APPENDIX

Acta Ciencia Indica Vol. XVII, M, 3, 549(1991)

FIXED POINT THEOREMS IN BANACH SPACE

P.L. SHARMA AND M.K. SAHU
Dr. Hari Singh Gour Vishwavidyalaya, Sagar (M.P.)

Received : 28 October, 1990

In this note we have tried to extend and generalize the theorems of Goebel and Zlotkiewicz[1], Iseki[2], Sharma & Rajput[3] and Singh and Chatterjee[4].

The object of this paper is to prove,

THEOREM 1. Let F be a mapping of a Banach Space X into itself and F satisfy the following conditions:

1. $F^2 = I$, where I is identify mapping.

2. $$||F(x)-F(y)|| \leq \frac{a||x-F(x)||\ ||y-F(y)||}{||x-F(y)|| + ||y-F(y)|| + ||x-y||} + \frac{b||x-F(x)||\ ||y-F(y)||}{||x-y||} + c[||x-F(x)|| + ||y-F(y)||] + d[||x-F(y)|| + ||y-F(x)||] + e||x-y|| .$$

For every x, y ε X and $x \neq y$, a, b, c, d; e > 0, a + 4b + 4c + 4d + e < 2, 2d + e < 1, Then F has a unique fixed point.

PROOF. Suppose x is a point in the Banach Space X.
Taking $y = 1/2\ (F+I)(x)$, $z = F(y)$, $u = 2y-z$, we have

$$||z-x|| = ||F(y)-F^2(x)||$$
$$\leq \frac{a||y-F(y)||\ ||F(x)-F^2(x)||}{||y-F^2(x)|| + ||F(x)-F^2(x)|| + ||y-F(x)||}$$
$$+ \frac{b||y-F(y)||\ \ ||F(x)-F^2(x)||}{||y-F(x)||}$$
$$+ c[||y-F(y)|| + ||F(x)-F^2(x)||]$$

$$+ \; d[\,\|y-F^2(x)\| + \|F(x)-F(y)\|\,] \; + \; e\,\|y-F(x)\|$$

$$= \frac{a\|y-F(y)\|\;\|x-F(x)\|}{\|y-x\|+\|F(x)-x\|+\|y-F(x)\|} + \frac{b\|y-F(y)\|\;\|F(x)-x\|}{\|y-F(x)\|}$$

$$+ \; c[\,\|y-F(y)\| + \|F(x)-x\|\,] + d\,[\,\|y-x\| + \|F(x)-F(y)\|\,]$$

$$+ \; e\|y-F(x)\|$$

$$\leqslant \frac{a\;\|y-F(y)\|\;\|F(x)-x\|}{\|F(x)-x\|} + \frac{b\|y-F(y)\|\qquad\|F(x)-x\|}{\|\frac{1}{2}\,(F+I)(x)-F(x)\|}$$

$$+ \; c[\,\|y-F(y)\| + \|F(x)-x\|\,] + d\;[\,\|\tfrac{1}{2}\,(F+I)(x)-x\| + \|F(x)-y\|$$

$$+ \; \|y-F(y)\| + e\,\|\tfrac{1}{2}\,(F+I)(x)-F(x)\|$$

$$= \left(\frac{a}{2} + 2\,b + c + d\right)\|y-F(y)\| + \left(c + d + \frac{e}{2}\right)\|F(x)-x\|$$

Therefore

$$\|z-x\| \leqslant \left(\frac{a}{2} + 2b + c + d\right)\|y-F(y)\| + \left(c + d + \frac{e}{2}\right)\|x-F(x)\|$$

and

$$\|u-x\| = \|2y-z-x\|$$

$$= \|(F+I)(x)-F(y)-x\|$$

$$= \|F(x)-F(y)\|$$

$$\leqslant \frac{a\|x-F(x)\|\;\|y-F(y)\|}{\|x-F(y)\|+\|y-F(y)\| + \|(x-y)\|} + \frac{b\|x-F(x)\|\;\|y-F(y)\|}{\|x-y\|}$$

$$+ \; c[\,\|x-F(x)\|+\|y-F(y)\|\,] + d[\,\|x-F(y)\|+\|y-F(x)\|\,] + e\|x-y\|$$

$$\leqslant \frac{a}{2}\;\|x-F(x)\| + 2b\|y-F(y)\| + c\;[\,\|x-F(x)\| + \|y-F(y)\|\,]$$

