APPENDIX
FIXED POINT THEOREMS IN BANACH SPACE

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In this note we have tried to extend and generalize the theorems of Goebel and Zlotkiewicz[1], Isekii[2], Sharma & Raju[3] and Singh and Chatterjee[4]. The object of this paper is to prove,

THEOREM 1. Let \( F \) be a mapping of a Banach Space \( X \) into itself and \( F \) satisfy the following conditions:

1. \( I^2 = I \), where \( I \) is identify mapping.

2. \[
\frac{a||x-F(x)|| + ||y-F(y)||}{||x-F(y)|| + ||y-F(y)|| + ||x-y||} + b||x-F(x)|| + e||y-F(y)|| + \frac{c||x-F(x)|| + ||y-F(y)||}{||x-y||} + d||x-F(x)|| + ||y-F(x)|| + e||x-y|| \]

For every \( x, y \in X \) and \( x \neq y \), \( a, b, c, d; e > 0, a + 4b + 4c + 4d + e < 2, 2d + e < 1 \), then \( F \) has a unique fixed point.

PROOF. Suppose \( x \) is a point in the Banach Space \( X \).

Taking \( y = 1/2 \ (F+I)(x), z = F(y), u = 2y-z \), we have

\[
||z-x|| = ||F(y)-F^2(x)|| \\
\leq \frac{a||y-F(y)|| + ||F(x)-F^2(x)||}{||y-F^2(x)|| + ||F(x)-F^2(x)|| + ||y-F(x)||} + \frac{b||y-F(y)|| + ||F(x)-F^2(x)||}{||y-F(x)||} + e||y-F(y)|| + ||F(x)-F^2(x)||
\]

\[
+ c||y-F(y)|| + ||F(x)-F^2(x)||
\]

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\[ a \| y - F(x) \| + \| F(x) - y \| + \frac{1}{2} \| y - F(x) \| \]

\[ a \| y - F(x) \| \| F(x) - x \| + \frac{1}{2} \| F(x) - y \| \]

\[ c \| y - F(x) \| \| F(x) - x \| + \frac{1}{2} \| F(x) - y \| \]

\[ (\frac{9}{2} + 2d + c + a) \| y - F(x) \| + (c + a + \frac{9}{2}) \| F(x) - y \| \]

Therefore

\[ |z - x| = \frac{9}{2} + 2d + c + a \]

and

\[ |w - x| = \frac{9}{2} + 2d + c + a \]

\[ \frac{9}{2} \| y - F(x) \| + \frac{9}{2} \| F(x) - y \| \]

\[ \frac{c}{2} \| x - F(x) \| + |y - F(y)| + \| x - F(x) \| + \| y - F(y) \| \]

\[ (\frac{9}{2} + 2c + 2d) \| y - F(y) \| + 2d \| y - F(y) \| + c \| x - F(x) \| + |y - F(y)| \]

\[ (\frac{9}{2} + 2c + 2d) \| y - F(y) \| + (\frac{9}{2} + c + a + \frac{9}{2}) \| x - F(x) \| \]

\[ \frac{9}{2} + 2c + 2d \| y - F(y) \| + (\frac{9}{2} + 2c + 2d + e) \| x - F(x) \| \]

Also

\[ |z - u| = \| F(y) - 2y + z \| = 2 \| F(y) - y \| \]

Combining (1) and (2) we have

\[ 2||y-F(y)|| \leq \left( \frac{5}{2} + 4a + 2c + 2d \right) ||y-F(y)|| + a \left( \frac{5}{2} + 2c + 2d + a \right)||y-F(y)|| \]

\[ a \leq \frac{5}{2} + 2c + 2d + a \]

where

\[ q = a \leq \frac{5}{2} + 2c + 2d + a \]

\[ a + 4b + 4c + 4d + a \leq 2 \]

Let \( g = \frac{1}{2} (F+1) \), then for any \( x \in X \)

\[ ||G^2(x)-G(x)|| = ||G(y)-y|| \]

\[ = \frac{1}{2} ||F+1|| ||y-y|| \]

\[ = \frac{1}{2} ||F(y)-y|| \]

\[ = \frac{1}{2} ||x-y|| \]

By the definition of \( q \), we claim that \( G^0(x) \) is a Cauchy sequence in \( X \). By the completeness, \( \{ G^0(x) \} \) converges to some element \( x_0 \) in \( X \), i.e.

\[ \lim_{n \to \infty} G^0(x) = x_0 \]

This implies \( G(x_0) = x_0 \).

