Chapter 7

**Bifurcation for the $p$-Laplacian**

In this chapter we study the existence of a bifurcation branch for the following type of equation:

$$-\Delta_p u = \lambda g |u|^{p-2}u + \lambda f r(u), \quad \text{in } D_0^{1,p}(\Omega). \quad (7.1)$$

where $\Omega$ is an open connected subset of $\mathbb{R}^N$ with $1<p<N$ and $g, f \in L^1_{\text{loc}}(\Omega)$. Further, we assume that $r \in C(\mathbb{R})$ and $r(0) = 0$.

As before we look for the weak solutions of (7.1). Note that for each $\lambda$, $u = 0$ is a trivial solution for (7.1). We say that a real number $\lambda_0$ is a bifurcation point for the equation (7.1), if for any $\varepsilon > 0$ there exist $u_\varepsilon \in D_0^{1,p}(\Omega) \setminus \{0\}$, $\lambda_\varepsilon \in \mathbb{R}$ such that $|u_\lambda| < \varepsilon, |\lambda_\varepsilon - \lambda_0| < \varepsilon$, and the pair $(\lambda_\varepsilon, u_\varepsilon)$ satisfies (7.1) in the weak sense, i.e,

$$\int_\Omega |\nabla u_\varepsilon|^{p-1} \nabla u_\varepsilon \cdot \nabla v = \lambda_\varepsilon \int_\Omega g |u_\varepsilon|^{p-2}u_\varepsilon v + f r(u_\varepsilon) v, \quad \forall v \in D_0^{1,p}(\Omega). \quad (7.2)$$

There are several results available in the literature on the existence of bifurcating branches of (7.1) for both bounded and unbounded domain $\Omega$, for example, [31, 34]. In all these earlier works, authors assumed that $f$ and $g$ are bounded and lie in $L^\infty(\Omega)$. Here we allow $f, g$ to be in certain weak Lebesgue spaces and they are not assumed to be bounded.

Here we make the following assumptions on the functions $r, g$ and $f$:

(B1) \begin{align*}
  r &\in C(\mathbb{R}), \quad |r(s)| \leq |s|^\gamma - 1, \quad \gamma \in [1, p^*), \text{ where } p^* = \frac{Np}{N-p}, \\
  \lim_{|s| \to 0} \frac{|r(s)|}{|s|^{p-1}} &\to 0, \text{ if } 1 \leq \gamma \leq p.
\end{align*}
Bifurcation for the $p$-Laplacian

(B2) \[ \begin{align*}
g & \in \mathcal{F}_N, \quad g^+ \neq 0, \\
f & \in \left\{ \begin{array}{ll} 
\mathcal{F}_p & \text{if } \gamma \geq p, \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1, \\
\mathcal{F}_{\frac{N}{p}} & \text{if } 1 \leq \gamma < p.
\end{array} \right.
\end{align*} \]

We use a topological degree argument as in [34] for proving the existence of a solution branch of (7.1) bifurcating from the trivial branch of zero solutions. Leray and Schauder extended the finite dimensional degree theory for certain maps from an infinite dimensional Banach space to itself. More precisely they define the degree for the compact perturbations of the identity. See [52, 35] for the definition and the important properties of the Leray-Schauder degree. We emphasize that the Leray-Schauder degree is defined for the maps from a Banach space to itself. Here, the functional framework for the equations of the type (7.1) leads to an equation involving certain maps between $D_{0}^{1,p}(\Omega)$ and its dual. In order to study the bifurcation property of (7.1) via topological degree theory, one needs to extend the definition of the degree for certain maps between a Banach space and its dual. In Appendix A, the degree is defined for certain classes of maps from a Banach space to its dual, see [71] for more details on this topic.

Under the assumptions (B1) and (B2), using Theorem A.0.23 we show that the first eigenvalue $\lambda_1$ of the following problem

\[ - \Delta_p u = \lambda g|u|^{p-2}u \quad \text{in } D_0^{1,p}(\Omega). \quad (7.3) \]

is a bifurcation point of equation (7.1).

In order to apply A.0.23 we need to introduce a suitable functional framework for equation (7.1). Throughout this chapter we assume that $\Omega$ is an open connected subset of $\mathbb{R}^N$ with $1 < p < N$ and $g, f \in L^1_{\text{loc}}(\Omega)$. We denote $D_0^{1,p}(\Omega)$ by $X$ and its dual by $X^*$. Under the assumptions (B1), (B2) recall the definition of the following functionals on $X$:

\[ J_p(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p, \]

\[ G_p(u) := \frac{1}{p} \int_{\Omega} g|u|^p. \]

It is easy to see that $J_p$ is differentiable and the derivative $J'_p : X \to X^*$ is given by

\[ \langle J'_p(u), v \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v, \quad \forall v \in X. \quad (7.4) \]
Similarly using the assumption $g \in \mathcal{F}_N$, one can show that the map $G_p$ is differentiable and the derivative $G'_p : X \to X^*$ is given by

$$\left\langle G'_p(u), v \right\rangle = \int_\Omega g |u|^{p-2} u v, \quad \forall v \in X. \quad (7.5)$$

Next we define a class of maps from $X$ to $X^*$, for which we can define the degree.

**Definition 7.0.3.** Let $A : X \to X^*$, we say that $A$ is of class $\alpha(X)$, if whenever $u_n \rightharpoonup u_0$ (weakly) in $X$ and

$$\lim_{n \to \infty} \langle Au_n, u_n - u_0 \rangle \leq 0 \quad (7.6)$$

then the sequence $\{u_n\}$ converges to $u_0$ (strongly).

**Remark 7.0.4.** The class $\alpha(X)$ is invariant under perturbations by compact functions. i.e. if $A$ is of class $\alpha(X)$ and if $K$ is compact map from $X$ to $X^*$, then $A + K$ is also of class $\alpha(X)$.

