CHAPTER-VII

ON FIXED POINTS OF ASYMPTOTICALLY REGULAR MAPPING

7.1 Many authors have extended the well known results of Jungck [198]. In addition to the authors specially cited in this chapter is Conserva [199], C.C. Yeh [200], Fisher [201] and [202], Khan [203], Khan and Imdad [204], Park [205], Park and Rhoades [206], Singh [207], have proved their results in complete metric space, Khan [208] in uniform space and C.C. Yeh [209] in L-space.

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[198] Jungck, G. (46)

[199] Conserva, V. (14)

[200] Yeh, C.C. (117)

[201] Fisher, B. (26)

[202] Fisher, B. (27)

[203] Khan, M.S. (52)

[204] Khan M.S. and Imdad, M. (55)

[205] Park, S. (71)

[206] Park, S. and Rhoades B .E. (75)

[207] Singh, S.L. (105)

[208] Khan, M.S. (54)

[209] Yeh, C.C. (118)

Sessa [210] has generalized the result of Das and Naik [211], considering two selfmaps A, S of a complete metric space (X, d) which are weakly commuting i.e.

(7.1.1) $d(ASx, SAx) \leqslant d(Sx, Ax)$

for any $x \in X$.

Example. Let X = [0, 1) equipped with Euclidean metric and $Sx = \frac{x}{x+8}$, $Ax = \frac{x}{4}$ for any $x \in X$.

We have for any $x \in X$

$$d(ASx, SAx) = \frac{x}{x+32} - \frac{x}{4x+32} = \frac{3}{x+32} \cdot \frac{x^2}{4x+32}$$

$$\leqslant \frac{x^2+4x}{4(x+8)} = \frac{x}{4} - \frac{x}{x+8} = d(Sx, Ax)$$

Thus S and A satisfy (1.1) but do not commute for any $x \neq 0$. Hardy and Rogers [212] proved a fixed point theorem for a self mapping T of a complete metric space (X, d) satisfying

(7.1.2) $$d(Tx,Ty) \leqslant a_1 d(x,Tx) + a_2 d(y, Ty) + a_3 d(x, Ty) + a_4 d(y,Tx) + a_5 d(x,y)$$

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[210] Sessa, S. (98)

[211] Das, K.M. and Naik (18)

[212] Hardy, G.E. and Rogers, T.D. (35)

We need the following definitions:

Definition 1. [213, 214] Let A and S be two self mappings of X and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is said to be asymptotically S- regular with respect to A if $d(Sn, Sx_n) \to 0$ as $n \to \infty$.

If A is identity map of X, Definition 1 become that of Engl [215].

Definition 2. [216] The pair {A, S} is said to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some point t in X.

A weak commuting pair is compatible pair but the converse is not necessarily true.

We need a well known result as lemma.

Lemma. Let (X, d) be a metric space and S, A: $X \to X$ be compatible and $\lim Ax_n = \lim_n Sx_n = t$ then $\lim_n SAx_n = At$, if A is continuous at t,

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[213] Rhoades,B.E.; Sessa,S;Khan,M.S. and Khan, M.D.(84)

[214] Rhoades, B.E; Sessa,S;Khan,M.S.and Swaleh,M.(85)

[215] Engl, H.W. (25)

[216] Jungck, G. (47)

Now we are in a position to prove:

Theorem. Let A, B, S and T be four self mappings of a complete metric space (X, d) satisfying:

(7.1.3) A and B are continuous,

(7.1.4) {S,A}, {T, A}, {T, B} are compatible pairs,

(7.1.5) there exists a sequence which is asymptotically B-regular and S-regular with respect to A and T-regular with respect to B;

(7.1.6)
$$\begin{aligned} d(Sx, Ty) \leq\ & a_1\, d(Sx, Ax) + a_2\, d(Ty, By) \\ & + a_3\, d(Sx, By) + a_4\, d(Ty, Ax) \\ & + a_5\, d(Ax, By) + a_6\, d(Tx, Ax) \\ & + a_7\, d(Tx, By) + a_8\, d(Sy, Ax) \\ & + a_9\, d(Sy, By) \end{aligned}$$

for all x, y in X, where $a_i \geq 0$, $\sum_{i=1}^{9} a_i < 1$.

Then A, B, S and T have a unique common fixed point.

<u>Proof</u>: From (7.1.6), for m, n is positive integer,

$$\begin{aligned} d(Ax_n, Ax_m) \leq\ & d(Ax_n, Sx_n) + d(Sx_n, Tx_m) \\ & + d(Tx_m, Bx_m) + d(Bx_m, Ax_m) \end{aligned}$$