Hence \( F(x_0) = x_0 \) i.e. \( x_0 \) is a fixed point \( X \).

For the uniqueness, if possible \( y_0 \) (\( \neq x_0 \)) be another fixed point of \( F \). Then

\[ ||x_0-y_0|| \leq \frac{b||x_0-F(x_0)|| + c||x_0-F(x_0)|| + d||y_0-F(y_0)||}{1-2a-c} \]

\[ + \frac{d||x_0-F(x_0)|| + c||y_0-F(y_0)|| + b||x_0-F(x_0)||}{1-2a-c} \]

\[ + e||x_0-y_0|| \]

\[ \leq \frac{1}{1-2a-c} \left( ||x_0-y_0|| \right) \]

\[ \leq 0 \]

\[ \Rightarrow x_0 = y_0 \] Hence the result.

REMARKS:

(1) If we put \( a = 0, b = 0, c = 0, d = 0 \) we get the result of K. Goebel and E. Zlotkiewicz[1].

(2) If we put \( a = 0, b = 0, c = 0 \), we get the result of K. Goebel[2].

(3) If we put \( a = 0, b = 0, c = 0 \), we get the result of P.L. Sharma and Rajput[3].

(4) If we put \( a = 0, b = 0, c = 0 \), we get the result of M.R. Singh and A.K. Chatterjee[4].
THEOREM 2. Let K be a closed and convex subset of a Banach Space X.
Let F: K -> K, G: K -> K satisfy the following conditions:

(2.1) F and G commute

(2.2) F^2 = I and G^2 = I, where I denotes the identity mapping

(2.3) \[ ||F(x) - F(y)|| \leq \frac{a||G(x) - F(x)|| + ||G(y) - F(y)||}{||G(x) - F(y)|| + ||G(y) - F(y)|| + ||G(x) - G(y)||} \]

\[ + \frac{b||G(x) - F(x)|| + ||G(y) - F(y)||}{||G(x) - G(y)||} \]

\[ + \frac{c||G(x) - F(x)|| + ||G(y) - F(y)||}{||G(x) - G(y)||} \]

\[ + \frac{d||G(x) - F(x)|| + ||G(y) - F(y)||}{||G(x) - G(y)||} \]

\[ + \frac{e||G(x) - G(y)||}{||G(x) - G(y)||} \]

For every x, y ∈ K and a, b, c, d, e > 0, a + b + c + d + e < 2. Then there exists at least one fixed point x_0 ∈ K such that F(x_0) = G(x_0) = x_0. Further if 2d + e < 1 then x_0 is the unique fixed point of F and G.

PROOF: From (2.1) and (2.2) it follows that (FG)^2 = I and (2.2) and (2.3) we have:

\[ ||FG(x) - FG(y)|| \leq \frac{a||GG^2(x) - FG^2(y)|| + ||GG^2(y) - FG^2(y)||}{||GG^2(x) - FG^2(y)|| + ||GG^2(y) - FG^2(y)|| + ||GG^2(x) - GG^2(y)||} \]

\[ + \frac{b||GG^2(x) - FG^2(y)|| + ||GG^2(y) - FG^2(y)||}{||GG^2(x) - GG^2(y)||} \]

\[ + \frac{c||GG^2(x) - FG^2(y)|| + ||GG^2(y) - FG^2(y)||}{||GG^2(x) - GG^2(y)||} \]

\[ + \frac{d||GG^2(x) - FG^2(y)|| + ||GG^2(y) - FG^2(y)||}{||GG^2(x) - GG^2(y)||} \]

\[ + \frac{e||GG^2(x) - FG^2(y)|| + ||GG^2(y) - FG^2(y)||}{||GG^2(x) - GG^2(y)||} \]

