The main objective of this thesis is to find a large class of weight functions that admits a positive principal eigenvalue for the weighted eigenvalue problems for the Laplacian and the $p$-Laplacian. More specifically, for a connected domain $\Omega$ in $\mathbb{R}^N$ with $N \geq 2$, we study sufficient conditions for a function $g \in L^1_{\text{loc}}(\Omega)$ to admit a pair $(\lambda, u)$, with $\lambda \in \mathbb{R}^+$ and $u > 0$ a.e. such that $u$ is a weak solution of the following problem:

$$-\Delta_p u = \lambda g |u|^{p-2}u, \quad \text{in } \Omega,$$

where $1 < p < N$ and $\Delta_p u := \text{div}(\nabla |u|^{p-2} \nabla u)$ is the $p$-Laplace operator. Such a $\lambda$, if exists, is called a principal eigenvalue of (1). In the literature, most of the sufficient conditions for the existence of a positive principal eigenvalue demand that the weight function $g$ or its positive part $g^+$ to be in $L^{\frac{N}{p}}(\Omega)$. However, in the field of applications one may need to consider weights that are not belonging to any of the Lebesgue spaces.

We look for a weak solution of (1) in $\mathcal{D}^{1,p}_0(\Omega)$, where

$$\mathcal{D}^{1,p}_0(\Omega) := \text{completion of } \mathcal{C}_c^\infty(\Omega) \text{ with respect to } \|\nabla \cdot\|_p \text{ norm}.$$ 

Now the existence of a positive principal eigenvalue for (1) is closely related with the existence of a minimizer for the functional $J(u) = \int_\Omega |\nabla u|^p$ on the level set $\mathcal{M}_p = \{ u \in \mathcal{D}^{1,2}_0(\Omega): \int_\Omega g |u|^p = 1 \}$. If the map $G, G(u) = \int_\Omega g^+ |u|^p$, is compact, then a direct variational method ensures the existence of a min-
imizer for $J$ on $\mathcal{M}_p$. If $g^+$ is in $L^N_\infty(\Omega)$, the dual of $L^{p^*}_\infty(\Omega)$, then the map $G$ is compact. This is mainly a consequence of three facts (i) the continuous embedding of $\mathcal{D}_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ (ii) the compactness of the embedding of $\mathcal{D}_0^{1,p}(\Omega)$ into $L^p_{loc}(\Omega)$ (iii) the density of $\mathcal{C}_c^\infty(\Omega)$ in $L^{p^*}_\infty(\Omega)$.

The main novelty of our results is that we allow weights that are not in any of the Lebesgue spaces, but only in certain weak Lebesgue spaces. For this we make use of the finest embedding of $\mathcal{D}_0^{1,p}(\Omega)$ into the Lorentz space $L^{(p^*;1)} = \{ u \in C(\mathbb{R}_+^*) : \int_0^{\infty} [t^{(1/p)\frac{1}{p^*}} u^*(t)]^p \frac{dt}{t} < \infty \}$, where $u^*$ denotes the one dimensional decreasing rearrangement of $u$. The Lorentz space $L(p^*,p)$ is a Banach space with a suitable norm and it is a proper subspace of the Lebesgue space $L^{p^*}(\Omega)$. However, when $g^+$ is in the Lorentz space $L(\frac{N}{p},\infty)$ (the dual space of $L(\frac{p^*}{p},\frac{1}{p})$), the map $G$ is not necessarily compact. For example, when $g(x) = \frac{1}{|x|^p}$ and $\Omega$ contains the origin, the map $G$ is not compact, indeed $\frac{1}{|x|^p}$ is in $L(\frac{N}{p},\infty)$. In this thesis, we find a large class of admissible weights in $L(\frac{N}{p},\infty)$ that admits a minimizer for $J$ on $\mathcal{M}_p$. Namely, the closure of $\mathcal{C}_c^\infty(\Omega)$ in $L(\frac{N}{p},\infty)$ that will henceforth be denoted by $\mathcal{F}_{\frac{N}{p}}$:

$$\mathcal{F}_{\frac{N}{p}} := \overline{\mathcal{C}_c^\infty(\Omega)}^{\| \cdot \| L(\frac{N}{p},\infty)} \subset L(\frac{N}{p},\infty).$$

We prove that $J$ admits a minimizer on $\mathcal{M}_p$, when $g^+$ is in $\mathcal{F}_{\frac{N}{p}}$.

We consider the case $p = 2$ separately, because of the simple and rich theory available for the Laplacian. Moreover, certain results that hold for the $p$-Laplacian hold good for the Laplacian under weaker assumptions. For $g^+ \in \mathcal{F}_{\frac{N}{p}}$, using a variant of the strong maximum principle, we show that $\lambda_1$, the minimum of $J$ on $\mathcal{M}_p$, is indeed a principal eigenvalue of (1). A necessary condition, namely a Pohozaev type identity for the existence of a principal eigenvalue is obtained for certain class of weight functions. The radial symmetry of the eigenfunctions corresponding to $\lambda_1$ are also discussed when $\Omega$ is $\mathbb{R}^N$ or a ball centred at the origin. The existence of a nontrivial
solution branch for certain types of nonlinear equations is studied as an application of the weighted eigenvalue problem using the bifurcation theory. The existence of an infinite sequence of eigenvalues is obtained using the Ljusternik-Schirelmann theory. The weighted eigenvalue problem for the Laplacian on bounded domains in $\mathbb{R}^2$ is also studied. We obtain various sufficient conditions for the existence of a positive principal eigenvalue by making use of the optimal embeddings of $H^1_0(\Omega)$ in the classes of Orlicz and Lorentz-Zygmund spaces.