CHAPTER VI

FIXED POINTS OF MEIR KEELER TYPE CONTRACTIVE MAPPINGS ON MENGES SPACE

6.1 Let $(X, d)$ be a complete metric space and $f$ is a mapping of $X$ into itself. The well known theorem of Banach [162] States.

**Theorem B.** If there exists a constant $k < 1$ such that for all $x, y$ in $X$

$$(6.1.1) \ d (fx, fy) \leq k \ d (x, y)$$

then $f$ has a unique fixed point.

Meir and Keeler [163] has defined weakly uniformly strict contraction

$$(6.1.2) \ \epsilon \leq d (x, y) < \epsilon + \delta \Rightarrow d (fx, fy) < \epsilon$$

and found a fixed point theorem analogous to Banach. Further Ciric [164] has replaced condition (6.1.2) by

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[162] Banach, S. \hspace{1cm} (2)
[163] Meir, A. and Keeler, E. \hspace{1cm} (65)
[164] Ciric, L.B. \hspace{1cm} (13)
(6.1.3) \( \varepsilon \leq \max \{d(x, y), d(x, fx), d(y, fy), \)
\[ \frac{1}{2} [d(x, fy) + d(y, fx)] < \varepsilon + \delta \]
implies \( d(fx, fy) < \varepsilon \).

This condition is named as weakly uniformly generalized strict contraction. We know that the mapping \( f \) in (6.1.2) is continuous, because it is obviously contractive.

In (6.1.3) \( f \) may not be continuous.

Maiti, Achari and Pal [165] and Maiti and Pal [166] has taken further assumption that functional \( F(x) = d(x, fx) \) is lower semi continuous. Further Park and Rhoades [167] has used some related contractive definitions:

Let \( \emptyset : \bar{p} \rightarrow [0, \infty] \) is upper semi continuous from the right on \( \bar{p} \) where \( p = \{d(x,y); x, y \text{ in } X\} \) and satisfy \( \emptyset(t) < t \) for each \( t > 0 \).

Boyd and Wong [168] considered map \( f \) satisfying,

[166] Maiti M; Pal, T.K. (60)
[167] Park, S. and Rhoades, B.E. (74)
(6.1.4) \( d(fx, fy) < \emptyset [d(x, y)] \)

With this idea Park and Rhoades (already mentioned) replaced (6.1.3) to

(6.1.5) \( d(fx, fy) \leq \max \{ \emptyset(d(x, y)), \emptyset(d(x, fx)), \emptyset(d(y, fy)), \frac{\emptyset(d(x, fy)) + \emptyset(d(y, fx))}{2} \} \)

We know that Jungck [169] has generalized Banach contraction principle using two maps. With this idea Park and Bac [170] replaced condition (6.1.2) by

(6.1.6) \( \varepsilon < d(fx, fy) < \varepsilon + \delta \Rightarrow d(gx, gy) < \varepsilon \)

where \( g(x) = f(x) \) and \( gx = gy \) whenever \( fx = fy \)

In 1984 Rhoades [171] had generalized the result of Park and Rhoades [172] involving a pair of mapping satisfying Meir-Keeler contraction to three mappings.


Theorem R.R. Let \((X, d)\) be a metric space, \(P, Q, T: X \times X\) be such that either \(PT = TP\) or \(QT = TQ\)

[170] Park, S. and Bac J.S. (72)
[171] Rhoades, B.E. (83)
[172] Park, S. and Rhoades, B.E. (74)
and \( P(X) \cup Q(X) \subseteq T(X) \), \( X \) be \( (P, Q, T) \) orbitally complete and also satisfy:

Given \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that

\[
(6.1.7) \quad \varepsilon < \max \{ d(Tx, Ty), \frac{1}{2} [d(Tx, Px) + d(Ty, Py)] \},
\]

\[
\frac{1}{2} [d(Tx, Qy) + d(Ty, Px)] < \varepsilon + \delta
\]

imply

\[
d(Px, Qy) < \varepsilon
\]

Further one of the following holds:

\[
(6.1.8) \quad d(Qx, Ty) \leq \max \{ d(y, Qx), d(y, Tx) \}
\]

for \( x, y \) with R.H.S. positive.

\[
(6.1.9) \quad d(Px, Ty) \leq \max \{ d(y, Tx), d(y, Px) \}
\]

for all \( x, y \) with R.H.S. positive.