Now if we put Gx = z, and Gy = w we get...
When \((FG)^2 = I\) and so by theorem 1, \(FG\) has at least one fixed point say \(x_0\) in \(K\) i.e.
\[
FGx_0 = x_0 \quad \text{(A)}
\]
and so \(FFGx_0 = Fx_0\)
or \(Gx_0 = Fx_0 \quad \text{(B)}
\]
Now
\[
||F(x_0) - x_0|| > ||Fx_0 - F^2x_0|| = ||Fx_0 - F(Fx_0)||
\]
\[
< \frac{a||Fx_0 - Fx_0|| + ||GFx_0 - FFx_0||}{||Fx_0 - Fx_0|| + ||GFx_0 - FFx_0|| + ||Gx_0 - GFx_0||}
\]
\[
+ \frac{b||Fx_0 - Fx_0|| + ||GFx_0 - FFx_0||}{||Fx_0 - Fx_0|| + ||GFx_0 - FFx_0|| + ||Gx_0 - GFx_0||}
\]
\[
+ c||Fx_0 - Fx_0|| + ||GFx_0 - FFx_0||
\]
\[
+ d||Fx_0 - Fx_0|| + ||GFx_0 - FFx_0|| + e||Gx_0 - GFx_0||
\]
\[
= \frac{a||Fx_0 - Fx_0|| + ||GFx_0 - FFx_0||}{||Fx_0 - Fx_0|| + ||GFx_0 - FFx_0|| + ||Gx_0 - GFx_0||}
\]
\[
+ c||Fx_0 - Fx_0|| + ||GFx_0 - FFx_0||
\]
\[
+ d||Fx_0 - Fx_0|| + ||GFx_0 - FFx_0|| + ||Gx_0 - GFx_0||
\]
\[
= (2d + e)||F(x_0) - x_0||
\]
Since \(2d + e < 1\), it follows that \(Fx_0 = x_0\) i.e. \(x_0\) is the fixed point of \(F\), but \(Fx_0 = Gx_0\) and so we have \(Gx_0 = x_0\) i.e. \(x_0\) is the common fixed point of \(F\) and \(G\). Now to show that \(x_0\) is unique fixed point of \(F\) and \(G\).

Now, on using (2.1), (2.2), (2.3) and (A), (B) we have
\[
||x_0 - y_0|| > ||F^2x_0 - F^2y_0|| = ||Fx_0 - FFy_0||
\]
\[
< \frac{a||Fx_0 - FFy_0|| + ||GFy_0 - FFy_0||}{||Fx_0 - FFy_0|| + ||GFy_0 - FFy_0|| + ||GFx_0 - GFy_0||}
\]
\[
\frac{b||GFx_0^* - FFx_0|| + ||GFy_0^* - FFy_0||}{||GFx_0^* - GFy_0||}
+ c(||GFx_0^* - FFy_0|| + ||GFy_0^* - FFx_0||)
+ d(||GFx_0^* - FFy_0|| + ||GFy_0^* - FFx_0||) + e||GFx_0^* - GFy_0||
\]
\[
\frac{a||x_0 - x_0^*|| + ||y_0 - y_0^*||}{||x_0 - y_0|| + ||y_0 - y_0^*||} + \frac{b||x_0 - x_0^*|| + ||y_0 - y_0^*||}{||x_0 - y_0||}
\]
\[
+ c(||x_0 - x_0^*|| + ||y_0 - y_0^*||) + d(||x_0^* - y_0|| + ||y_0 - x_0^*||)
+ e||x_0^* - y_0^*||
\]
\[
= (2d + e) ||x_0^* - y_0^*||
\]

Since \(2d + e < 1\), it follows \(x_0 = y_0\), proving the uniqueness of \(x_0\). This completes the proof of theorem 2.

**REMARKS:**
(1) If we put \(G = I\) in theorem 2, we get theorem 1.
(2) If we put \(e = 0, b = 0, c = 0, d = 0\), in theorem 2, we get the theorem by Khan[5].
(3) If we put \(e = 0, b = 0, d = 0\) in theorem 2, we get the theorem by Sharma and Bajaj[6].

**REFERENCES**

A UNIQUE FIXED POINT THEOREM IN COMPLETE METRIC SPACE

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In the present paper we shall establish the following unique fixed point theorem which is the generalization of the result due to Pachpatte[2].