The definition of degree for the maps in $\alpha(X)$ and its properties are given in Appendix A. Next we prove the the following proposition:

**Proposition 7.0.5.** The map $J'_p$ is of class $\alpha(X)$.

**Proof.** First we show that $J'_p$ is monotone. For $u, v \in X$, we have

$$\left\langle J'_p(u), v \right\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \leq \|u\|^{p-1} \|v\|.$$

$$\left\langle J'_p(u) - J'_p(v), u - v \right\rangle = \|u\|^p + \|v\|^p - \left\langle J'_p(v), u \right\rangle - \left\langle J'_p(u), v \right\rangle \geq \|u\|^p + \|v\|^p - \|u\| \|v\|^{p-1} - \|u\|^{p-1} \|v\|$$

$$= \left(\|u\| - \|v\|\right) \left(\|u\|^{p-1} - \|v\|^{p-1}\right). \quad (7.7)$$

Next we show that $J'_p$ is of class $\alpha(X)$. Let $u_n \rightharpoonup u_0$ in $\mathcal{D}_0^{1,p}(\Omega)$ and

$$\lim \left\langle J'_p(u_n), u_n - u_0 \right\rangle \leq 0. \quad \text{Note that}$$

$$\lim \left\langle J'_p(u_n) - J'_p(u_0), u_n - u_0 \right\rangle \leq \lim \left\langle J'_p(u_n), u_n - u_0 \right\rangle - \lim \left\langle J'_p(u_0), u_n - u_0 \right\rangle.$$
Also we have \( \lim \langle J_p'(u_0), u_n - u_0 \rangle = 0 \), since \( u_n \to u_0 \). Therefore

\[
\lim \langle J_p'(u_n) - J_p'(u_0), u_n - u_0 \rangle \leq 0. \tag{7.8}
\]

From the monotonicity of the map \( J_p' \), we have

\[
\lim \langle J_p'(u_n) - J_p'(u_0), u_n - u_0 \rangle \geq 0. \tag{7.9}
\]

Thus from (7.8) and (7.9) we get \( \langle J_p'(u_n) - J_p'(u_0), u_n - u_0 \rangle \to 0 \). Now using (7.7), we conclude that \( u_n \to u_0 \) (strongly).

Next we study the compactness of certain nonlinear maps.

**Theorem 7.0.6.** Let \( \gamma \in [p, p^*] \) and let \( r \in C_0(\mathbb{R}) \) such that \( |r(s)| \leq c|s|^\gamma - 1 \), for some \( c > 0 \). Let \( g \in \mathcal{F}_{p^*} \), where \( p^* \) is the conjugate exponent of \( \frac{p}{\gamma} \). Then the map \( N : D_0^{1,p}(\Omega) \to [L(p^*, p)]^* \) defined by

\[
\langle N(u), v \rangle = \int_{\Omega} g r(u) v
\]

is compact.

**Proof.** First we show that \( N \) is well defined. Since \( p \leq \gamma \), using the Lorentz-Sobolev embedding and the monotonicity of the Lorentz spaces in the second index, we get \( D_0^{1,p}(\Omega) \to L(p^*, \gamma) \). Thus, for \( u \in D_0^{1,p}(\Omega) \), using property (iv) of Proposition (2.2.7), we get \( |u|^{\gamma - 1} \in L \left( \frac{p^*}{\gamma - 1}, \frac{\gamma}{\gamma - 1} \right) \). Now from the growth condition of \( r \) we obtain

\[
r(u) \in L \left( \frac{p^*}{\gamma - 1}, \frac{\gamma}{\gamma - 1} \right).
\]

Note that

\[
\frac{1}{p} + \frac{\gamma - 1}{p^*} + 1 = 1, \quad \frac{\gamma}{\gamma - 1} + \frac{1}{\gamma} = 1. \tag{7.10}
\]

Thus for \( v \in [L(p^*, p)] \) by using the Hölder inequality and the growth assumption on \( r \), we get the following:

\[
| \langle N(u), v \rangle | \leq \int_{\Omega} |g| |r(u)| |v| \\
\leq C_1 \|g\|_{(\frac{p^*}{p}, \infty)} \|u\|_{(p^*, p)}^{\gamma - 1} \|v\|_{(p^*, \gamma)} , \tag{7.11}
\]
where the constant $C_1$ depends only on $N, p$ and $\gamma$. The well definedness and the continuity of $N$ is evident from (7.11). Next we show that $N$ is compact. Let $v \in L(p^*, p)$,

$$
| \langle N(u_n) - N(u), v \rangle | \leq \int_{\Omega} |g||r(u_n) - r(u)||v|
$$

$$
= \int_{\Omega} \left\{ |g|^{\frac{\gamma - 1}{\gamma}} |r(u_n) - r(u)| \right\} \left\{ |g|^{|\frac{1}{\gamma}} |v| \right\}. \quad (7.12)
$$

Note that

$$
|g|^{\frac{\gamma - 1}{\gamma}} \in L \left( \frac{\gamma \cdot \frac{p^*}{p}}{\gamma - 1}, \infty \right),
$$

$$
|r(u_n) - r(u)| \in L \left( \frac{p^*}{\gamma - 1}, \frac{\gamma}{\gamma - 1} \right),
$$

$$
\frac{\gamma - 1}{\gamma \cdot \frac{p^*}{p}} + \frac{\gamma - 1}{p^*} = \frac{\gamma - 1}{\gamma} \left[ \frac{1}{p} + \frac{\gamma}{p^*} \right] = \frac{\gamma - 1}{\gamma}.
$$

Thus using the Hölder inequality (Proposition (2.2.7)), we get

$$
|g|^{\frac{\gamma - 1}{\gamma}} |r(u_n) - r(u)| \in L \left( \frac{\gamma}{\gamma - 1}, \frac{\gamma}{\gamma - 1} \right) = L^{\gamma - 1}(\Omega). \quad (7.13)
$$

