CHAPTER-V

COMMON FIXED POINT THEOREM FOR TWO SYSTEMS OF TRANSFORMATIONS IN 2-METRIC SPACE

5.1 The idea of 2-metric space have been investigated by Gahler and he introduced the concept of 2-metric space in a series of his papers [119, 120, 121] and also White [122] has developed the same idea further. Abstract properties of 2-metric space were suggested by the area function for a triangle determined by a triplet in Euclidean space, associated with a given 2-metric was a natural topology.

We need the definitions:

Definition 1. A 2- metric space is a space X with a non-negative real valued function d on X x X x X satisfying:

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[119] Gahler, S. (30)

[120] Gahler, S. (31)

[121] Gahler, S. (32)

[122] White, A.G. (116)

(5.1.1) $d(x, y, z) \neq 0$ for x, y, z in X,

(5.1.2) $d(x, y, z) = 0$ if at least two of x, y, z are equal,

(5.1.3) $d(x, y, z) = d(x, z, y) = d(y, z, x)$

(5.1.4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$.

<u>Definition</u> 2. A sequence $\{x_n\}$ in 2-metric space X is called convergent sequence if there is an x in X, such that

(5.1.5) $\lim_{n} d(x_n, x, a) = 0$ for all a in X.

<u>Definition</u> 3. A sequence $\{x_n\}$ in a 2- metric space X is called ;cauchy sequence, if

(5.1.6) $\lim_{n,m} d(x_n, x_m, a) = 0$ for all a in X.

<u>Definition</u> 4. A 2-metric space X in which every cauchy sequence is convergent to a point x, is a complete metric space.

We would like to give some references.

Andelafte [123], Borsan [124], Cho [125], [126] and [127], Diminnie [128], [129], [130], [131], [132], [133], Franic [134], Hsiago [135], Iseki [136], [137],

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[123]	Andelafte, E. and Freese, R.	(1)
[124]	Borsan, D.	(3)
[125]	Cho, J.	(6)
[126]	Cho, J.	(7)
[127]	Cho, J; Freese, R.U.	(8)
[128]	Diminnie, C; Gahler, S. and White,A.	(19)
[129]	Diminnie, C; Gahler, S. and White,A.	(20)
[130]	Dimminie, C; Gahler S. and White,A.	(21)
[131]	Diminnie, C; Gahler S. and White,A.	(22)
[132]	Dimminie, C; White,A.	(23)
[133]	Diminnie, C; White, A.	(24)
[134]	Franic, I.	(29)
[135]	Hsiago, C.	(37)
[136]	Iseki, K.	(40)
[137]	Iseki, K.	(41)

[138], [139], [140], [141], Kim [142], Newton [143] and Ram [144].

Jungck's fixed point theorem [145] was proved in 1976 and improved by Conserva[146], Das-Naik [147], Fisher [148], Jungck [149], Park [150], Rhoades [151], Sessa [152] and Tiwari- Singh [153] is generally called as Jungck contraction principle. On the other hand, with a view to generalize the Banach contraction principle, Matkowski [154] proved a fixed point theorem for a system of n-transformations on a product of n-metric spaces. This result has been extended and generalized by

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[138] Iseki, K. (42)
[139] Iseki, K. (43)
[140] Iseki, K. (44)
[141] Iseki, K. (45)
[142] Kim, S.S. (57)
[143] Newton, M. (68)
[144] Ram, B. (76)
[145] Jungck, G. (46)
[146] Conserva, V. (14)
[147] Das, K.M. and Naik, K.V. (18)
[148] Fisher, B. (26)
[149] Jungck, G. (47)
[150] Park, S. (71)
[151] Rhoades, B.E.; Sessa, S; Khan, M.S. and Swaleh, M. (85)
[152] Sessa, S. (98)
[153] Tiwari, B.M.L and Sing S.L. (114)
[154] Matkowski, J. (63)

Czerwick [155], [156], Kominek [157], Reddy and Subrahamanyam [158], Singh and Gairola [159], and Sing-Kulshreshtha [160].

