	Czewick, S.	(16)
[65]	Czerwick, S.	(17)
[66]	Kominck, Z.	(59)
[67]	Reddy,K.B. and Subrahmanyam,P.V.	(80)
[68]	Singh, S.L. and Gairola,V.C.	(107)
[69]	Singh, S.L. and Kulshrestha, C.	(108)
[70]	Nadler, S.B.	(67)

1.3 Many authors have extended the well known result of Jungck [71]. In addition to the authors specially cited in the chapter is conserva [72], Yeh [73] [73-A] & Fisher [74], [75], Khan [76], Khan and Imdad [77], Park [78], Park and Rhoades, B.E. [79], Singh [80] have proved their results in metric, uniform and L-spaces, Sessa [81] has generalized the result.

Das and Naik [82] using weakly commuting mapping.

In this chapter V we shall use a new type of inequality for four self mappings in complete metric space, some pairs of mappings compatible, using

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[71] Jungck, G. (46)

[72] Conserva, V. (14)

[73] Yeh, C.C. (117) [73A] Yeh, C.C. (118)

[74] Fisher, B. (26)

[75] Fisher, B. (27)

[76] Khan, M.S. (52)

[77] Khan and Imdad (55)

[78] Park, S. (71)

[79] Park, S. and Rhoades, B.E. (75)

[80] Singh, S.L. (105)

[81] Sessa, S. (98)

[82] Das and Naik (18)

asymptotically regularity in the generalized sense to study common fixed point.

1.4 Kasahara [83] has introduced L-space, then Yeh [84] has given few fixed point theorem in L-space. It seems that the notion of metric is not necessary in Banach [85] contraction theorem and some of its generalization.

Denote By N. the set of all non-negative integers. A pair $(x, \rightarrow)$ of a set X and subset $\rightarrow$ of the set $X^n \times X$ is called a L. space if the following conditions are satisfied.

(1.4.1) If $x_n = x$ in X for all n in N,

then $(\{x_n\}_{n \in N}, x) \in \rightarrow$,

(1.4.2) If $(\{x_n\}_{n \in N}, x) \in \rightarrow$, then $(\{x_{ni}\}_{i \in N}, x) \in \rightarrow$

for every subsequence $\{x_{ni}\}_{i \in N}$ of $\{x_n\}_{n \in N}$.

(for the rest definition etc please refer to chapter II and III)

Ciric [86] has obtained the result regarding the non-unique fixed points for orbitally continuous self

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[83] Kasahara, S. (49)

[84] Yeh, C.C. (118)

[85] Banach, S. (2)

[86] Ciric, L.B. (12)

mapping on a orbitally complete metric space. Taskovic[87], [88] proved some fixed point theorems generalizing the above result.

In Chapter II we shall prove many theorems. In theorem 1, we shall use nine mappings generalizing the above results.

Theorem 2 deals with the product of two mappings.

Theorem 4 deals with the product of three mappings for the first time.

Theorem 5 deals with common fixed point and theorem 6 deals with using four point in space in place of two points. In Chapter III we extend the result of Pal and Maiti [89] and [90] in various maps by proving five theorem.

(1.5) During the last fifteen years or so many results on fixed or common fixed point theorem were studied for certain expansion mappings. Fisher [91] proved:

Theorem F. Let T be a self mapping of a metric space X such that, for all x, y $\in$ X;

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[87] Taskovic, M.R. (112)

[88] Taskovic, M.R. (113)

[89] Pal, T.K. and Maiti, M. (61)

[90] Pal, T.K. and Maiti, M. (60)

[91] Fisher, B. (28)

$d(Tx, Ty) \geq \frac{1}{2} [d(x, Tx) + d(y, Ty)]$,

Then T is identity mapping.

Wang, Li; Gao and Iseki [92] obtained:

Theorem W1: If there is a constant $\alpha > 1$, such that

(1.5.1) $d(fx, fy) \geq \alpha d(x, y)$

for any x, y in X and f is onto, then f has a unique fixed point.

Theorem W_2: If there exist non-negative real numbers a, b, c with $a + b + c > 1$ and $a > 1$ such that

(1.5.2) $d(fx, gy) \geq a\, d(x, fx)$

$+ b\, d(y, fy) + c\, d(x, y)$

for every x, y in X with $x \neq y$ and f is onto, then f has a fixed point.

In 1985 Rhoades [93] generalize the above result for pair of maps for common fixed point. In fact he proved:

Theorem R_1. Let f, g be surjective self map of a complete metric space (x, d). Suppose, there exists a constant $a > 1$, such that

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[92] Wang; S.Z.; Li, B.Y; Gao, Z.M. and IsekiK.(115)

[93] Rhoades, B.E. (82)

(1.5.3) $d(fx, gy) \geq a\, d(x, y)$

for each x, y in X. Then f and g have a unique common fixed point.

Theorem R_2. Let f, g be surjective continuous self maps of a complete metric space X. If there exists a real number a > 1 such that

(1.5.4) $d(fx, gy) \geq a \min\{d(x,fx), d(x,gy), d(x,y)\}$

for each x, y in X, then f or g has a fixed point or f and g have a common fixed point.

In Chapter IV, theorem 1 deals with geometric mean type of result.

Remark. Khan [94] has proved.

Theorem K. Let T be a self mapping of a complete metric space (X, d) satisfying:

(1.5.5) $d(Tx, Ty) \leq \alpha\{d(x, Tx)\, d(y, Ty)\}^{\frac{1}{2}}$

for all x, $y \in X$ and $0 \leq \alpha < 1$, then T has a unique fixed point.

We knew that Kannan [95] mapping is like Arithametic mean type and Khan's mapping is like G.M. Since A.M. is greater than Geometric mean, this result is

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[94] Khan, M.S. (53)

[95] Kannan, R. (48)

not interesting for contraction. However, it is better for expansive. Intheorem 1 we prove a better theorem than the results awaited above.

1.6 For theorem 2, we first define generalized mapping definition. A pair of mappings (f, g) on X into itself is said to have generalized Kannan property if there exist constants.

(1.6.1) $K_2 + K_4 + K_5 + K_6 > 1$

(1.6.2) $1- K_1 + K_4 > 0$

and

(1.6.3) $d(fx, gy) \geqslant K_1 d(x, fx) + K_2 d(y, gy) + K_3 d(x, gy) + K_4 d(y,fx) + K_5 d(x,y)$

$$+ K_6 \frac{d(y,gy)[1+\{d(x,y)\ d(y, gy)\}^{\frac{1}{2}}]}{1 + d(x, y)}$$

for all x, y in X.

Then we prove the theorem for common fixed point using expansion mapping.This theorem also generalizes and theorems of Gao and Rhoades as particular cases.

For theorem 3, we are concerned with three surjective mapping in place of two. Theorem 4 deals with harmonic mean type mapping. In the last (theorems) we extend the above result of Rhoades to 2- metric space and prove:

Theorem 5. Let f and g be surjective sequentially continuous mappings from a complete 2-metric space (X, d) into itself. Suppose that there exists $\phi \in F$ such that

(1.6.4) $\phi[d(fx, gy, a)] \geqslant \min\{d(x,fx,a), d(y,gy,a), d(x, gy, a), d(y, fx, a), d(x,y)\}$

for all x, y and a in X where

$$\sum_{n=o}^{\infty} \phi^{n}(t) < \infty \quad \text{for all } t > 0$$

Then f or g has a fixed point or f and g have a common fixed point.

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