Similarly by noting that

$$
|g|^{\frac{1}{\gamma}} \in L(\gamma \cdot \frac{p^*}{p}, \infty), \quad |v| \in L(p^*, \gamma),
$$

$$
\frac{1}{\gamma \cdot \frac{p^*}{p}} + \frac{1}{p^*} = \frac{1}{\gamma} \left[ \frac{1}{p} + \frac{\gamma}{p^*} \right] = \frac{1}{\gamma},
$$

we get $|g|^{\frac{1}{\gamma}} |v| \in L(\gamma, p) \subset L^{\gamma}(\Omega)$ and

$$
\left\| |g|^{\frac{1}{\gamma}} |v| \right\|_{\gamma} \leq C_2 \left\| |g|^{\frac{1}{\gamma}} \right\|_{(\gamma \cdot \frac{p^*}{p}, \infty)} \left\| |v| \right\|_{(p^*, \gamma)}
$$

$$
\leq C_3 \left\| |g|^{\frac{1}{\gamma}} \right\|_{(\gamma \cdot \frac{p^*}{p}, \infty)} \left\| |v| \right\|_{(p^*, p)}, \quad (7.14)
$$

where the constants $C_2, C_3$ depends only on $N, p$ and $\gamma$. Now by the classical
Hölder inequalities available for the Lebesgue spaces, we obtain
\[ | \langle N(u_n) - N(u), v \rangle | \leq \left\| g \right\|_{\frac{\gamma}{(\bar{p}, \infty)}} \left\| |r(u_n) - r(u)|^{\frac{\gamma - 1}{\gamma}} \right\|_{\frac{1}{\gamma - 1}} \left\| |v|^{\frac{1}{\gamma}} \right\|_{\gamma} \]
\[ \leq C_3 \left\| g \right\|_{\frac{1}{(\bar{p}, \infty)}} \left\| |r(u_n) - r(u)|^{\frac{\gamma - 1}{\gamma}} \right\|_{\frac{1}{\gamma - 1}} \left\| \|v\|_\gamma \right\|_{(p^*, p)}. \]

Thus
\[ \| N(u_n) - N(u) \| \leq C_4 \left\| |r(u_n) - r(u)|^{\frac{\gamma - 1}{\gamma}} \right\|_{\frac{1}{\gamma - 1}}, \]
where the constant \( C_4 \) depends only on \( N, p, \gamma \) and \( \|g\|_{(\bar{p}, \infty)} \). Now for the compactness of \( N \), it is enough to prove the following:
\[ \lim_{n \to \infty} \left\| |r(u_n) - r(u)|^{\frac{\gamma - 1}{\gamma}} \right\|_{\frac{1}{\gamma - 1}} = 0. \tag{7.15} \]

Using the assumption \( g \in \mathcal{F}_{\bar{p}} \), we show that
\[ \lim_{n \to \infty} \int_{\Omega} |g|^\gamma |r(u_n) - r(u)|^{\gamma-1} = 0. \tag{7.16} \]

Let \( \varepsilon > 0 \) and \( h \in C_c^\infty(\Omega) \) be arbitrary. Now we write
\[ \int_{\Omega} |g|^\gamma |r(u_n) - r(u)|^{\gamma-1} = \int_{\Omega} h |r(u_n) - r(u)|^{\gamma-1} + \int_{\Omega} (|g| - h) |r(u_n) - r(u)|^{\gamma-1} \tag{7.17} \]

First we estimate the second integral on the right hand side of the above equation, using the Hölder inequality and the Lorentz-Sobolev embedding.
\[ \int_{\Omega} (|g| - h) |r(u_n) - r(u)|^{\gamma-1} \leq C \||g| - h\|_{(\bar{p}, \infty)} \left\| |r(u_n) - r(u)|^{\gamma-1} \right\|_{\left(\frac{\bar{p}^*}{\gamma}, \frac{\bar{p}}{\gamma}\right)}. \tag{7.18} \]

Further,
\[ \left\| |r(u_n) - r(u)|^{\gamma-1} \right\|_{\left(\frac{\bar{p}^*}{\gamma}, \frac{\bar{p}}{\gamma}\right)} \leq \left\| |r(u_n) - r(u)|^{\gamma-1} \right\|_{\left(\frac{\bar{p}^*}{\gamma - 1}, \frac{\bar{p}}{\gamma - 1}\right)} \]
\[ \leq \left\{ \left\| |r(u_n)|^{\frac{\gamma}{\gamma - 1}} \right\|_{\left(\frac{\bar{p}^*}{\gamma - 1}, \frac{\bar{p}}{\gamma - 1}\right)} + \left\| r(u) \right\|_{\left(\frac{\bar{p}^*}{\gamma - 1}, \frac{\bar{p}}{\gamma - 1}\right)} \right\}^{\frac{\gamma - 1}{\gamma}} \]
\[ \leq C_6 \left\{ \||u_n\|_{(\bar{p}, p)}^{\gamma - 1} + \|u\|_{(\bar{p}, p)}^{\gamma - 1} \right\}. \tag{7.19} \]
Note that \( m = \sup_n \left\{ \|u_n\|_{(p^*, p)}^{-1} + \|u\|_{(p^*, p)}^{-1} \right\} \) is finite, since the sequence \( \{u_n\} \) is bounded in \( D_{0}^{1, p}(\Omega) \) and the embedding of \( D_{0}^{1, p}(\Omega) \) into \( L(p^*, p) \) is continuous. Thus from (7.19) into (7.18), we get
\[
\int_{\Omega} (|g| - h) |r(u_n) - r(u)|^{\frac{\gamma}{\gamma - 1}} \leq C \|g| - h| \|_{(\tilde{p}, \infty)}, \quad \forall n \in \mathbb{N}. \tag{7.20}
\]
where the constant \( C \) depends only on \( N, p \) and \( \gamma \). Now since \( g \in \mathcal{F}_{\tilde{p}} \), we have \( |g| \in \mathcal{F}_{\tilde{p}} \). Thus we can choose \( g_{\varepsilon} \in C_c^\infty(\Omega) \) such that
\[
\| g| - g_{\varepsilon} \|_{(\tilde{p}, \infty)} < \frac{\varepsilon}{C}.
\]
Now by taking \( h = g_{\varepsilon} \) in (7.20), we obtain
\[
\int_{\Omega} (|g| - g_{\varepsilon}) |r(u_n) - r(u)|^{\frac{\gamma}{\gamma - 1}} < \varepsilon, \quad \forall n \in \mathbb{N}. \tag{7.21}
\]
Next we estimate the first integral on the right hand side of (7.17). As \( \gamma < p^* \), the embedding of \( D_{0}^{1, p}(\Omega) \) into \( L_{loc}^{\gamma}(\Omega) \) is compact. Thus we have \( u_n \to u \) in \( L_{loc}^{\gamma}(\Omega) \). Since \( r \) is continuous, using the generalized dominated convergence theorem and the growth condition on \( r \), one can easily deduce that
\[
r(u_n) \to r(u) \text{ in } L_{loc}^{\gamma}(\Omega).
\]
Now since \( g_{\varepsilon} \in C_c^\infty(\Omega) \), we conclude:
\[
\lim_{n \to \infty} \int_{\Omega} g_{\varepsilon} |r(u_n) - r(u)|^{\frac{\gamma}{\gamma - 1}} = 0. \tag{7.22}
\]
Therefore, by taking \( h = g_{\varepsilon} \) (7.17), (7.21) together with (7.22) yields
\[
\int_{\Omega} |g| |r(u_n) - r(u)|^{\frac{\gamma}{\gamma - 1}} < 2 \varepsilon,
\]
for large \( n \in \mathbb{N} \). This completes the proof.