Following Matkowski, we write

$$(5.1.7)\quad c'_{i,k} = \begin{cases} c_{i,h} & \text{for } i \neq k,\ i,k = 1,2,---n \\ 1- c_{i,k} & \text{for } i = k \end{cases}$$

and $c^{s+1}_{i,k}$ are defined recursevely by

$$(5.1.8)\quad c^{s+1}_{i,h} = \begin{cases} c^{s}_{1,1}\, c^{s}_{i+1\,,\,k+1} + c^{s}_{i+1,1}\, c^{s}_{i,k+1}, & \text{for } i \neq k \\ c^{s}_{1,1}\, c^{s}_{i+1,k+1} - c^{s}_{i+1,1}\, c^{s}_{i,k+1} & \text{for } i = k \end{cases}$$

$s = 1, --- n-1,\ i,\ k = 1, ---- n-s.$

In fact $[\ c^{s}_{i,k}]$ is an (n-s) x (n-s) matrix.

The following lemma is due to Matkowski;

Lemma, Let $c'_{i,k} > 0$; i, k = 1, --- n, then the system of inequalities

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[155] Czerwick, S. (16)

[156] Czeřwick, S. (17)

[157] Kominek (59)

[158] Reddy, K.B. and Subrahmanyam,P.V. (80)

[159] Singh, S.L. and Gairola,V.C. (107)

[160] Singh, S.L. and Kulshreshtha, C. (108)

(5.1.9) $\sum_{k=i}^{n} c_{i,k} \; r_h < r_i$, i = 1, 2, - - - n

has a solution $r_i > 0$, i = 1, 2, --- n iff

(5.1.10) $c^s_{i,i} > 0$, s = i, ---; i = 1, --- n+1-s.

Suppose that $r_i > o$, i = 0, i = 0, 1, -- n is the solution of the system of inequality (5.1.9).

We define

(5.1.11) $h = \max_i \left(r_i^{-1} \sum_{h=1}^{n} c_{ik} \; r_h \right)$, $r_h > 0$, $0 < h < 1$.

Let A_1 , A_2, ---- An be arbitrary sets.

$A = A_1 \times A_2 \times$ - - - X An and X_1, X_2, - - - x_n

be 2-metric spaces with metric di.

Let (X, d) be a metric space, we write following Nadler[161].

(5.1.12) cL(x) = { c : c is a non empty closed subset of X},

(5.1.13) N ($\in$, c) = {x $\in$ X: d(x, c) < $\in$ for c $\in$ c } , $\in > 0$, c $\in$ c L(x).

[161] Nadler, S.B. (67)

(5.1.4)

$$H(A,B) = \begin{cases} \inf \{\epsilon > 0: ACN(\epsilon, B) \text{ and } BCN(\epsilon, A)\} \text{ if the inf. exists otherwise } A,B \in CL(x) \\ \infty \end{cases}$$

The function H is called the Hausdorff distance for CL(x) induced by di. D (x, A) will denote the ordinary distance between $x \in X$ and $A \in CL(x)$.

Let (Xi, di), i = 1, 2, ---- n be 2-metric spaces.

Hi(A,B) i = 1, -----n will denote the generalized Hausdorff distance between two elements of CL(Xi),

i = 1, --- n Obtained from di, i = 1, --- n and Di(x,A) will denote the ordinary distance between $x \in Xi$,

$A = CL(x_i)$, i = 1 , ---- n

In this chapter, we present common fixed point theorem for two systems of multivalued transformations in 2-metric space.

In fact we prove:

<u>Theorem 1</u>. Let (Xi, di), i = 1, --- n be complete 2-metric spaces and $X = X_1 \times \cdots \times X_n$.

