APPENDIX A

DEGREE THEORY

In this section we define the degree for certain nonlinear map defined on a subset $D \subset X$, with values in $X^*$. First we give definitions of certain classes of functions useful for defining the degree for mapping between $X$ and $X^*$. For more detailed readings in this topic we refer to the book of Skrypnik [71].

**Definition A.0.8.** Let $X$ be a Banach space and let $A$ be a map defined on a subset $D \subset X$, with values in $X^*$. Then

(a) $A$ is called monotone mapping, if the following inequality

$$\langle Au - Av, u - v \rangle \geq 0;$$

holds for all $u, v \in D$

(b) The map $A$ is said to be demicontinuous on $D$, if for any sequence $u_n \in D$ converging strongly to $u_0 \in D$ then

$$\lim_{n \to \infty} \langle Au_n, v \rangle = \langle Au_0, v \rangle, \forall v \in X.$$

(c) $A$ is said to be bounded if it carries bounded subsets of $D$ into bounded subsets of $X^*$.

**Definition A.0.9.** Let $X$ be a Banach space, $A$ be a map defined on a subset $\overline{D} \subset X$, with values in $X^*$ and let $F \subset \overline{D}$. Then

(a) $A$ is said to satisfy the condition $\alpha_0(F)$, if $u_n \in F$ such that

$$u_n \rightharpoonup u, \quad Au_n \rightharpoonup 0$$
and
\[ \lim_{n \to \infty} \langle Au_n, u_n - u_0 \rangle \leq 0 \] (A.1)
then the sequence \( u_n \) converges strongly to \( u_0 \).

(b) \( A \) satisfies the condition \( \alpha(F) \), if \( u_n \in F \) such that \( u_n \rightharpoonup u \) and
\[ \lim_{n \to \infty} \langle Au_n, u_n - u_0 \rangle \leq 0 \] (A.2)
then the sequence \( u_n \) converges strongly to \( u_0 \).

Note that (b) is a stronger condition than (a).

**Definition A.0.10.** Let \( F \subset \overline{D} \). Then
\[ A_0(D, F) = \{ A : \overline{D} \to X^* : A \text{ bounded, demicontinuous, satisfies } \alpha_0(F) \} \]
\[ A(D, F) = \{ A : \overline{D} \to X^* : A \text{ bounded, demicontinuous, satisfies } \alpha(F) \} \]

**Definition of the Degree**

We define \( \text{Deg}(A, \overline{D}, 0) \)-the degree of a mapping \( A \) on the set \( \overline{D} \) with respect to the origin of the space \( X^* \)- under the following conditions:

(a) \( A \in A_0(D, \partial D) \);

(b) \( Au \neq 0 \) for any \( u \in \partial D \).

First we define the degree when \( X \) is real separable reflexive Banach space. Let \( \{ v_i : i \in \mathbb{N} \} \) be a Schauder basis for \( X \) and let \( F_n \) be the linear span of \( v_1, \ldots, v_n \).

Now for each \( n \in \mathbb{N} \), we define the finite-dimensional approximation \( A_n \) of the mapping \( A \) in the following way:
\[ A_n u = \sum_{i=1}^{n} \langle Au, v_i \rangle v_i, \text{ for } u \in \overline{D_n}, \quad D_n = D \cap F_n. \] (A.3)

Note that \( A_n \) is mapping from \( \overline{D_n} \subset F_n \) into \( F_n \). Next theorem shows that under the assumptions (a) and (b) one can define the Brouwer degree of \( A_n \).

**Theorem A.0.11.** Let \( A \) be a mapping satisfying conditions (a) and (b). Then there exists \( n_0 \) such that for \( n \geq n_0 \) the following assertions hold:

1. the equation \( A_n u = 0 \) has no solutions belonging to \( \partial D_n \),
2. The degree \( \deg(A_n, D_n, 0) \) of the mapping \( A_n \) on the set \( D_n \) with respect to 0 \( \in F_n \) is defined and independent of \( n \).

For a proof see Theorem 2.1 of [71].

By the above theorem, \( \lim_{n \to \infty} \deg(A_n, D_n, 0) \) exists and we denote it by \( \deg(A, D, 0, \{v_i\}) \). Next we show that \( \deg(A, D, 0, \{v_i\}) \) is independent of the choice of the Schauder basis. For a proof of next theorem, see Theorem 2.2 [71].

**Theorem A.0.12.** Suppose that the conditions (a) and (b) are satisfied. Then the limit

\[
\deg(A, D, 0, \{v_i\}) = \lim_{n \to \infty} \deg(A_n, D_n, 0)
\]

does not depend on the choice of the Schauder basis \( \{v_i : i \in \mathbb{N}\} \).