As a consequence of the above theorem we have the following corollary:

**Corollary 7.0.7.** Under the assumption \((B2)\), the map \( G_p' : X \to X^* \) is compact.
Proof. Note that $G_p' = i^* \circ N$, where $i^*$ is the adjoint of the inclusion map

$$i : X \to L(p^*, p),$$

given by the Lorentz Sobolev embedding and $N$ as in the above theorem with $r(t) = |t|^{p-2}t$. Now the compactness of $G_p'$ is immediate from the continuity of $i^*$ and the compactness of $N$. \qed

**Proposition 7.0.8.** Let $(B1), (B2)$ hold and let $H : X \to X^*$ be defined as

$$\langle H(u), v \rangle = \int_{\Omega} f(r(u)) v.$$

Then the map $H$ is compact. Moreover

$$\frac{\|H(u)\|_{X^*}}{\|u\|_{X}^{p-1}} \to 0 \quad \text{as} \quad \|u\|_{X} \to 0. \quad (7.23)$$

Proof. For $\gamma \in [p, p^*)$, the compactness of $H$ follows from Theorem 7.0.6. For $\gamma \in [1, p)$, using $(B1)$, $|r(s)| \leq C|s|^{p-1}$, for some $C > 0$. Also by $(B2)$, $f \in \mathcal{F}_p^p$ and hence $H$ is compact, by Theorem 7.0.6.

The proof of the second part of the theorem is divided into two cases:

**Case 1:** $\gamma \in (p, p^*)$

In this case, by the same calculations that yields (7.11), we obtain

$$|\langle H(u), v \rangle| \leq C \|f\|_{(\tilde{p}, \infty)} \|u\|_{(p^*, p)}^{\gamma-1} \|v\|_{(p^*, p)}. $$

Hence by the Lorentz-Sobolev embedding, we get

$$\|H(u)\|_{X^*} \leq \tilde{C} \|f\|_{(\tilde{p}, \infty)} \|u\|_{X}^{\gamma-1}.$$ 

Therefore,

$$\frac{\|H(u)\|_{X^*}}{\|u\|_{X}^{p-1}} \leq \tilde{C} \|f\|_{(\tilde{p}, \infty)} \|u\|_{X}^{\gamma-p}. $$

Since $\gamma > p$, the above inequality shows that,

$$\frac{\|H(u)\|_{X^*}}{\|u\|_{X}^{p-1}} \to 0 \quad \text{as} \quad \|u\|_{X} \to 0.$$

**Case 2:** $\gamma \in [1, p]$

In this case we have,
\[
\lim_{|s| \to 0} \frac{|r(s)|}{|s|^{p-1}} = 0.
\]

Thus for a given \(\varepsilon > 0\), there exists \(s_0 > 0\), such that
\[
|r(s)| \leq \frac{\varepsilon}{\|f\|_{(N, \infty)}^{p-1}} |s|^{p-1}, \; \forall \; |s| \leq s_0.
\]

Moreover, using \((B1)\), there exist constants \(C_1, C_2\) depending on \(s_0\), such that
\[
|r(s)| \leq C_1 |s|^{p-1}, \; \forall \; |s| \leq s_0.
\]
\[
|r(s)| \leq C_2 |s|^{p-1}, \; \forall \; |s| \geq s_0.
\]

Let \(A = \{x : |u(x)| \leq s_0\}\) and \(B = A^c \cap \Omega\). Now for \(v \in D_0^{1,p}(\Omega)\), we compute the following integral:
\[
\langle H(u), v \rangle = \int_{\Omega} f r(u)v = \int_{A} f r(u)v + \int_{B} f r(u)v. \quad (7.24)
\]

The first integral in the right hand side of \((7.24)\) can be estimated using the Hölder inequality as below:
\[
\left| \int_{A} f r(u)v \right| \leq \int_{A} |f| |r(u)| |v|
\]
\[
\leq \|f\|_{(N, \infty)}^{p-1} \int_{A} |v| |r(u)|
\]
\[
\leq \|f\|_{(N, \infty)}^{p-1} \|u\|_{(p^*, p)}^{p-1} \|v\|_{(p^*, p)}
\]
\[
\leq \varepsilon \|u\|_{(p^*, p)}^{p-1} \|v\|_{(p^*, p)}. \quad (7.25)
\]

To estimate the second integral we note that \(f \in F_{N}^{p}\). Thus for \(\varepsilon > 0\), we choose \(f_{\varepsilon} \in C_c^\infty(\Omega)\) such that \(\|f - f_{\varepsilon}\|_{(N, \infty)} < \varepsilon\). Thus
\[
\left| \int_{B} f r(u)v \right| \leq \int_{B} |f| |r(u)| |v|
\]
\[
= \int_{B} f_{\varepsilon} |r(u)| |v| + \int_{B} (|f| - f_{\varepsilon}) |r(u)| |v|. \quad (7.26)
\]