Let there be two systems of multivalued transformation Fi; Gi; $X \to CL(X_i)$, i = 1, ----n, --- n. If there exists non- negative numbers b, c, and $a_{i,k}$;

i, k = 1, ---- n such that

(5.1.15) $Hi\,[Fi(x_1 ---- x_n),\ Gi\,(y_1, - - - y_n),\ a_i]$

$$\leq \sum_{h=1}^{n} a_{i,k}\ dh\,(xh,\ yh,\ ah)$$

$$+\ b\,[Di\,\{x_i,\ Fi\,(x_1, -- x_n),\ a_i\}$$

$$+\ Di\,\{y_i\ ,\ Gi\,(y_1, -- x_n),\ a_i\}$$

$$+\ c\ [Di\,\{\ x_i,\ Gi\,(y_1, - - - y_{2n}),\ ai\}$$

$$+\ Di\,\{y_i,\ Fi(x_1, - - x_n),\ a_i\}$$

for all $xh,\ yh,\ a_h \in Xh$; $i,\ k = 1,\ --- n$ and the numbers $c^s_{i,k}$, $s = 1$, --- n; $i,\ k = 1$, --- $n+1-s$ defined in (5.1.7) and (5.1.8) satisfies (5.1.10) where b and c satisfies

(5.1.16) $0 \leq\ 2\,b + 2\,c\ < 1 - h$

Where h is defined in (5.1.11) and $0 < h < 1$. Then there exists an $\bar{x}_i$, $i = 1$, --- n such that

$\bar{x}_i \in\ Fi\ (\bar{x}_1, --- \bar{x}_n)\ \cap\ Gi\ (\bar{x}_1, -- \bar{x}_n)$, $i = 1,---n$.

Proof: Choose $x_i^o \quad X_i$, i = 1 , --- n and let a sequence $\{x_i^m\}$ in X_i such that

$$x_i^{2m+1} \quad Fi\,(x_1^{2m}\ , --- \ x_n^{2m}\ , \ a_i)$$

$$x_i^{2m+2} \quad Gi(x_1^{2m+1}\ ,--- \ x_n^{2m+1}\ , a_i)$$

$$di(x_i^{2m+1},\ x_i^{2m+2}\ ,\ a_i) < e\ Hi\ [Fi(x_1^{2m} -- x_n^{2m}),$$

$$Gi(x_1^{2m+1} --- x^{2m+1}),\ a_i]$$

m = 0 , 1, 2, ----- for some $e > 1$.

From (5.1.7), (5.1.8), (5.1.10), (5.1.11) and the lemna we may choose a system of positive number r_1 -- r_n such that

$$\sum_{h=1}^{n} c_{i,h}\ r_n < hr_i\ ; \quad i = 1 -----n.$$

We may assume (From the homogenicity of the system) that

$di\ (x_i^o,\ x_i\ ,\ a_i) < r_i$ and $r_i > 1$ for i = 1 , ---- n

Then from the hypothesis, we write,

$$di\ (x_i^1,\ x^2\ ,\ \ a_i) < e \sum^{n} a_{i,k}\ \ r_h$$

$$+ e\ (b+c)\ [d_i(x_i^o\ ,\ x_i^1\ ,\ a_i)$$

$$+ d_i\ (x_i^1,\ x_i^2\ ,\ a_i)]$$

$$< e\ (h+b+c)\ r_i + e(b+c)\ d_i(x_i^1,x_1^2,a_i),$$

Now, as observed by Czerwick [161A], h exists and $0 < h < 1$, also $0 \leqslant 2b + 2c + h < 1$, therefore

(5.1.17) $0 \leqslant 2\,eb + 2\,ec + eh < 1$

Hence

$$d_i\,(x_i^1, x_i^2, a_i) \leqslant \frac{e(h+b+c)}{1-e(b+c)}\, r_i$$

$$= q\, r_i, \quad i = 1, 2, ---n.$$

where

$$0 < q = \frac{e\,(h+b+c)}{1-\,e(b+c)} < 1$$

follows from (5.1.17)