If \( T \) is a mapping of the complete metric space \( X \) in to itself satisfying the inequality,

\[
\left[ d(Tx, Ty) \right]^2 \leq \alpha \left[ d(x, Tx) d(y, Ty) + d(x, Ty) d(y, Tx) \right] + \beta \left[ d(x, Tx) d(y, Ty) + d(x, Ty) d(y, Tx) \right] + \gamma \left[ \left( d(y, Ty) \right)^2 + \left( d(y, Ty) \right)^2 \right]
\]

for all \( x, y \) in \( X \), where \( \alpha, \beta, \gamma \geq 0 \) and \( \alpha + 2\beta + \gamma < 1 \), then \( T \) has a unique fixed point.

Introduction:- Let \((X, d)\) be a metric space. A mapping \( T: X \rightarrow X \) is called a contraction mapping. If there exists a real number \( k, 0 < k < 1 \) such that

\[
d(Tx, Ty) \leq kd(x, y) \text{ for all } x, \ y \in X.
\]

The well known Banach contraction theorem is given below:-

THEOREM 1:- If \( T \) is a mapping of a complete metric space \( X \) in to itself such that, \( d(Tx, Ty) \leq kd(x, y) \), for all \( x, \ y \in X \) where \( 0 \leq k < 1 \), then \( T \) has a unique fixed point.

In 1976, Fisher (1976) proved the following

THEOREM 2:- If \( T \) is a mapping of the complete metric space \( X \) into itself satisfying the inequality

\[
\left[ d(Tx, Ty) \right]^2 \leq \alpha d(x, Tx) d(y, Ty) + \beta d(x, Ty) d(y, Tx)
\]

for all \( x, y \) in \( X \) where \( 0 \leq \alpha < 1 \) and \( 0 \leq \beta \) then \( T \) has a fixed point. Further, if \( 0 \leq \alpha, \beta < 1 \) then the fixed point of \( T \) is unique.

B.G. Pachpatte (Pachpatte 1980), in 1980 proved the following theorem:

THEOREM 3:- If \( T \) is a mapping of the complete metric space \( X \) into itself satisfying inequality.
\[ |d(Tx, Ty)|^2 \leq a \{ d(x, Tx) \ d(y, Ty) \ + d(x, Ty) \ d(y, Tx) \} + \beta \{ d(x, Tx) \ d(y, Ty) + d(x, Ty) \ d(y, Tx) \}
\]
for all \( x, y \) in \( X \) where \( a, \beta \geq 0 \) and \( a + 2 \beta < 1 \) then \( T \) has a unique fixed point.

Now we wish to establish a theorem which includes (theorem 1) of Rachev.

THEOREM 4:- If \( T \) is a mapping of the complete metric space \( X \) into itself satisfying inequality,
\[ |d(Tx, Ty)|^2 \leq a \{ d(x, Tx) \ d(y, Ty) + d(x, Ty) \ d(y, Tx) \} + \beta \{ d(x, Tx) \ d(y, Ty) + d(x, Ty) \ d(y, Tx) \} + \gamma \{ d(y, Ty) |^2 + d(y, Ty) |^2 \}
\]
for all \( x, y \) in \( X \) where \( a, \beta, \gamma \geq 0 \) and \( a + 2 \beta + \gamma < 1 \) then \( T \) has a unique fixed point.

PROOF: Let \( x \) be an arbitrary point in \( X \). Then
\[
|d(T^n x, T^n y)|^2 \leq a \{ d(T^n x, T^n y) |^2 + d(T^n x, T^n y) |^2 \} + \beta \{ d(T^n x, T^n y) + d(T^n x, T^n y) \}
\]
\[ + \gamma \{ d(T^n y, T^n y) |^2 + d(T^n y, T^n y) |^2 \}
\]
\[ = a \{ d(T^n x, T^n y) + d(T^n x, T^n y) \}
\]
\[ + \beta \{ d(T^n x, T^n y) + d(T^n y, T^n y) \}
\]
\[ + \gamma \{ d(T^n y, T^n y) + d(T^n y, T^n y) \}
\]
\[ = (1 - \beta - \gamma) \{ d(T^n x, T^n y) \} + (a + \beta) \{ d(T^n x, T^n y) \}
\]
\[ + \gamma \{ d(T^n x, T^n y) \}
\]
for \( n = 1, 2, 3 \ldots \) and since \( \frac{a + \beta}{1 - \beta - \gamma} < 1 \) it follows that
\[ \{ T^n x \} \] is a Cauchy sequence in the complete metric space \( X \), and so has a limit point \( z \). Now we have
\[ d(T^nx, Tz)^2 \leq a \{ d(T^{n-1}x, T^nx) d(z, Tz) + d(T^{n-1}z, Tz) \ d(z, T^nx) \} \\
+ b \{ d(T^{n-1}x, T^nx) d(z, T^nx) + d(T^{n-1}z, Tz) \ d(z, Tz) \} \\
+ \gamma \{ d(z, T^nx)^2 + d(z, Tz)^2 \} \]