Now
\[
\int_{B} f_{\varepsilon} |r(u)| |v| \leq C_1 \int_{B} f_{\varepsilon} |u|^{p^*-1} |v|.
\]
Note that \(|u|^{p^*-1} \in L\left(\frac{p^*}{p^*-1}, \frac{p}{p-1}\right) \subset L\left(\frac{p^*}{p^*-1}, \frac{p}{p-1}\right)\) and \(q(p^*-1) > p\), where \(q\) is the conjugate exponent of \(p\). Now by the Hölder inequality and the monotonicity of the Lorentz spaces in the second index yields:

\[
\int_B f_\varepsilon |r(u)| \leq C_3 \|f_\varepsilon\|_\infty \left\| |u|^{p^*-1}\right\|_\left(\frac{p^*}{p^*-1}, \frac{p}{p-1}\right) \|v\|_{(p^*, p)} \\
\leq C_3 \|f_\varepsilon\|_\infty \|u\|_{(p^*, q(p^*-1))} \|v\|_{(p^*, p)} \\
\leq C_4 |f_\varepsilon|_\infty \|u\|_{(p^*, p)} \|v\|_{(p^*, p)},
\]

where all the constants depends only on \(N, p\) and \(\gamma\). Next we estimate

\[
\int_B (|f| - f_\varepsilon) |r(u)| \leq C_2 \int_B (|f| - f_\varepsilon)|u|^{p-1} |v| \\
\leq C_5 \|f| - f_\varepsilon\|_\left(\frac{p}{p-1}, \infty\right) \|u|^{p-1}\right\|_\left(\frac{p}{p-1}, \frac{p}{p-1}\right) \|v\|_{(p^*, p)} \\
\leq C_6 \varepsilon \|u\|_{(p^*, p)} \|v\|_{(p^*, p)} .
\]

Therefore by substituting (7.27), (7.28) in (7.26) we obtain

\[
\left| \int_B f r(u) \right| \leq C_7 \left\{ \varepsilon + \|u\|_{(p^*, p)}^{p^*-p} \right\} \|u\|_{(p^*, p)} \|v\|_{(p^*, p)} .
\]

Now (7.25) and (7.28) together implies that

\[
\frac{\|H(u)\|_X}{\|u\|_X^{p^*-1}} \leq C_8 \left\{ \varepsilon + \|u\|_{(p^*, p)}^{p^*-p} \right\} .
\]

Since \(\varepsilon\) is arbitrary, the right hand side of the above inequality goes to zero as \(\|u\|_X \to 0\). This completes the proof.

Henceforth, for convenience, we denote \(J_{p^*}, G_{p^*}\) by \(J, G\) respectively.

**Remark 7.0.9.** In conclusion, under the assumptions \((B1), (B2)\) the maps \(G'\) and \(H\) are compact. Since the map \(J'\) is of class \(\alpha(X)\), by the Remark 7.0.4, for each \(\lambda \in \mathbb{R}\), the maps

\[
A_\lambda = J' - \lambda(G' + H), \quad \tilde{A}_\lambda = J' - \lambda G'
\]

are in \(\alpha(X)\). Further, both \(A_\lambda\) and \(\tilde{A}_\lambda\) are continuous. Thus the degree can be defined for \(A_\lambda\) and \(\tilde{A}_\lambda\) (see Appendix A).

Let \(\lambda_1\) be the first eigenvalue of (7.3). Using Theorem A.0.23 we prove
that \( \lambda_1 \) is a bifurcation point of (7.1). For this we need to calculate the index of \( A_\lambda \) at zero, for \( \lambda \) in some neighbourhood of \( \lambda_1 \) (see Appendix A). First we calculate the \( \text{Ind}(A_\lambda, 0) \) for \( \lambda \) in some neighbourhood of \( \lambda_1 \) and then we use the homotopy invariance property of degree to calculate the \( \text{Ind}(A_\lambda, 0) \).

In order to define \( \text{Ind}(A_\lambda, 0) \), we must prove that the trivial solution, \( u \equiv 0 \), is an isolated zero of \( A_\lambda \). In the next proposition we show that, for each \( \lambda \in (0, \lambda_1) \), \( u \equiv 0 \) is an isolated zero of \( A_\lambda \).

**Proposition 7.0.10.** Let \((B1), (B2)\) hold and let \( \lambda_1 \) be the first eigenvalue of (7.3). Then for each \( \lambda \in (0, \lambda_1) \), \( 0 \) an isolated zero of \( A_\lambda \) and

\[
\text{Ind}(A_\lambda, 0) = 1.
\]

**Proof.** Let \( \lambda \in (0, \lambda_1) \). Since, \( \lambda_1 \) is the minimum of the Rayleigh quotient,

\[
\lambda < R(u) = \frac{J(u)}{G(u)}, \quad u \in D^+(g),
\]

where

\[
D^+(g) = \left\{ u \in D^{1,p}_0(\Omega) : \int_\Omega g|u|^p > 0 \right\}.
\]

Further,

\[
\frac{\langle J'(u), u \rangle}{\langle G'(u), u \rangle} = \frac{J(u)}{G(u)}, \quad \forall u \in D^+(g).
\]

Therefore

\[
\langle A_\lambda(u), u \rangle = \langle J'(u) - \lambda G'(u), u \rangle > 0, \quad \forall u \in D^+(g).
\]

(7.31)

Also for \( u \not\in D^+(g) \cup \{0\} \), note that \( G(u) \leq 0 \) and \( J(u) > 0 \). Thus, for each \( u \in D^+(g) \) with \( u \neq 0 \), we have

\[
\langle A_\lambda(u), u \rangle = \langle J'(u) - \lambda G'(u), u \rangle = p(J(u) - \lambda G(u)) > 0.
\]

(7.32)

Now from (7.31) and (7.32), for \( \lambda \in (0, \lambda_1) \), it is clear that,

\[
\langle A_\lambda(u), u \rangle > 0, \quad \forall u \in D^{1,p}_0(\Omega), u \neq 0.
\]

Hence \( u \equiv 0 \) is the only zero of \( A_\lambda \) and hence is isolated. Further, using
Theorem A.0.18, we conclude that,

\[ \text{Ind}(\tilde{A}_{\lambda}, 0) = 1, \quad \forall \lambda \in (0, \lambda_1). \]

Now we need to show that \( u = 0 \) is an isolated zero of \( \tilde{A}_{\lambda} \) for each \( \lambda \in (\lambda_1, \lambda_1 + \delta) \), for some \( \delta > 0 \). For this, first we prove the isolatedness of the first eigenvalue \( \lambda_1 \) of (7.3). When \( g \) is in suitable Lebesgue spaces, this has been proved by several authors, for example see [27, 34]. Here we adapt the idea of [34] for the weights in \( F_{\frac{N}{p}} \).