$$+ \; d\;[\,\|x-y\| + \|y-F(y)\| + \|\tfrac{1}{2}\,(F+I)(x)-F(x)\|\,]$$

$$+ \; e\;\|x - \tfrac{1}{2}\,(F+I)(x)\|$$

$$= (2b + c + d)\;\|y-F(y)\| + \left(\frac{a}{2} + c + d + \frac{e}{2}\right)\|x-F(x)\|$$

$$\therefore\; \|z-u\| \leqslant \|z-x\| + \|u-x\|$$

$$\leqslant \left(\frac{a}{2} + 4b + 2c + 2d\right)\|y-F(y)\|$$

$$+ \left(\frac{a}{2} + 2c + 2d + e\right)\|x-F(x)\| \qquad \text{------------ (1)}$$

Also

$$\|z-u\| = \|F(y)-2y+z\| = 2\;\|F(y)-y\| \qquad \text{------------ (2)}$$

Acta Ciencia Indica Vol. XVII, M, 3 (1991)

Combining (1) and (2) we have

$$2||y-F(y)|| \leq \left(\frac{a}{2} + 4b + 2c + 2d\right) ||y-F(y)|| + \left(\frac{a}{2} + 2c + 2d + e\right)||y-...$$

$$\therefore \; ||y-F(y)|| \leq q\, ||x-F(x)||$$

where $$q = \frac{\frac{a}{2} + 2c + 2d + e}{2 - \frac{a}{2} - 4b - 2c - 2d} <$$

$$a + 4b + 4c + 4d + e < 2$$

Let $G = \frac{1}{2}(F+I)$, then for and $x \in X$

$$||G^2(x)-G(x)|| = ||G(y)-y||$$

$$= ||\tfrac{1}{2}(F+I)(y)-y||$$

$$= \tfrac{1}{2}||F(y)-y||$$

$$\leq \tfrac{q}{2}||x-F(x)||$$

By the definition of q, we claim that $\{G^n(x)\}$ is a cauchy sequence in X. By the completeness, $\{G^n(x)\}$ converges to some element x_0 in X, i:e

$$\lim_{n\to\infty} G^n(x) = x_0$$

This implies $G(x_0) = x_0$.

Hence $F(x_0) = x_0$ i:e x_0 is a fixed point X.

For the uniqueness, if possible y_0 ($\neq x_0$) be another fixed point of .

Then

$$||x_0-y_0|| = ||Fx_0-F(y_0)||$$

$$\leq \frac{a||x_0-F(x_0)||\;||y_0-F(y_0)||}{||x_0-F(y_0)||+||y_0-F(y_0)||+||x-y||} + \frac{b||x_0-F(x_0)||\;||y_0-F(y_0)||}{||x_0-y_0||}$$

$$+ c[||x_0-F(x_0)||+||y_0-F(y_0)||] + d[||x_0-F(y_0)||+||y_0-F(x_0)||]$$

$$+ e||x_0-y_0||$$

$$\therefore \; (1-2d-c)\,||x_0-y_0|| \leq 0$$

$\therefore \; x_0 = y_0$ Hence the result.

REMARKS:

(i) If we put $a = 0, b = 0, c = 0, d = 0$ we get the result of K. Goebel and E. Zlotkiewicz[1].

(ii) If we put $a = 0$, $b = 0$, $c = 0$, we get the result of K. Iseki[2].

(iii) If we put $a = 0$, $d = 0$, we get the result of P.L. Sharma and Rajput[3].

(iv) If we put $a = 0$, we get the result of M.R. Singh and A.K. Chatterjee[4].

THEOREM 2. Let K be a closed and convex sub set of a Banach Space X. Let $F:K \to K$, $G:K \to K$ satisfy the following conditions;

(2.1) F and G commutes

(2.2) $F^2 = I$ and $G^2 = I$, where I denote the identity mapping

$$(2.3)\quad \|F(x)-F(y)\| \leqslant \frac{a\|G(x)-F(x)\|\ \|G(y)-F(y)\|}{\|G(x)-F(y)\|+\|G(y)-F(y)\|+\|G(x)-G(y)\|}$$

$$+\frac{b\|G(x)-F(x)\|\ \|G(y)-F(y)\|}{\|G(x)-G(y)\|}$$

$$+\ c[\|G(x)-F(x)\| + \|G(y)-F(y)\|]$$

$$+\ d[\|G(x)-F(y)\| + \|G(y)-F(x)\|]$$

$$+\ e\|G(x)-G(y)\|$$

For every x, $y \in K$ and a, b, c, d, $e \geqslant 0$, $a+4b+4c+4d+e < 2$. Then there exists at least one fixed point $x_o \in K$ such that $F(x_o) = G(x_o) = x_o$. Further if $2d+e<1$ then x_o is the unique fixed point of F and G.