The \( P, Q, T \) have a unique common fixed point \( z \in X \) and for and \( x_0 \in X \), every sequence \( \{T_{x_n}\} \) defined by

\[
Tx_{2n+1} = Px_{2n+2} = Qx_{2n+1}, \quad n = 0, 1 --- \text{converges to } z.
\]

Further Ganguli [175] has also proved a theorem for three maps.

We shall give some more references like

[175] Ganguli, A. (33)
Chung [176], Matkowski-Wegrzyk [177], Park-Moon [178], Shisheng [179], Singh [180], Singh and Virendra [181] and Yen-Chung [182].

In this Chapter we introduce Meir and Keeler type contractive conditions for three mappings in Menger spaces and prove fixed point theorem. Also mild continuity condition is used.

In 1942, Karl Menger [183] generalized the metric axioms by associating a distributive function with each pair of points of an abstract set X. (A distributive function is a mapping \( F: R \rightarrow R^+ \) which is nondecreasing, left continuous, with \( \inf F = 0 \) and \( \sup F = 1 \). Thus for any pair of point \( u, v \) of \( X \) we associate a distributive function denoted by \( F(u,v) \) (or \( F(u,v) \)) and for any positive number \( x, Fu, v(x) \) or \( F(u,v;x) \) as the probability that the distance between \( u \) and \( v \) is less than \( x \). This new theory of probabilistic metric spaces started to develop after the paper of Schweizer

[176] Chung, K.G. (10)
[177] Matkowski, J. and Wegrzyk, R. (64)
[178] Park, S. and Moon, K.P. (73)
[179] Shisheng, Z. (102)
[180] Singh, S.L. (103)
[181] Singh, S.L. and Virendra (106)
[182] Yen, C.L. and Chung, K.J. (119)
and Sklar [184]. For basic work in this direction we refer [185], [186], [187], [188], [189], [190], [191], [192], [193], [194] and [195].

Probabilistic metric space

**Definition 1.** A mapping $F: \mathbb{R} \to \mathbb{R}^+$ is called distributive function if it is non decreasing left continuous with $\inf F = 0$ and $\sup F = 1$. We shall denote by $L$ the set of all distribution functions.

**Definition 6.1.** A probabilistic metric space (P.M. space) is an ordered pair $(X, J)$ where $X$ is non-empty set and $J$ is a mapping from $X \times X$ to $L$. The value of $J$ at $(u, v) \in X \times X$ is denoted by $F_{u, v}$ (or $F(u, v)$). They satisfy

(P1) $F(u, v; x) = 1$ iff $u = v$, $x > 0$;

(P2) $F(u, v; 0) = 0$

(P3) $F(u, v) = F(v, u)$

(P4) if $F(u, v; x) = 1$, $F(v, w; y) = 1$ then $F(u, w; x + y) = 1$.

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Definition 6.2. A mapping \( t; [0, 1] \times [0, 1] \rightarrow [0, 1] \)
is called a t-norm if it satisfies

\[
\begin{align*}
(t \, 1) & \quad t(a, b) = t(b, a); \\
(t \, 2) & \quad t(a, b) \geq t(c, d) \text{ iff } a \geq c, \quad b \geq d; \\
(t \, 3) & \quad t(a, 1) > \text{ where } a > 0 \text{ and } t(1,1) = 1 \\
\end{align*}
\]

for all \( a, b, c, d \in [0, 1] \).

Definition 6.3. A menger space is a triplet \((X, J, t)\) where \((X, J)\) is a P.M. space and the t-norm \( t \) is such that

\[
F(u, w, x+y) \geq t[F(u, v; x), F(v, w; y)]
\]

\[u, v, w \in X, \quad x, y > 0\]

This is called Menger's triangular inequality.