**Proposition 7.0.11.** Let \( \Omega \) be a connected domain in \( \mathbb{R}^N \) with \( 1 < p < N \). Let \( g \in F_{\frac{N}{p}} \setminus \{0\} \). Then the first eigenvalue \( \lambda_1 \) of (7.3) is isolated.

**Proof.** Assume that \( \lambda_1 \) is not isolated. Let \( \{\mu_n\} \) be a sequence of eigenvalues of (7.3) such that \( \mu_n \to \lambda_1 \). Let \( v_n \) be an eigenfunction corresponding to \( \mu_n \) normalized as \( \int g|v_n|^p = 1 \). Thus

\[
\int |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla v = \mu_n \int g|v_n|^{p-2}v_n v, \quad \forall v \in X. \tag{7.33}
\]

Let \( \Omega_n^- = \{x: v_n(x) < 0\} \). From Remark 6.2.9, \( v_n \) changes sign and hence \( |\Omega_n^-| \neq 0 \). Now by taking \( v = v_n^- \) in (7.33) and using Hölder inequality, we obtain

\[
\int_{\Omega_n^-} |\nabla v_n^-|^p = \mu_n \int_{\Omega_n^-} g|v_n^-|^p
\leq C_1 \mu_n \left\| g\chi_{\Omega_n^-} \right\|_{(\frac{N}{p}, \infty)} \left\| v_n^- \right\|_{(p', p)}^p
\leq C_2 \mu_n \left\| g\chi_{\Omega_n^-} \right\|_{(\frac{N}{p}, \infty)} \int_{\Omega_n^-} |\nabla v_n^-|^p. \tag{7.34}
\]

The last inequality is obtained using Lorentz-Sobolev embedding. Now from (7.34), it is clear that, since \( \mu_n \lambda_1 > 0 \),

\[
\left\| g\chi_{\Omega_n^-} \right\|_{(\frac{N}{p}, \infty)} > C_3, \tag{7.35}
\]

where \( C_3 \) depends only on \( p \) and \( \Omega \). Now using Lemma 3.0.16, there exists
a bounded set \( \Omega \subset \Omega \) such that
\[
\left\| g \chi_{\Omega \setminus \tilde{\Omega}} \right\|_{L^1_p} < \frac{C_3}{2}. \tag{7.36}
\]
Therefore
\[
\left\| g \chi_{\Omega_n \setminus \tilde{\Omega}} \right\|_{L^1_p} \leq \left\| g \chi_{\Omega_n \setminus \tilde{\Omega}} \right\|_{L^1_p} + \left\| g \chi_{\Omega_n \setminus \tilde{\Omega}^c} \right\|_{L^1_p} \leq \frac{C_3}{2}.
\]
Substituting (7.35) in the above inequality yields:
\[
\left\| g \chi_{\Omega_n \setminus \tilde{\Omega}} \right\|_{L^1_p} > \frac{C_3}{2}, \quad \forall n \in \mathbb{N}. \tag{7.37}
\]
Now as the norm \( \| \cdot \|_{L^1_p} \) is absolutely continuous in \( \mathcal{F}_{\frac{N}{p}} \) with respect to the Lebesgue measure in \( \mathbb{R}^N \), we must have
\[
|\Omega_n \cap \tilde{\Omega}| > C, \tag{7.38}
\]
for some positive constant \( C \) independent of \( n \).

Note that
\[
J(v_n) = \mu_n \to \lambda_1 \text{ and } \int \Omega g v_n^{\frac{1}{p}} = 1.
\]
Now by the same argument as in Theorem 6.2.2, up to a subsequence, denoted by \( v_n \) itself, \( v_n \to \pm \phi_1 \) in \( D_0^{1,p}(\Omega) \), where \( \phi_1 \) is the positive eigenfunction corresponding to \( \lambda_1 \) such that \( \int \Omega g \phi_1^p = 1 \). Further,
\[
\|v_n\| = \mu_n^{\frac{1}{p}} \to \lambda_1^{\frac{1}{p}} = \|\phi_1\|.
\]
Thus, as \( D_0^{1,p}(\Omega) \) is uniformly convex, we have
\[
v_n \to \pm \phi_1 \text{ strongly in } D_0^{1,p}(\Omega).
\]
Without loss of generality, we assume that \( v_n \to \phi_1 \) strongly in \( D_0^{1,p}(\Omega) \). Thus we obtain a subsequence \( \{v_{n_k}\} \) of \( \{v_n\} \) such that
\[
v_{n_k} \to \phi_1, \text{ a.e. in } \Omega,
\]
since \( D_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \). Now as \( |\tilde{\Omega}| < \infty \), by Egoroff’s theorem (see page...
73 of [69], \( v_n \to \phi_1 \) uniformly on \( \overline{\Omega} \) except on a set of arbitrarily small measure. Thus for the \( C \) in (7.38), there exists \( \Omega_0 \subset \Omega \) such that \( v_n \to \phi_1 \) uniformly on \( \Omega \setminus \Omega_0 \) and \( |\Omega_0| < C \). Hence for large \( n \), \( v_n \) must be positive in \( \Omega \setminus \Omega_0 \) and therefore
\[
|\Omega_n \cap \overline{\Omega}| \leq |\Omega_0| < C.
\]
This is a contradiction to (7.38). Hence \( \lambda_1 \) must be isolated. \( \square \)

**Remark 7.0.12.** Since \( \lambda_1 \) is isolated there exists \( \delta > 0 \) such that (7.3) does not admit an eigenvalue in \( (\lambda_1, \lambda_1 + \delta) \). Further, if \( A_\lambda \) has a nontrivial zero, then \( \lambda \) is an eigenvalue of (7.3). Thus we conclude that \( u \equiv 0 \) is the unique zero of \( A_\lambda \), for \( \lambda \in (\lambda_1, \lambda_1 + \delta) \).

Next we compute the index \( \widetilde{A}_\lambda \) at zero, for each \( \lambda \in (\lambda_1, \lambda_1 + \delta) \).