PROOF: From (2.1) and (2.2) it follows that $(FG)^2=I$ and (2.2) and (2.3) we have

$$\|FG.Gx-FG.Gy\| \leqslant \frac{a\|GG^2x-FG^2x\|\ \|GG^2y-FG^2y\|}{\|GG^2x-FG^2y\|+\|GG^2y-FG^2y\|+\|GG^2x-GG^2y\|}$$

$$+\frac{b\|GG^2x-FG^2x\|\ \|GG^2y-FG^2y\|}{\|G.G^2x-G.G^2y\|}$$

$$+\ c[\|GG^2x-FG^2x\| + \|G.G^2y-FG^2y\|]$$

$$+\ d[GG^2x-FG^2y\| + \|GG^2y-FG^2x\|] + e\|GG^2x-GG^2y\|$$

$$= \frac{a\|G(x)-FGG(x)\|\ \|Gy-FGGy\|}{\|Gx-FGGy\|+\|Gy-FGGy\|+\|Gx-Gy\|}$$

$$+\frac{b\|G(x)-FGG(x)\|\ \|Gy-FGGy\|}{[illegible]}$$

$$+\ c[\|Gx-FG.Gx\| + \|Gy-FG.Gy\|]$$

$$+\ d[\|Gx-FG.Gy\| + \|Gy-FGGy\|] + e\|Gx-Gy\|$$

Now if we put $Gx = z$, and $Gy = w$ we get

$$||GFz-FGw|| \leq \frac{a||z-FGz||\ ||w-FGw||}{||z-FGw||+||w-FGw||+||z-w||} + \frac{b||z-FGz||\ ||w-FGw||}{||z-w||}$$

$$+ \ c[||z-FGz|| + ||w-FGw||]$$

$$+ \ d[||z-FGw|| + ||w-FGz||] + e\ ||z-w||$$

When $(FG)^2 = I$ and so by theorem 1, FG has at least one fixed point say x_o in K i.e.

$$FGx_o = x_o \quad \text{----------} \ (A)$$

and so $FFGx_o = Fx_o$

$$\text{or } Gx_o = Fx_o \quad \text{----------} \ (B)$$

Now

$$||F(x_o)-x_o|| = ||Fx_o-F^2x_o|| = ||Fx_o-F(Fx_o)||$$

$$\leq \frac{a||Gx_o-Fx_o||\ ||GFx_o-FFx_o||}{||Gx_o-FFx_o||+||GFx_o-FFx_o||+||Gx_o-GFx_o||}$$

$$+ \frac{b||Gx_o-Fx_o||\ ||GFx_o-FFx_o||}{||Gx_o-GFx_o||}$$

$$+ \ c[||Gx_o-Fx_o|| + ||GFx_o-FFx_o||]$$

$$+ \ d[||Gx_o-FFx_o|| + ||GFx_o-Fx_o||] + e\ Gx_o-GFx_o||$$

$$= \frac{a||Fx_o-Fx_o||\ ||GGx_o-FFx_o||}{||Fx_o-FFx_o||+||GGx_o-FFx_o||+||Gx_o-GGx_o||} + \frac{b||Fx_o-Fx_o||\ ||GGx_o-FFx_o||}{||GGx_o-GGx_o||}$$

$$+ \ c[||Fx_o-Fx_o|| + ||GGx_o-FFx_o||]$$

$$+ \ d[||Fx_o-FFx_o|| + ||GGx_o-Fx_o||] + e\ ||F(x_o-GGx_o||$$

$$= (2d + e)\ ||F(x_o)-x_o||$$

Since $2d + e < 1$, it follows that $Fx_o = x_o$ i.e x_o is the fixed point of F, but $Fx_o = Gx_o$ and so we have $Gx_o = x_o$ i.e x_o is the common fixed point of F and G. Now to show that x_o is unique fixed point of F and G.