$$\leqslant d(Ax_n, Sx_n) + a_1 d(Sx_n, Ax_n) + a_2 (Tx_m, Bx_m)$$

$$+ a_3 [d(Sx_n, Ax_n) + d(Ax_n, Ax_m)$$

$$+ d(Ax_m, Bx_m)]$$

$$+ a_4 [d(Tx_m, Bx_m) + d(Bx_m, Ax_m)$$

$$+ d(Ax_m, Ax_n)$$

$$+ a_5 [d(Ax_n, Ax_m) + d(Ax_m, Bx_m)]$$

$$+ a_6 [d(Tx_n, Bx_n) + d(Bx_n, Ax_n)]$$

$$+ a_7[d(Tx_n, Bx_n) + d(Bx_n, Ax_n)$$

$$+ d(Ax_n, Ax_m)$$

$$+ d(Ax_m, Bx_m)]$$

$$+ a_8[d(Sx_m, Ax_m) + d(Ax_m, Ax_n)]$$

$$+ a_9[d(Sx_m, Ax_m) + d(Ax_m, Bx_n)]$$

$$+ d(Tx_m, Bx_m) + d(Bx_m, Ax_m)$$

or $(1- a_3 - a_4 - a_5 - a_7 - a_8) d(Ax_n, Ax_m)$

$$\leqslant (1 + a_1 + a_3) d(Sx_n, Ax_n)$$

$$+ (a_6+a_7) d(Tx_n, Bx_n) + (1+a_2+a_4)d(Tx_m,Bx_m)$$

$$+ (1+a_3+a_4+a_5+a_7+a_9) d(Ax_m, Bx_m)$$

$$+ (a_6+a_7) d(Ax_n, Bx_n)+(a_8+a_9)d(Sx_m,Ax_m)$$

which on letting $n, m \to \infty$ and using the condition (7.1.5) implies that $\{Ax_n\}$ is a cauchy sequence and hence by completeness of X, there is z in X such that $Ax_n \to z$.

Also we have

$d(Sx_n, z) \leqslant d(Sx_n, Ax_n) + d(Ax_n, z)$.

Taking the limit as $n \to \infty$, we obtain $Sx_n \to z$.

Similarly

$$Tx_n \to z, \quad Bx_n \to z.$$

Now using the continuity of A and B and lemma we have

$$A^2x_n \to Az, \; SAx_n \to Az$$

$$B^2x_n \to Bz, \; TBx_n \to Bz$$

and

$$TAx_n \to Az.$$

Now consider

$$d(SAx_n, TBx_m) \leqslant a_1 d(SAx_n, A^2x_n)$$
$$+a_2d(TBx_n, B^2x_m) + a_3d(SAx_n, B^2x_m)$$
$$+ a_4d(TBx_m, A^2x_n) + a_5d(A^2x_n, B^2x_m)$$
$$+ a_6 d(TAx_n, A^2xn) + a_7 d(TAx_n, B^2x_m)$$
$$+ a_8 d(SBx_m, A^2x_n) + a_9 d(SBx_m, B^2x_m)$$

Taking the limit as m, n $\to \infty$, we obtain

$$d(Az, Bz) \leq (a_3 + a_4 + a_5 + a_7 + a_8)\, d(Az, Bz)$$

which implies that

$$Az = Bz.$$

Next we shall show that Az = Tz. For this

$$\begin{aligned} d(SAx_n, Tz) \leq\ & a_1\, d(SAx_n, A^2x_n) + a_2\, d(Tz, Bz) \\ & + a_3\, d(SAx_n, Bz) + a_4\, d(Tz, A^2x_n) \\ & + a_5\, d(A^2x_n, Bz) + a_6 d(TAx_n, T^2x_n) \\ & + a_7 d(TAx_n, Bz) + a_8\, d(Sz, A^2x_n) \\ & + a_9\, d(Sz, Bz) \end{aligned}$$

On taking the limits as n $\to \infty$, we get

$$d(Az, Tz) \leq (a_2 + a_4)\, d(Az, Tz)$$

which gives

$$Az = Tz.$$

Similarly we can prove Bz = Sz.

Thus

$$Az = Bz = Sz = Tz$$

Again

$$d(Sx_n, TBx_n) \leq a_1 d(Sx_n, Ax_n)$$

$$+ a_2 d(TBx_n, B^2x_n) + a_3 d(Sx_n, B^2x_n)$$

$$+ a_4 d(TBx_n, Ax_n) + a_5 d(Ax_n, B^2x_n)$$

$$+ a_6 d(Tx_n, Ax_n) + a_7 d(Tx_n, B^2x_n)$$

$$+ a_8 d(SBx_n, Ax_n) + a_9 d(SBx_n, B^2x_n)$$

Taking $n \to \infty$, we write

$$d(z, Bz) \leq (a_3+a_4+a_5+a_7+a_8+a_9)\, d(z, Bz)$$

$$\therefore \quad z = Bz$$

Thus $z = Az = Bz = Sz = Tz$.

Now we shall prove the uniqueness of z, suppose z and w are common fixed point of A, B, S and T, we write

$$d(z, w) = d(Sz, Tw)$$

$$\leq a_1 d(Sz, Az) + a_2 d(Tw, Bw)$$

$$+ a_3 d(Sz, Bw) + a_4 d(Tw, Az)$$

$$+ a_5 d(Az, Bw) + a_6 d(Tz, Az)$$

$$+ a_7 d(Tz, Bw) + a_8 d(Sw, Az)$$

$$+ a_9 d(Sw, Bw) = (a_3+a_4+a_5+a_7+a_8)\, d(z,w)$$

Thus $z = w$.

This complete the proof of the theorem.

Finally we give an example for the validity of our theorem.

Example 2. Let $X = (0, \infty)$ and $d(x, y) = |x-y|$

for $x, y \in X$

Define $Ax = x^2$, $Sx = 4x^2$,

$Bx = 9x^2$ $Tx = 16x^2$

for all x in X.

$|Sx-Ax| = 3x^2 \to 0$ if $x \to 0$ and

$|Asx - SAx| = 4x^2 \to 0$ if $x \to 0$

Thus $d(Ax, Sx) \to 0$ only if $x \to 0$ in which case

$d(SAx, SAx) \to 0$, So A and S are compatible.

But they are not weakly commuting and hence not commuting pair. Also it is easy to prove the condition (7.1.4) and (7.1.5).

For the choice $a_1 = a_2 = a_4 = a_6 = a_8 = \frac{1}{2}$ and

$a_3 = a_5 = a_7 = a_9 = 1/3$

Condition (7.1.6) is satisfied. Clearly 0 is the unique common fixed point of A, B, S and T.

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