Definition 6.4: The \((\varepsilon, \lambda)\)-neighboorhood of a point \( u \) in P.M. space \( X \) is a set

\[
\text{Nu}(\varepsilon, \lambda) = \{u \in X : F(u, v; \varepsilon) > 1 - \lambda\}, \quad \varepsilon, \lambda > 0
\]

Definition 6.5. A sequence \( \{u_n\} \) in a P.M. space is said to converge to \( u \) if for every \( \varepsilon, \lambda > 0 \), there exists an integer \( N \) \((\varepsilon, \lambda)\) such that

\[
F(u_n, u, \varepsilon) > 1 - \lambda \quad \text{for all } n \geq N
\]

Definition 6.6. A sequence \( \{u_n\} \) in a P.M. space is cauchy it for every \( \varepsilon, \lambda > 0 \), there exists an
integer $N ( \in , \lambda )$ such that for all $m, n \geq N$,

$$F (U_n, U_m, \in ) > 1 - \lambda$$

A P.M. Space is complete if every cauchy sequence in it converges to some point of the space.

6.2 Let $f, g, h$ be the self mappings on $X$. Consider the following conditions:

(6.2.1) For a given $\in > 0$, there exists a $\delta = \delta (t) > 0$

such that for $u, v \in X$,

$$\in \leq \max \{d(hu, hv), [d(fu, hu) d(gv, hv)]^{1/2},$$

$$[d(fu, hv) d(gv, hu)]^{1/2}\} < \in + \delta$$

implies $d(fu, gv) < \in$

Further we also write in place of (6.2.1)

(6.2.2) $\in < \min\{F(hu, hv, x), F(fu, hu; x) F(gv, hv; x)^{1/2}\}$

$$[F(fu, hv; x) F(gv, hu, x)]^{1/2}$$

We need some more definitions:

**Definition 6.7.** Three selfmapping $f, g, h$ on $X$ will be called Meir Keeler type contractive triplet if given $\in > 0$ there exists a $\delta = \delta (t) > 0$ such that

(6.2.3) $0 = k(u, v; t) < k (u, v; \in + \delta ) = 1$ implies

$$F (fu, gv; \in ) = 1.$$
Definition 6.8. Let $f$, $g$, $h$ be selfmappings on $X$, if there exists a point $u_0$ in $X$ and a sequence $\{u_n\}$ in $X$ such that

$$hu_{2n+1} = fu_{2n}; \quad hx_{2n+2} = gx_{2n+1}, \quad n = 0, 1, 2 \ldots$$

then the space $X$ is called $(f, g; h(u_0))$-orbitally complete with respect to $u_0$ or simply $(f, g, h)$ orbitally complete if the closure of $\{hu_n; n = 1, 2 \ldots\}$ is complete.

Definition 6.9. The mapping $h$ is called $(f, g, h(u_0))$ orbitally continuous if the restriction of $h$ on the closure of $\{hu_n; n = 1, 2 \ldots\}$ is continuous.

Definition 6.10. Two mappings $f$ and $h$ on $X$ will be called weakly commuting if

$$F(fhu, hfu; x) \geq F(hu, fu; x)$$

for all $u \in X$ and $x > 0$.

The following lemma is motivated by S.S. Chang [196] and proved by S.L. Singh and other [197].

Lemma. Let $\{y_n\}$ be a sequence in $X$ such that

$$\lim_{n \to \infty} F(y_n, y_{n+1}; \varepsilon) = 1 \quad \text{for any } \varepsilon > 0.$$

If $\{y_n\}$ is not cauchy then there exists $\varepsilon_0 > 0$ and two sequences of positive integers $\{m_i\}$ and $\{n_j\}$ such that

[196] Chang, S.S.