and, on letting \( n \) tend to infinity, we see that

\[ d(z, Tz)^2 \leq (b + \gamma) \{ d(z, Tz)^2 \} \]

Since \( b + \gamma < 1 \), follows that \( Tz = z \). Hence \( z \) is a fixed point of \( T \).

Now suppose that \( T \) has a second fixed point \( z' \). Then

\[ d(z, z')^2 \leq a \{ d(z, Tz) d(z', Tz') + d(z, Tz') d(z', Tz) \} \\
+ b \{ d(z, Tz) d(z', Tz) + d(z, Tz') d(z', Tz') \} \\
+ \gamma \{ d(z, Tz)^2 + d(z, Tz')^2 \} \\
= (a + \gamma) \{ d(z, z') \}^2 \]

Since \( a + \gamma < 1 \), it follows that \( z = z' \), and so the fixed point is unique. This complete the proof of theorem.

**REMARK:** If we put \( \gamma = 0 \) then we get the result of B.G. Pachpatte (1980).

Now we establish the following fixed point theorem on a compact metric space, which includes theorem (3) of Pachpatte.

**THEOREM 5:** If \( T \) is a continuous mapping of the compact metric space \( X \) into itself satisfying the inequality.

\[ d(Tx, Ty)^2 \leq a \{ d(x, Tx) d(y, Ty) + d(x, Ty) d(y, Tx) \} \\
+ b \{ d(x, Tx) d(y, Tx) + d(x, Ty) d(y, Ty) \} \\
+ \gamma \{ d(y, Tx)^2 + d(y, Ty)^2 \} \]

for all distinct \( x, y \) in \( X \), where \( a, b, \gamma > 0 \) and \( a + 2b + \gamma = 1 \), \( \varepsilon + \gamma < 1 \) then \( T \) has a unique fixed point.

**PROOF:** Define a function \( f \) on \( X \) by

\[ f(x) = d(x, Tx) \]

for all \( x \) in \( X \). Since \( d \) and \( T \) are continuous functions, it follows that \( f \) is a continuous function on \( X \). Since \( X \) is compact there exist a point \( z \) in \( X \) such that

\[ f(z) = \inf \{ f(x) : x \in X \} \]
We will now suppose that \( Tz \neq z \). Then by hypothesis
\[
\frac{d(Tz, Tz^2)}{2} < a \frac{d(z,Tz) + d(Tz, Tz^2)}{2} + b \frac{d(z, Tz^2)}{2} + \gamma \frac{d(Tz, Tz^2)}{2}
\]

which implies
\[
d(Tz, Tz^2) < a d(z, Tz) + b d(Tz, Tz^2) + \gamma d(Tz, Tz^2)
\]

and so
\[
d(Tz, Tz^2) < \frac{a + b + \gamma}{1 - b - \gamma} d(z, Tz) = d(z, Tz)
\]

or \( f(Tz) < f(z) \).

This contradiction the definition of \( z \), so that we must have \( Tz = z \) and \( z \) is a fixed point of \( T \).

Now suppose that \( T \) has a second distinct fixed point \( z' \). Then
\[
\frac{d(z, z')}{2} = \frac{d(Tz, Tz')}{2}
\]

\[
< a \frac{d(z, Tz) + d(z, Tz')}{2} + b \frac{d(z, Tz')}{2} + \gamma \frac{d(Tz, Tz')}{2}
\]

\[
= \left( a + \gamma \right) \frac{d(z, z')}{2}
\]

giving a contradiction since \( a + \gamma \neq 1 \). It follows that the fixed point is unique, completing the proof of theorem.

REMARK: If we put \( \gamma = 0 \), then we get the result of B.G. Pachpatte (1980).

REFERENCES