Now, on using (2.1), (2.2), (2.3) and (A), (B) we have

$$||x_o-y_o|| = ||F^2x_o-F^2y_o|| = ||FFx_o-FFy_o||$$

$$\leq \frac{a||GFx_o-FFx_o||\ ||GFy_o-FFy_o||}{||GFx_o-FFy_o||+||GFy_o-FFy_o||+||GFx_o-GFy_o||}$$

$$+ \frac{b||GFx_o-FFx_o|| \ ||GFy_o-FFy_o||}{||GFx_o-GFy_o||}$$

$$+ c[||GFx_o-FFy_o|| + ||GFy_o-FFy_o||$$

$$+ d[||GFx_o-FFy_o|| + ||GFy_o-FFx_o||] + e||GFx_o-GFy_o||$$

$$= \frac{a||x_o-x_o|| \ ||y_o-y_o||}{||x_o-y_o||+||y_o-y_o||+||x_o-y_o||} + \frac{b||x_o-x_o|| \ ||y_o-y_o||}{||x_o-y_o||}$$

$$+ c[||x_o-x_o||+||y_o-y_o||] + d[||x_o-y_o|| + ||y_o-x_o||]$$

$$+ e \ ||x_o-y_o||$$

$$= (2d + e) \ ||x_o-y_o||$$

Sincen $2d + e < 1$, it follows $x_o = y_o$, proving the uniqueness of x_o. This completes the proof of theorem 2.

REMARKS:

(i) If we put $G = I$ in theorem 2, we get theorem 1.

(ii) If we put $a = 0$, $b = 0$, $c = 0$, $d = 0$, in theorem 2, we get the theorem by Khan[5].

(iii) If we put $a = 0$, $b = 0$, $d = 0$ in theorem 2, we get the theorem by Sharma and Bajaj[6].

REFERENCES

[1] Goeble, K. and Zlotkiewicz, E. (1971); Some fixed point theorem in Banach Space, Collo. Math. 23, 103-106.

[2] Iseki K. (1974): Fixed point theorem in Banach Space, Math. Sem. Not. Kobe Univ. Vol.2, 11-13.

[3] Sharma, P.L. and Rajput, S.S. (1983); Fixed point theorem in Banach Space, Vikram Mathematical Journal Vol.4, 35-38.

[4] Singh M.R. and Chatterjee, A.K. (1987); Fixed point theorems in Banach Space, Pure Maths. Manuscript Vol.6, 53-61.

[5] Khan, M.S. (1982); Fixed point and their approximation in Banach Space for certain commuting mapping, Glasgow Math. Journal 23.

[6] Sharma, P.L. and Bajaj, N. (1983); Fixed point theorem in Banach Space for commuting mappings, Journal of M.A.C.T. Vol.16, 11-13.

Acta Ciencia Indica Vol. XVII, M, 4, 685 (1991)

A UNIQUE FIXED POINT THEOREM IN COMPLETE METRIC SPACE

P.L. SHARMA AND M.K. SAHU
Department of Mathematics
Dr. Harisingh Gour Vishwavidhyalaya, Sagar (M.P.)

Received : 24 August, 1990

In the present paper we shall establish the following unique fixed point theorem which is the generalization of the result due to Pachpatte[2].

If T is a mapping of the complete metric space X in to itself satisfying the inequality.

$$[d(Tx, Ty)]^2 \leq \alpha\,[d(x,Tx)\,d(y,Ty) + d(x,Ty)\,d(y,Tx)] + \beta\,[d(x,Tx)\,d(y,Tx) + d(x,Ty)\,d(y,Ty)] + \gamma\,[\{d(y,Tx)\}^2 + \{d(y,Ty)\}^2]$$

for all x, y in X, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + \gamma < 1$, then T has a unique fixed point.

Introduction:- Let (X, d) be a metric space. A mapping $T: X \to X$ is called a contraction mapping. If there exists a real number k, $0 \leq k < 1$ such that

$$d(Tx, Ty) \leq k\, d(x,y) \text{ for all } x, y \in X.$$

The well know Banach contraction theorem is given below:-

THEOREM 1:- If T is a mapping of a complete metric space X in to itself such that, $d(Tx, Ty) \leq k\, d(x,y)$, for all $x, y \in X$ where $0 \leq k < 1$, then T has a unique fixed point.