[197] Singh, S.L.; Pant, B.D. and Chamola, K.P.
\[ (i) \quad m_i > n_{i+1}, \quad i \to \infty, \quad n_i \to \infty, \]

\[ (ii) \quad F(y_m, y_n; \in_0) < 1-, \quad i = 1, 2, \ldots \]

\[ (iii) \quad F(y_{m-1}, y_n; \in_0) > - , \quad i = 1, 2 \ldots \]

Now we are in a position to prove:

**Theorem 1.** Let \( f, g, h \) be a Meir-Keeler type contractive triplet on \( X \). If there exists a point \( u_0 \) in \( X \) such that \( X \) is \( (f, g, h(u_0)) \)-orbitally complete then \( fz = yz = hz, \ z \in X, \) Further

\[ (6.2.4) \quad F(gu, hv; x) > \min\{F(v, gu; x), F(v, hu; x)\} \]

or

\[ (6.2.5) \quad F(fu, hv; x) > \min\{F(v, hu; x), F(v, fu; x)\} \]

then

\( f, g, h \) have a unique common fixed point and \( \{hu_n\} \) converges to a fixed point.

**Proof:** Condition (6.2.3) is equivalent to

\[ (6.2.6) \quad F(fu, gv; x) > \min\{F(hu, hv; x), \quad [F(hu, fu; x) F(hu, gv;x)]^{\frac{1}{2}}, \quad [F(hu, gv;x) F(hv, fu; x)]^{\frac{1}{2}} \]

Case I. Suppose \( hu_n = hu_{n+1} \) and \( hu_{n+1} \neq hu_{n+2} \) for some odd integer. By (6.2.6)
\[ F(h_{n+2}, h_{n+1}; x) = F(f_{n+1}, g_u; x) \]
\[ \geq \min \{ F(h_{n+1}, h_n; x), \]
\[ [F(h_{n+1}, h_{n+2}; x) F(h_n, h_{n+1}; x)]^{\frac{1}{2}}, \]
\[ [F(h_{n+1}, h_{n+1}; x) F(h_n, h_{n+2}; x)]^{\frac{1}{2}} \}

This implies

\[ h_{n+1} = h_{n+2} \text{ i.e. in general} \]
\[ h_{n+i} = h_n, \text{ i = 1, 2, \ldots} \]

Taking \( u_n = z \), we have

\[ h_u = h_{n+1} = g_u \]
\[ i.e. \ h_z = g_z \]

From the hypothesis,

let \( f_z \neq g_z \), then

\[ F(f_z, g_z; x) > F(f_z, g_z; x) \]
gives a contradiction,

But

\[ f_z = g_z \]

\[ \therefore \ P_z = Q_z = h_z \]

For the case, \( n \) is even, is dealt similarly.
Case II Suppose \( h_n \neq h_{n+1} \) for all integer. From the hypothesis, let \( Y = \inf d(h_n, h_{n+1}) \) and

\[
F(h_{n+1}, h_{n+2}; x) = F(f'_n, y_{n+1}; x) \\
< \max\{F(h_n, h_{n+1}; x), [F(h_{n+1}, h_n; x) \\
F(h_{n+2}, h_{n+1}; x)]^{\frac{1}{2}}, [F(h_{n+1}, h_{n+1}; x) \\
F(h_{n+1}, h_n; x)]^{\frac{1}{2}} \}
\]

which implies

\[
F(h_{n+1}, h_{n+2}; x) < F(h_n, h_{n+1}; x)
\]

Thus

\[
F(h_n, h_{n+1}; x)
\]

is monotonic increasing in \( n \) and converge to a number \( 0 < Y < 1 \). Let \( Y < 1 \). From this we infer that; every \( s \in (0, Y) \) there exists a positive integer \( N \) such that \( Y - s < \frac{1}{2} \)

\[
F(h_n, h_{n+1}; x) < Y \quad \text{for all} \quad n > N.
\]

Since the \( 2k > N \), then by hypothesis

\[
F(h_{2k+1}, h_{2k+2}; x) = F(f_{2k}, u_{2k+1}; x)
\]
\[ \min \{ F(h_{2k}, h_{2k+1}; x) \}
\]

\[ \left[ F(h_{2k+1}, h_{2k}; x) F(h_{2k+2}, h_{2k+1}; x) \right]^{\frac{1}{2}} \]

\[ \left[ F(h_{2k+1}, h_{2k+1}; x) F(h_{2k+2}, h_{2k}; x) \right]^{\frac{1}{2}} \]

\[ > f(h_{2k}, h_{2k+1}; x) \]

Thus

\[ F(h_{2k+1}, h_{2k+2}; x) > r, \text{ a contradiction} \]

imply \( r = 1 \).