**Proposition 7.0.13.** Let \( (B1), (B2) \) hold. Then, for \( \lambda \in (\lambda_1, \lambda_1 + \delta) \),
\[
\text{Ind}(\widetilde{A}_\lambda, 0) = -1.
\]

**Proof.** Here we adapt the idea of the proof of Theorem 4.1. of [34] for the weights in \( F_\frac{p}{p} \). For a fixed \( k > 0 \), we define a \( C^1 \) function \( \psi : \mathbb{R} \to \mathbb{R} \) by
\[
\psi(t) = \begin{cases} 
0, & t \leq k, \\
\frac{2\delta}{\lambda_1}(t - 2k), & t \geq 3k.
\end{cases}
\] (7.39)
and \( \psi \) is strictly convex in \( (k, 3k) \). For \( \lambda \in (\lambda_1, \lambda_1 + \delta) \), let
\[
\psi_\lambda(u) = J(u) - \lambda G(u) + \psi(J(u))
\]
be a functional defined on \( X \). Note that critical points of \( \psi_\lambda \) are given by the solutions of the following equation:
\[
\dot{J}'(u) - \lambda G'(u) + \psi'(J(u))J'(u) = 0.
\]
Let \( u_0 \) be a nontrivial critical point of \( \psi_\lambda \). Then \( u_0 \) must satisfy the following:
\[
\dot{J}'(u_0) - \frac{\lambda}{1 + \psi'(J(u_0))} G'(u_0) = 0. \tag{7.40}
\]
From the above equation, it is clear that \( u_0 \) is an eigenfunction of (7.3) corresponding to the eigenvalue \( \frac{\lambda}{1 + \psi'(J(u_0))} \) of (7.3). Since \( \lambda_1 \) is the unique
eigenvalue of $(7.3)$ in $(0, \lambda_1 + \delta)$ and $0 < \frac{\lambda}{1 + \psi'(J(u_0))} < \lambda_1 + \delta$, we must have

\[
\frac{\lambda}{1 + \psi'(J(u_0))} = \lambda_1.
\]

Therefore,

\[
u_0 = r\phi_1, \quad \psi'(J(u_0)) = \frac{\lambda}{\lambda_1} - 1,
\]

for some $r \in \mathbb{R}$, where $\phi_1$ is the first eigenvalue of $(7.3)$ with $\int_\Omega g|\phi_1|^p = 1$ and $\phi_1 > 0$. Note that

\[
0 < \frac{\lambda}{\lambda_1} - 1 < \frac{2\delta}{\lambda_1} \quad \text{and} \quad \psi'(t) = \begin{cases} 0, & t \leq k, \\ \frac{2\delta}{\lambda_1}, & t \geq 3k. \end{cases}
\]

Thus we must have $J(u_0) \in (k, 3k)$. Since $\psi$ is strictly convex in $(k, 3k)$, there exist a unique $t_0$ such that $\psi'(t_0) = \frac{\lambda}{\lambda_1} - 1$. Hence there exists a unique $r > 0$ such that $u_0 = \pm r\phi_1$, since $J$ is even. Thus, for $\lambda \in (\lambda_1, \lambda_1 + \delta)$, $\psi'_\lambda$ has precisely three isolated zeros $-u_1, 0, u_1$, where $u_1$ is a positive eigenfunction corresponding to $\lambda_1$ and normalized as $J(u_1) \in (k, 3k)$.

If we show that $\pm u_1$ are local minima of $\psi_\lambda$, then $\text{Ind}(\psi'_\lambda, \pm u_1)$ can be obtained using Theorem A.0.21. Since $\pm u_1$ are the only nontrivial critical points of $\psi_\lambda$, if we prove that $\psi_\lambda$ attains its minimum, then it is necessary that $\pm u_1$ are the minimizers of $\psi_\lambda$. Thus it is enough to prove that $\psi_\lambda$ is bounded below, weakly sequentially lower semi-continuous and coercive.

(i) $\psi_\lambda$ is weakly sequentially lower semi-continuous: Let $u_n \rightharpoonup u$ in $X$. Since $G$ is compact and $J$ is lower semi-continuous, we get

\[
\lim_n \psi_\lambda(u_n) = \lim_n (J(u_n) - \lambda G(u_n)) + \lim_n \psi(J(u_n)) \\
\geq J(u) - \lambda G(u) + \lim_n \psi(J(u_n)).
\]

Since $\psi$ is increasing and continuous we have, $\lim_n \psi(J(u_n)) = \psi(J(u))$ and hence

\[
\lim_n \psi_\lambda(u_n) \geq \psi_\lambda(u).
\]

(ii) $\psi_\lambda$ is coercive:

\[
\psi_\lambda(u) = J(u) - \lambda G(u) + \psi(J(u)) \\
= J(u) - \lambda_1 G(u) + (\lambda_1 - \lambda)G(u) + \psi(J(u)) \quad (7.41)
\]
Bifurcation for the \( p \)-Laplacian

Since \( J(u) - \lambda_1 G(u) \geq 0 \), we get

\[
\psi_\lambda(u) \geq (\lambda_1 - \lambda)G(u) + \psi(J(u))
\]

\[
\geq \frac{\lambda_1 - \lambda}{\lambda_1} J(u) + \psi(J(u)) \quad \vdots \quad \frac{J(u)}{\lambda_1} \geq G(u), \quad \lambda_1 - \lambda < 0 \quad (7.42)
\]

\[
\geq -\frac{\delta}{\lambda_1} J(u) + \frac{2\delta}{\lambda_1} (J(u) - 2k), \quad \text{for large } J(u)
\]

\[
= \frac{\delta}{\lambda_1} J(u) - \frac{4\delta k}{\lambda_1} \rightarrow \infty \text{ as } \|u\|_X \rightarrow \infty.
\]

(iii) \( \psi_\lambda \) is bounded below:

From (7.41) we obtain,

\[
\psi_\lambda(u) \geq -\frac{\delta}{\lambda_1} J(u) + \psi(J(u)).
\]

Now we use the definition of \( \psi_\lambda \) to obtain a lower bound for \( \psi_\lambda \). If \( J(u) \geq 3k \), then

\[
\psi_\lambda(u) \geq \frac{\delta}{\lambda_1} J(u) - \frac{4\delta k}{\lambda_1} \geq -\frac{\delta k}{\lambda_1}.
\]