In 1976, Fisher (1976) proved the following

THEOREM 2:- If T is a mapping of the complete metric space X into itself satisfying the inequality

$$[d(Tx, Ty)]^2 \leq \alpha\, d(x,Tx)\, d(y,Ty) + \beta\, d(x,Ty)\, d(y,Tx)$$

for all x, y in x where $0 \leq \alpha < 1$ and $0 \leq \beta$ then T has a fixed point. Further, if $0 \leq \alpha, \beta < 1$ then the fixed point of T is unique.

B.G.Pachpatte (Pachpatte 1980), in 1980 proved the following theorem:

THEOREM 3:- If T is a mapping of the complete metric space X into itself satisfying inequality.

7160

$$[d(Tx, Ty)]^2 \leq \alpha\, [d(x,Tx)\, d(y,Ty) + d(x,Ty)\, d(y,Tx)]$$
$$+ \beta\, [d(x,Tx)\, d(y,Tx) + d(x,Ty)\, d(y,Ty)]$$

for all x, y in X where $\alpha, \beta \geq 0$ and $\alpha + 2\beta < 1$ than T has a unique fixed point.

Now we wish to establish a theorem which includes (theorem 1) of Pachpatte.

THEOREM 4:- If T is a mapping of the complete metric space X into itself. Satisfying inequality.

$$[d(Tx, Ty)]^2 \leq \alpha\, [d(x,Tx)\, d(y,Ty) + d(x,Ty)\, d(y,Tx)]$$
$$+ \beta\, [d(x,Tx)\, d(y,Tx) + d(x,Ty)\, d(y,Ty)]$$
$$+ \gamma\, [\{d(y,Tx)\}^2 + \{d(y,Ty)\}^2]$$

for all x, y in X where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + \gamma < 1$ than T has a unique fixed point.

PROOF: Let x be an arbitrary point in X. Then

$$[d(T^n x, T^{n+1}x)]^2 \leq \alpha\, [d(T^{n-1}x, T^n x)\, d(T^n x, T^{n+1}x) + d(T^{n-1}x, T^{n+1}x)\, d(T^n x, T^n x)]$$
$$+ \beta\, [d(T^{n-1}x, T^n x)\, d(T^n x, T^n x) + d(T^{n-1}x, T^{n+1}x)]$$
$$+ \gamma\, [\{d(T^n x, T^n x)\}^2 + \{d(T^n x, T^{n+1}x)\}^2]$$
$$= \alpha\, [d(T^{n-1}x, T^n x)\, d(T^n x, T^{n+1}x)]$$
$$+ \beta\, [d(T^{n-1}x, T^{n+1}x)\, d(T^n x, T^{n+1}x)]$$
$$+ \gamma\, [\{d(T^n x, T^{n+1}x)\}^2]$$

$$[d(T^n x, T^{n+1}x)] \leq \alpha\, [d(T^{n-1}x, T^n x)]$$
$$+ \beta\, [d(T^{n-1}x, T^n x) + d(T^n x, T^{n+1}x)]$$
$$+ \gamma\, [d(T^n x, T^{n+1}x)]$$

$$(1-\beta-\gamma)\, [d(T^n x, T^{n+1}x)] \leq (\alpha+\beta)\, [d(T^{n-1}x, T^n x)]$$

$$\therefore [d(T^n x, T^{n+1}x)] \leq \frac{(\alpha+\beta)}{(1-\beta-\gamma)}\, [d(T^{n-1}x, T^n x)]$$

for $n = 1, 2, 3 \ldots\ldots$ and since $\frac{\alpha+\beta}{1-\beta-\gamma} < 1$ it follows that $\{T^n x\}$ is a Cauchy sequence in the complete metric space X, and so has a limit point z. Now we have

$$[d(T^n x, Tz)]^2 \leq \alpha\ [d(T^{n-1}x, T^n x)\ d(z, Tz) + d(T^{n-1}, Tz)\ d(z, T^n x)]$$
$$+ \beta\ [d(T^{n-1}x, T^n x)\ d(z, T^n x) + d(T^{n-1}x, Tz)\ d(z, Tz)]$$
$$+ \gamma\ (\{d(z, T^n x)\}^2 + \{d(z, Tz)\}^2]$$

and, on letting n tend to infinite. We see that

$$[d(z, Tz)]^2 \leq (\beta + \gamma)\ [d(z, Tz)]^2$$

Since $\beta + \gamma < 1$ follows that $Tz = z$. Hence z is a fixed point of T.