To prove that the sequence \( \{h_n\} \) is a Cauchy sequence, suppose the contradictory to be true. Then by lemma, there exists \( \varepsilon_0 > 0 \) and two sequences of positive integers,

\( \{m_i\} \) and \( \{n_i\} \), such that

(i) \( m_i > n_{i+1}, n \to \infty, i \to \infty \);

(ii) \( F(h_{m_i}, h_{n_i}; \varepsilon_{0/2}) \leq 1 - \lambda, i = 1, 2, \ldots \)

(iii) \( F(h_{m_i-1}, h_{n_i}; \varepsilon_{0/2}) > 1 - \lambda, i = 1, 2, \ldots \)

With the help of (ii) and (iii) and taking limits, we have

\[ F(h_{m_i}, h_{n_i}; \varepsilon_0) \geq \min \{ F(h_{m_i}, h_{m_i-1}; \varepsilon_{0/2}) \} \]
\[ F(h_{u_i}^{m_l}, h_{u_i}^{n}, \subset_{0/2}) > \min(1, 1 - \lambda) = 1 - \lambda, \]

a contradiction.

Therefore \( \{h_{u_n}\} \) is a cauchy sequence since \( X \) is \((f, g, h(u_0))\)-orbitally complete, there exists \( z \) in \( X \) such that \( T_{u_n} \rightarrow z \).

Now if \((6.2.4)\) holds,
\[
F(g_{u_{2n+1}}, hz; x) > \min \{F(z, g_{u_{2n+1}}; x), F(z, h_{u_{2n+1}}; x)\}
\]
implies
\[
Tz = z.
\]

Similarly for \((6.2.5)\), \( Tz = z \).

Also since \( h_{u_m} \neq h_{u_{m+1}} \), we get by hypothesis
\[
F(z, gz; x) > F(z, gz; x)
\]
gives \( z = gz \), Similarly we get \( z = fz \).

Thus \( fz = gz = hz = z \).

**Theorem 2.** Let \( f, g, h : X \rightarrow X \) be a Meir-Keer type contractive triplet. If there exists a point \( u_0 \) in \( X \) such that \((f, g, h(u_0))\) be orbitally complete. If there exists a \( z \) in \( X \) such that \( f \) commutes with \( g \) and \( h \), then \( z \) is unique common fixed point of \( f, g, h \).
Proof: From theorem 1, we have

\[ fz = gz = hz = v \text{ and } hv = v \text{ then} \]

\[ h^2z \neq hz \text{ and so by hypothesis and the fact that } f \text{ commutes with } g \text{ and } h, \text{ we write} \]

\[ F(hz, h^2z; x) = F(fz, ghz; x) \]

\[ > \min \{ F(hz, hhz; x), \]

\[ ( F(fz, hz; x) F(ghz, hhz; x) \}^{\frac{1}{2}}, \]

\[ ( F(fz, hhz; x) F(ghz, hz; x) \}^{\frac{1}{2}} \]

\[ > F(hz, h^2z; z) \]

We get a contradiction and therefore \( hz = z \), which implies \( fu = fhz = hhz = hv = v \) and also \( gv = v \).

Therefore \( v \) is a common fixed point of \( f, g \) and \( h \).

Suppose \( v' \) is another fixed point of \( f, g \) and \( h \), \( v \neq v' \).

Thus by hypothesis

\[ F(v, v'; x) > F(v, v', x) \]

Thus \( v = v' \).