Similarly

$$d_i\,(x_i^2, x_i^3, a_i) < e\,Hi\,[Fi(x_1^2, ---x_n^2),$$

$$Gi\,(x_1^1, ---x_n^1), a_i]$$

$$\leqslant e \sum_{h=1}^{n} a_{i,k}\, d_h\,(x_h^2, x_h^1, ah)$$

$$+ eb\,[Di\,\{x_i^2, Fi\,(x_1^2, ---x_n^2), a_i\}$$

$$+ Di\,\{x_i^1, Gi\,(x_1^1, ---x_n^1), a_i\}]$$

$$+ ec\,[Di\{x_i^2, Gi(x_1^1\, ---x_n^1), a_i\}$$

[161A] Czerwik (17)

$$+ Di \{x_i^1 , Fi (x_1^2, --- x_n^2) , a_i\}]$$

$$\leqslant eq \sum_{n=1}^{n} a_{i,h} r_h + e (b+c) q r_i$$

$$+ e (b+c) di (x_i^2 , x_i^3 , a_i)$$

So that

$$d_i(x_i^2 , x_i^3 , a_i) \leqslant q^2 r_i , \quad i = 1 , ---n$$

Inductively

$$di (x_i^m , x_i^{m+1} , a_i) \leqslant q^m r_i , i = 1, ----n.$$

Thus

$\{x_i^m\}$ is a cauchy sequence and

$$x_i^m \to Ui \in X_i , i = 1, --- n.$$

For each $1 \leqslant i \leqslant n$,

$$Di [x_i, Fi (x_1 , --- x_n), a_i]$$

$$\leqslant di (x_i, x_i^{2m+2} , a_i)$$

$$+ Di [Fi (\bar{x}_1 , ----\bar{x}_n), x_i^{2m+2} , a_i]$$

$$+ Di [x_1, Fi(\bar{x}_1, --- \bar{x}_n) , x_i^{2m+2}]$$

$$\leq d_i(\bar{x}_i, x_i^{2m+2}, a_i)$$

$$+ Hi\,[Fi(\bar{x}_1 --- \bar{x}_n), Gi(x_1^{2m+1}, ---- x_n^{2m+1}), a_i)$$

$$+ Hi\,[\bar{x}_i, Fi(\bar{x}_1, --- \bar{x}_n), Gi(x_1^{2m+1} --- x_n^{2m+1})]$$

$$\leq di(\bar{x}_i, x_i^{2m+2}, a_i)$$

$$+ \sum_{h=1}^{n} a_{i,h}\, dh(xh, xh^{2m+1}, ah)$$

$$+ b\,[Di\{\bar{x}_i, Fi(\bar{x}_1 --- \bar{x}_n), a_i\}$$

$$+ d_i(x_i^{2m+1}, x_i^{2m+2}, a_i)]$$

$$+ c\,[di(\bar{x}_i, x_i^{2m+2}, a_i)$$

$$+ Di\{x_i^{2m+1}, F(\bar{x}_1 ---- \bar{x}_n), a_i\}]$$

$$+ Di\{\bar{x}_i, Fi(\bar{x}_1, ---- \bar{x}_n)\; x_i^{2m+2}\}$$

Making $m \to \infty$ we obtain

$$\bar{x}_i \in Fi(\bar{x}_1, ---- \bar{x}_n), i = 1, ----- n$$

Since,

$Fi(\bar{x}_1, ---- \bar{x}_n)$ is closed, $i = 1, 2, --- n$.

Similarly

$$\bar{x}_i \in Gi\ (\bar{x}_1\ ,-----\bar{x}_n),\ i = 1,\ 2,\ ----n.$$

Thus the proof is complete.

Remark. For metric space we arrive at the result of Nadler, Reich, Iseki.

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