Now suppose that T has a second fixed point Z'. Then

$$[d(z, z')]^2 = [d(Tz, Tz')]^2$$
$$\leq \alpha\ [d(z,Tz)\ d(z', Tz') + d(z,Tz')\ d(z', Tz)]$$
$$+ \beta\ [d(z,Tz)\ d(z', Tz) + d(z, Tz')\ d(z', Tz')]$$
$$+ \gamma\ [\{d(z', Tz)\}^2 + \{d(z', Tz')\}^2]$$
$$= (\alpha + \gamma)\ [d(z,z')]^2$$

Since $\alpha + \gamma < 1$, it follows that $z=z'$, and so the fixed point is unique. This complete the proof of theorem.

REMARK: If we put $\gamma = 0$ then we get the result of B.G. Pachpatte (1980).

Now we establish the following fixed point theorem on a compact metric space. Which includes theorem (3) of Pachpatte.

THEOREM 5: If T is a continuous mapping of the compact metric space X into itself satisfying the inequality.

$$[d(Tx, Ty)]^2 < \alpha\ [d(x,Tx)\ d(y,Ty)+d(x,Ty)\ d(y,Tx)]$$
$$+ \beta\ [d(x,Tx)\ d(y,Tx) + d(x,Ty)\ d(y,Ty)]$$
$$+ \gamma\ [\{d(y,Tx)\}^2 + \{(y, Ty)\}^2]$$

for all distinct x, y in X. where $\alpha, \beta, \gamma > 0$ and $\alpha + 2\beta + \gamma = 1$, $\alpha + \gamma < 1$ than T has a unique fixed point.

PROOF: Define a function f on X by

$f(x) = d(x,Tx)$, for all x in X. Since d and T are continuous function, it follows that f is a continuous function on X. Since X is compact there exist a point z in x such that -

$$f(z) = \text{Int}\ \{f(x): x \in X\}$$

We will now suppose that $Tz \neq z$. Then by hypothesis

$$[d(Tz,T_z^2)]^2 < \alpha\, [d(z,Tz)\, d(Tz, T_z^2) + d(z,T_z^2)\, d\,(Tz, Tz)$$
$$+ \beta\, [d(z,Tz)\, d\,(Tz,Tz) + d(z,T_z^2)\, d\,(Tz, T^2z)$$
$$+ \gamma\, [\, \{d(Tz, Tz)\}^2 + \{d(Tz, T_z^2)\}^2]$$
$$= \alpha\, [d(z,Tz)\, d\,(Tz, T_z^2)] + \beta\, [d(z, T_z^2)\, d\,(Tz, T_z^2)]$$
$$+ \gamma\, [d(Tz, T_z^2)]^2$$

Which implies

$$d\,(Tz, T_z^2) < \alpha\, d(z, Tz) + \beta\, [d(z, Tz) + d\,(Tz, T_z^2)]$$
$$+ \gamma [d(Tz, T_z^2)]^2$$

and so

$$d\,(Tz, T^2) < \frac{\alpha + \beta}{1 - \beta - \gamma}\, d\,(z, Tz) = d\,(z, Tz)$$

or $f(Tz) < f(z)$

This contradiction the definition of z, so that we must have Tz=z and z is a fixed point of T.

Now suppose that T has a second distict fixed point z'. Then

$$[d(z,z')]^2 = [d(Tz, Tz')]^2$$
$$< \alpha\, [d(x,Tz)\, d(z',Tz') + d(z,Tz')\, d\,(z', Tz)]$$
$$+ \beta\, [d(z,Tz)\, d(z', Tz) + d(z,Tz')\, d(z', Tz')]$$
$$+ \gamma\, [\,\{d(z', Tz)\}^2 + \{d(z', Tz')\}^2\,]$$
$$= (\alpha + \gamma)\ [d\,(z, z')]^2$$

giving a contradiction since $\alpha + \gamma \leq 1$ it follows that the fixed point is unique, completing the proof of theorem.

REMARK: If we put $\gamma = 0$, then we get the result of B.G. Pachpatte (1980).

REFERENCES

1. Fisher B. (1976), Fixed point and constant mapping on metric space, Atti Accad, Naz., Lincci Rend. Cl. Sci. Mat. Natur. 61, 329-332.

2. Pachpatte B.G. (1980), On Certain Fixed point mapping in metric space, Journal of M.A.C.T. volume 13, 59-63.