Using the above two theorems one can justify the following definition of degree when \( X \) is a reflexive separable Banach space.

**Definition A.0.13.** Let \( A \) be a map satisfying conditions (a) and (b). Then the degree on the set \( D \) with respect to the point 0 \( \in X^* \) is defined as

\[
\deg(A, D, 0, 0) = \lim_{n \to \infty} \deg(A_n, D_n, 0),
\]

where \( A_n, D_n \) are as defined in (A.3) and this limit is denoted by \( \text{Deg} (A, D, 0) \).

Next we define degree when \( X \) is nonseparable reflexive Banach space. We denote the set of all finite-dimensional subspaces of \( X \) by \( F(X) \). Let \( F \in F(X) \) and let \( v_1, v_2, \ldots, v_n \) be a basis for \( F \). We define the finite-dimensional mapping

\[
A_F(u) = \sum_{i=1}^{n} \langle Au, v_i \rangle v_i, \quad D_F = (D \cap F).
\]

Next we state a theorem without a proof, see Theorem 3.1 in [71] for a proof.

**Theorem A.0.14.** Let \( A : D \to X^* \) be a demicontinuous operator satisfying condition \( \alpha(\partial D), \) where \( \partial D \) is the boundary of a bounded open set \( D \subset X \) and \( Au \neq 0 \) for \( u \in \partial D \). Then there exists a subspaces \( F_0 \in F(X) \) such that any subspace \( F \in X \) containing \( F_0 \) satisfies the properties:

1. the equation \( A_F(u) = 0 \) has no solution on \( \partial D_F \),
2. \( \deg(A_F, \overline{D_F}, 0) = \deg(A_{F_0}, \overline{D_{F_0}}, 0) \), where \( \deg \) is the degree of the finite-dimensional mapping.

The above theorem justifies the following definition of degree.

**Definition A.0.15.** Under the conditions of Theorem 3.1, the number

\[
\operatorname{Deg}(A, \overline{D}, 0) = \deg(A_{F_0}, \overline{D_{F_0}}, 0)
\]

is called the degree of the mapping \( A \) on the set \( \overline{D} \) with respect to the point \( 0 \in X^* \). Here \( A_F, D_F \) are defined according to (A.4) and \( F_0 \) is the finite dimensional subspace determined by Theorem 3.1.

**Properties of the degree**

The degree of a mapping, introduced above, possesses all the natural properties of Brouwer degree of finite dimensional mapping and Leray-Schauder degree for the compact perturbation of the identity in the infinite dimensional spaces. In this sections we discuss some important properties that are useful for proving the existence of bifurcation branches for certain nonlinear maps.

**Theorem A.0.16.** Let \( A : \overline{D} \rightarrow X^* \) be a mapping of class \( \alpha_0(D) \) and suppose that

\[
Au \neq 0 \text{ for } u \in \partial D.
\]

Then \( \operatorname{Deg}(A, \overline{D}, 0) = 0 \).

For a proof of theorem see Theorem 4.3 of [71]. From this theorem one can obtain the following corollary which states sufficient conditions for the solvability of the equation \( Au = 0 \).

**Corollary A.0.17.** Let \( A : \overline{D} \rightarrow X^* \) be a mapping of class \( \alpha_0(D) \). If \( Au \neq 0 \) for \( u \in \partial D \) and \( \operatorname{Deg}(A, \overline{D}, 0) \neq 0 \), then equation \( Au = 0 \) has at least one solution in \( \overline{D} \).

The above corollary answers the solvability of nonlinear operator equations and nonlinear boundary value problems for which one can show that the degree is nonzero. Next theorem gives a condition under which the degree is nonzero.

**Theorem A.0.18.** Let \( A : \overline{D} \rightarrow X^* \) be a mapping of class \( \alpha_0(D, \partial D) \). Suppose that \( 0 \in \overline{D} \setminus \partial D \) and

\[
\langle Au, u \rangle \geq 0, \quad Au \neq 0 \tag{A.5}
\]
for $u \in \partial D$. Then $\text{Deg}(A, \overline{D}, 0) = 1$.

**INDEX OF A MAP**

Let $D$ be a bounded open set in a separable reflexive Banach space $X$, $A : D \to X^*$ a mapping of class $A_0(D)$.

**Definition A.0.19.**

(i) A point $u_0 \in D$ is a zero of the mapping $A$ if $Au_0 = 0$ and it is called isolated zero, if there exist $r_0 > 0$ such that the ball $B_{r_0}(u_0)$ do not contain any other zeros of $A$.