If \( J(u) \leq 3k \), then

\[
\psi_\lambda(u) \geq -\frac{\delta}{\lambda_1} 3k + \psi(J(u)) \geq -\frac{3\delta k}{\lambda_1}.
\]

Thus by a standard variational argument we conclude that \( \psi_\lambda \) attains its minimum. Now by Theorem A.0.21 we have

\[
\text{Ind}(\psi'_\lambda, u_1) = \text{Ind}(\psi'_\lambda, -u_1) = 1. \quad (7.43)
\]

Next we show that

\[
\langle \psi'_\lambda(u), u \rangle > 0, \quad \text{for } \|u\| = r.
\]

for sufficiently large \( r \).

\[
p \langle \psi'_\lambda(u), u \rangle = J(u) - \lambda G(u) + \psi(J(u)) J(u)
\]

\[
= -\frac{\delta}{\lambda_1} J(u) + \psi(J(u)) J(u)
\]

\[
\geq \left( \frac{2\delta}{\lambda} - \frac{\delta}{\lambda} \right) J(u) > 0, \quad \text{if } J(u) \geq 3k.
\]
Now for \( r > 3k \), using Theorem A.0.18, we get
\[
\text{Deg}[\psi'_\lambda, B_r(0), 0] = 1.
\]
As \( k \) is arbitrary, we can choose \( r_0 > 0 \), so that both \( u_1 \) and \( -u_1 \) are in \( B_{r_0}(0) \) and \( r_0 > 3k \). Thus
\[
\text{Deg}[\psi'_\lambda, B_{r_0}(0), 0] = 1.
\]
Further, \(-u_0, 0, u_0\) are the only critical points of \( \psi_\lambda \), we get
\[
\text{Ind}(\psi'_\lambda, 0) = -1.
\]
From the definition of \( \psi_\lambda \), it is clear that
\[
\text{Deg}[\psi'_\lambda, B_r(0), 0] = \text{Deg}[\tilde{\Lambda}_\lambda, B_r(0), 0] \quad (7.44)
\]
for small \( r > 0 \). Thus we obtain \( \text{Ind}(\tilde{\Lambda}_\lambda, 0) = -1 \). \qed

**Remark 7.0.14.** From Proposition 7.0.10 and Proposition 7.0.13 we have,
\[
\text{Ind}(\tilde{\Lambda}_\lambda, 0) = \begin{cases} 1, & \lambda \in (0, \lambda_1), \\ -1, & \lambda \in (\lambda_1, \lambda_1 + \delta). \end{cases}
\]

Next we prove that \( \text{Ind}(\tilde{\Lambda}_\lambda, 0) = \text{Ind}(A_\lambda, 0) \).

**Proposition 7.0.15.** Let \((B1), (B2)\) hold. Let \( \lambda_1 \) be the first eigenvalue of (7.3). Then
\[
\text{Ind}(A_\lambda, 0) = \text{Ind}(\tilde{\Lambda}_\lambda, 0), \quad \forall \lambda \in (0, \lambda_1 + \delta) \setminus \{\lambda_1\}.
\]

**Proof.** In view of homotopy invariance of degree, we show that \( \tilde{\Lambda}_\lambda \) and \( A_\lambda \) are homotopic for \( \lambda \in (0, \lambda_1 + \delta) \setminus \{\lambda_1\} \) in \( B_r(0) \) and on \( \partial B_r(0) \) for sufficiently small \( r > 0 \). For a fixed \( \lambda \in (0, \lambda_1 + \delta) \setminus \{\lambda_1\} \), we define
\[
A^t_\lambda = \tilde{\Lambda}_\lambda - tH, \quad t \in [0, 1],
\]
where \( H \) is the map defined in Proposition 7.0.8. Note that \( A^0_\lambda = \tilde{\Lambda}_\lambda \) and \( A^1_\lambda = A_\lambda \). Next we show that for each \( t \in [0, 1] \), \( A^t_\lambda \) does not vanish on \( \partial B_r(0) \) for sufficiently small \( r > 0 \). Suppose not, then there exist \( t_n \in [0, 1], u_n \neq 0 \) such that \( u_n \to 0 \) and
\[
A^t_n(u_n) = J'(u_n) - \lambda G'(u_n) - \lambda t_n H(u_n) = 0.
\]
Let $\tilde{u}_n = \frac{u_n}{\|u_n\|}$. Since $J'$ and $G'$ are $p - 1$ homogeneous, by dividing the above equation with $\|u\|^{p-1}$, we get

$$J'(\tilde{u}_n) - \lambda G'(\tilde{u}_n) - \lambda t_n\frac{H(u_n)}{\|u_n\|^{p-1}} = 0.$$ 

By Proposition 7.0.8 $\frac{H(u_n)}{\|u_n\|^{p-1}} \to 0$ and hence $J'(\tilde{u}_n) - \lambda G'(\tilde{u}_n) \to 0$. Since $\tilde{u}_n$ is bounded, up to a subsequence $\tilde{u}_n \to u_0$. Therefore, by the compactness of $G'$ and the continuity of $J'$ we get

$$\lim_{n \to \infty} J'(\tilde{u}_n) = \lambda G'(\tilde{u}_0).$$

$$\lim_{n \to \infty} u_n = (J')^{-1}(\lambda G'(\tilde{u}_0)) = u_0.$$ 

Thus $J'(\tilde{u}_0) - \lambda G'(\tilde{u}_0) = 0$, a contradiction as $\lambda$ is not an eigenvalue for (7.3). Thus by the homotopy invariance of the degree we get

$$\deg [A_{\lambda}, B_r(0), 0] = \begin{cases} 1, & \lambda \in (0, \lambda_1), \\ -1, & \lambda \in (\lambda_1, \lambda_1 + \delta). \end{cases}$$

this completes the proof.

Now we have the following result:

**Theorem 7.0.16.** Let $\Omega$ be an open connected subset of $\mathbb{R}^N$ with $1 < p < N$. Let $r, f$ and $g$ satisfy $(B1), (B2)$. Then the first eigenvalue $\lambda_1$ of (7.3) is a bifurcation point of (7.1).

The proof follows from the above proposition, using Theorem A.0.23.