(ii) The map $A$ is called a potential operator if $A$ is the gradient of a functional.

If $u_0$ is an isolated zero, then one can establish that the equality

$$\text{Deg} \left( A, B_{r_0}(u_0), 0 \right) = \text{Deg} \left( A, B_r(u_0), 0 \right)$$

holds for $0 < r < r_0$. Thus we have the following definition:

**Definition A.0.20.** The number

$$\lim_{r \to 0} \text{Deg}(A, \overline{B_r(u_0)}), 0)$$

is called the index of the mapping $A$ at the isolated zero $u_0$ and it is denoted by $\text{Ind}(A, u_0)$.

**Theorem A.0.21.** Suppose that a functional $F : X \to \mathbb{R}$ has a local minimum at $u_0$ and it is an isolated critical point of $F$. If the derivative $F'$ is of class $\alpha(X)$ then $\text{Ind}(F', u_0) = 1$.

See Theorem 6.1 of [71] The index of the mapping is useful concept to study the existence of solution branches of certain nonlinear operator equations. Let $U$ be a neighbourhood of the origin in a separable reflexive Banach space $X$, $A, T : X \to U \to X^*$ nonlinear mappings satisfying the conditions:

(a) $A$ is a mapping of class $A(U)$ and $A(0) = 0$,

(b) $T$ is weakly compact and $T(0) = 0$. 

Bifurcation theorem

We study the existence of bifurcation point for the following nonlinear operator equation

\[ Au + \lambda T u = 0. \tag{A.6} \]

**Definition A.0.22.** A real number \( \lambda_0 \) is called a bifurcation point for the equation \((A.6)\) if for any \( \varepsilon > 0 \) there exist \( u_\varepsilon \in U, \lambda_\varepsilon \in \mathbb{R} \) such that \( |\lambda_\varepsilon - \lambda_0| < \varepsilon, \ 0 < \|u_\varepsilon\| < \varepsilon \) and

\[ Au_\varepsilon + \lambda_\varepsilon T u_\varepsilon = 0. \]

Without loss of generality one may assume that there exists a \( \delta_0 > 0 \) such that zero is an isolated critical point of the mapping \( A + \lambda T \) for \( |\lambda - \lambda_0| < \delta_0 \), since otherwise \( \lambda_0 \) itself would be a bifurcation point. Then the index at zero, \( \Ind(A + \lambda T, 0) \) is well defined for the mapping \( A + \lambda T \) for \( |\lambda - \lambda_0| < \delta_0 \). Let

\[ i_0^\pm = \limsup_{\lambda \to \lambda_0^\pm} \Ind(A + \lambda T, 0). \]

Next theorem states sufficient conditions under which \( \lambda_0 \) is a bifurcation point for the equation \((A.6)\):

**Theorem A.0.23.** Let mappings \( A, T \) satisfy the conditions a),b) and assume that at least two of the numbers

\[ i_0^-, i_0^+, i_0^+, i_0^+, \quad \Ind(A + \lambda T, 0) \]

are distinct. Then \( \lambda_0 \) is a bifurcation point of the Eq: \((A.6)\).
Appendix B

Orlicz spaces

Here we recall the definition and some of the properties of an N-function, Orlicz function, Orlicz spaces etc. For more detailed discussions on these topics, we refer to [2].

Definition B.0.24. (N-function): An N-function is a real valued function A defined on $[0, \infty)$ satisfying the following conditions:

(i) $A$ is continuous on $[0, \infty)$;

(ii) $A$ is strictly increasing;

(iii) $A$ is convex;

(iv) $\lim_{t \to 0} \frac{A(t)}{t} = 0$ and $\lim_{t \to \infty} \frac{A(t)}{t} = \infty$;

(v) if $s > t > 0$, then $\frac{A(s)}{s} > \frac{A(t)}{t}$.

Example B.0.25. The following are some of the classical examples of N-functions:

(i) $A_p(t) = \frac{t^p}{p}$, for $1 < p < \infty$,

(ii) $t \exp(t) - t$,

(iii) $\exp(t^2) - 1$.

Definition B.0.26. (Orlicz function): A continuous and convex real-valued function A, defined on $[0, \infty)$ is called an Orlicz function if,

(i) $A(0) = 0$,

(ii) $\lim_{t \to \infty} \frac{A(t)}{t} = \infty$.

A is said to be degenerate, if $A(t) = 0$, for some $t > 0$. 

151
Example B.0.27. The function \( A(t) = \exp(t) - 1 \), is not an \( N \)-function as condition (iv) is not satisfied. Similarly \( B(t) = t \log^+(t) \) is not an \( N \)-function, since condition (ii) is violated. However one can easily verify that \( A \) and \( B \) satisfy all the conditions of an Orlicz function. Further, \( B \) is an example of a degenerate Orlicz function.

Definition B.0.28. (Conjugate convex function): Let \( A \) be an Orlicz function, we define its conjugate convex function \( \tilde{A} \) as below:

\[
\tilde{A}(s) = \max_{t \geq 0} \{st - A(t)\}. \tag{B.1}
\]

One can verify that \( \tilde{A} \) also satisfies all the conditions for an Orlicz function. Further, from the definition of the conjugate convex function, one can obtain the following Young's inequality:

Definition B.0.29. (Young's inequality): Let \( \tilde{A} \) be the conjugate convex function of the Orlicz function \( A \). Then

\[ st \leq A(t) + \tilde{A}(s), \quad s, t > 0. \]

Definition B.0.30. (Dominance and Equivalence of Orlicz functions): Let \( A \) and \( B \) be two Orlicz functions, we say that \( B \) dominates \( A \) globally if there exists a positive constant \( k \) such that

\[ A(t) \leq B(kt) \tag{B.2} \]

holds for all \( t \geq 0 \). Similarly, \( B \) dominates \( A \) near infinity if there exist positive constants \( t_0 \) and \( k \) such that (B.2) holds for all \( t \geq t_0 \). The two Orlicz functions \( A \) and \( B \) are equivalent globally (resp. near infinity) if each dominates the other one globally (resp. near infinity).

Definition B.0.31. (The \( \Delta_2 \) Condition): An Orlicz function, \( A \) is said to satisfy global \( \Delta_2 \)-condition if there exists a positive constant \( k \) such that for every \( t \geq 0 \),

\[ A(2t) \leq kA(t). \tag{B.3} \]

Similarly, \( A \) satisfies a \( \Delta_2 \) condition near infinity if there exists \( t_0 > 0 \) such that (B.3) holds for all \( t \geq t_0 \).

Definition B.0.32. (The Orlicz Class \( K_A(\Omega) \)): Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and let \( A \) be an \( N \)-function. The Orlicz class \( K_A(\Omega) \) is the set of all (equivalence classes modulo equality a.e. in \( \Omega \) of) measurable functions \( u \).
defined on $\Omega$ that satisfy
\[ \int_\Omega A(|u(x)|)dx < \infty. \]

Since $A$ is convex, $K_A(\Omega)$ is always a convex set of functions but it may not be a vector space; for instance, there may exist $u \in K_A(\Omega)$ and $\lambda > 0$ such that $\lambda u \notin K_A(\Omega)$.

**Definition B.0.33. (\(\Delta\)-regular):** We say that the pair $(A, \Omega)$ is $\Delta$-regular if either

(a) $A$ satisfies a global $\Delta_2$-condition, or

(b) $A$ satisfies a $\Delta_2$-condition near infinity and $|\Omega|$ is finite.

**Definition B.0.34. (Orlicz space $L_A(\Omega)$):** The Orlicz space $L_A(\Omega)$ is the linear hull of the Orlicz class $K_A(\Omega)$. Let
\[ k_u := \inf \left\{ k > 0 : \int_\Omega A\left( \frac{|u(x)|}{k} \right) dx \leq 1 \right\}. \]

The functional $k_u$ is a norm (Luxemburg norm) on $L_A(\Omega)$, and $L_A(\Omega)$ is complete with respect to the Luxemburg norm.

**Definition B.0.35. (The Space $E_A(\Omega)$):** $E_A(\Omega)$ is the closure of $C_c(\Omega)$ in $L_A(\Omega)$.

**Proposition B.0.36** (Some Properties of Orlicz spaces). Let $A$ be an Orlicz function and let $A$ be its conjugate Orlicz function.

(i) $K_A(\Omega) = E_A(\Omega) = L_A(\Omega)$ if and only if $(A, \Omega)$ is $\Delta$-regular.

(ii) Hölder inequality: Let $f \in L_A(\Omega), g \in L_\tilde{A}(\Omega)$. Then
\[ \int_\Omega |fg| \leq 2\|f\|_A\|g\|_{\tilde{A}}. \]

(iii) $C^\infty(\Omega)$ is dense in $E_A(\Omega)$.

(iv) $E_A(\Omega)$ is separable.

**Theorem B.0.37** (An Embedding Theorem for Orlicz Spaces). The embedding
\[ L_B(\Omega) \to L_A(\Omega) \]
holds if and only if either

(a) $B$ dominates $A$ globally, or

(b) $B$ dominates $A$ near infinity and $|\Omega| < \infty$. 