Chapter 6

Weighted eigenvalue problems for the $p$–Laplacian

In this chapter, we consider a nonlinear analogue of the linear weighted eigenvalue problem for the Laplacian, that we discussed in Chapter 4. Here, instead of the Laplace operator, we consider the $p$-Laplace operator $\Delta_p$, where

$$\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u).$$

For an open connected subset $\Omega$ in $\mathbb{R}^N$ with $p \in (1, N)$, we study the sufficient conditions for a weight function $g$ to ensure the existence of $\lambda \in \mathbb{R}$ and $u \in \mathcal{D}^{1,p}_0(\Omega) \setminus \{0\}$ such that

$$- \Delta_p u = \lambda g |u|^{p-2} u, \quad \text{in} \quad \Omega. \quad (6.1)$$

We say that $u \in \mathcal{D}^{1,p}_0(\Omega)$ solves (6.1), if:

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v = \lambda \int_{\Omega} g |u|^{p-2} u v, \quad \forall v \in \mathcal{D}^{1,p}_0(\Omega). \quad (6.2)$$

Note that, problem (6.1) is a nonlinear analogue and a natural generalization of the linear weighted eigenvalue problem for the Laplacian that we considered in Chapter 4.

There are several sufficient conditions on the weight function $g$ available in the literature for the existence of a principal eigenvalue for (6.1). In [56] Lindqvist proved the existence, uniqueness and simplicity of a principal eigenvalue for general $p$, when $g \equiv 1$ and $\Omega$ is bounded. Many authors
have given sufficient conditions on \( g \) for the existence of a positive principal eigenvalue for (6.1), when \( \Omega = \mathbb{R}^N \) : for example Huang [45], Allegretto and Huang [5], Fleckinger et al. [39], studied the problem (6.1) for general \( p \). All these earlier results assume that either \( g \) or \( g^+ \) is in \( L^\frac{N}{p}(\mathbb{R}^N) \). In [73], Willem and Szulkin enlarged the class of weight functions beyond the Lebesgue space \( L^\frac{N}{p}(\Omega) \) by proving the existence of an eigenvalue of (6.1), for the weights whose positive part has a faster decay than \( \frac{1}{|x|^p} \) at infinity and at all the points in the domain.

Here we prove a result analogous to Theorem 4.3.15. More precisely, we prove that for the weights whose positive part is in \( \mathcal{F}_N \), (6.1) admits a positive principal eigenvalue. Further, we prove that this principal eigenvalue is the unique principal eigenvalue of (6.1). The results presented in this chapter have appeared [9] in Electronic Journal of Differential Equations.

As in the case of Laplacian, here also we use a direct variational principal to prove the existence of eigenvalues. Indeed, there is a one to one correspondence between the eigenvalues of (6.1) and the critical values of the following Rayleigh quotient

\[
R_p(u) = \frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} g|u|^p},
\]

on the set

\[
\mathcal{D}_p^+(g) = \left\{ u \in \mathcal{D}_0^{1,p}(\Omega) : \int_{\Omega} g|u|^p > 0 \right\}.
\]  

(6.3)

Under suitable assumptions on the weight function \( g \), one can arrive at equation (6.2) as the Euler-Lagrange equation for the critical points of \( R_p \) on \( \mathcal{D}_p^+(g) \) with eigenvalues taken to be the corresponding critical values. Let

\[
\mathcal{M}_p = \left\{ u \in \mathcal{D}_0^{1,p}(\Omega) : \int_{\Omega} g|u|^p = 1 \right\},
\]

\[
J_p(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p,
\]

\[
\lambda_1 = \inf \left\{ J_p(u) : u \in \mathcal{M}_p \right\}.
\]

Due to the homogeneity of the Rayleigh quotient \( R_p \), a critical value of \( R_p \) on \( \mathcal{D}_p^+(g) \) is a critical value of \( J_p \) on \( \mathcal{M}_p \) and vice versa. For the existence of an eigenvalue of (6.1), we show the existence of a minimizer of \( J_p \) on \( \mathcal{M}_p \). Later we prove the existence of infinitely many critical values for \( J_p \) on \( \mathcal{M}_p \), using the Ljusternik Schnirelmann theorem due to Szukin [72].
This chapter is organized as follows. In Section 1, we give examples and counterexamples of weights for which (6.1) admits a positive principal eigenvalue and we relate our sufficient conditions with various sufficient conditions available in the literature. The existence and other qualitative properties such as the simplicity, the uniqueness of the first eigenvalue are discussed in Section 2. The radial symmetry of the first eigenfunctions of (6.1), under certain symmetry assumption on the weight function $g$ is given in Section 2. In Section 3, we discuss the Ljusternik-Schnirelmann theory on $C^1$ Banach manifold and a proof for the existence of infinitely many eigenvalues of (6.1) is given. Further extensions and applications are indicated in Section 4.

6.1 Examples and counterexamples.

First we give some examples of classes of weight functions for which the functional $J_p$ admits a minimizer on $M_p$. Further, we relate our results with various sufficient conditions available in the literature, for the existence of a positive principal eigenvalue for (6.1). The nonexistence of eigenvalues of (6.1) for certain weights are obtained using a Pohozaev type identity that we derive here.

6.1.1 Examples

In this chapter, we will prove that, for $g$ such that $g^+ \in F_N$, $J_p$ admits a minimizer on $M_p$ (Theorem 6.2.2) and this minimizer is an eigenfunction of (6.1) corresponding to $\lambda_1$. Thus Proposition 3.0.13 shows that our result subsumes results of [5, 39, 45]. In the next lemma, we show that weights considered by Szulkin and Willem in [73] for the weighted eigenvalue problems for the $p$-Laplacian are in $F_N$. More specifically, they considered $g$ satisfying the following conditions:

$$
\begin{align*}
g \in L^1_{\text{loc}}(\Omega), & \quad g^+ = g_1 + g_2 \neq 0, & \quad g_1 \in L^N_p(\Omega), \\
\lim_{|x| \to \infty, x \in \Omega} |x|^p g_2(x) = 0, & \quad \lim_{x \to a, x \in \Omega} |x-a|^p g_2(x) = 0 & \quad \forall a \in \Omega.
\end{align*}
$$

In the next lemma we show that the positive part of a function satisfying (6.4) belongs to the space $F_N$. Thus our results imply the results of [73].

**Lemma 6.1.1.** Let $\Omega$ be a domain in $\mathbb{R}^N$ and let $1 < p < N$. Let $g$ satisfies
condition (6.4). Then \( g^+ \in \mathcal{F}^N_p \).

Proof. Let \( g \) be a function satisfying (6.4) with \( g^+ = g_1 + g_2 \). Clearly \( g_1 \in \mathcal{F}^N_p \), since \( L^p_N(\Omega) \subset \mathcal{F}^N_p \) (see Proposition 3.0.13). Further, by Theorem 3.0.24, \( g_2 \in \mathcal{F}^N_p \). Hence the result.

In the next example, we show that \( \mathcal{F}^N_p \) contains weights that fail to satisfy (6.4).

Example 6.1.2. Let \( 1 < p < N \). In the cube \( \Omega = \{(x_1, \cdots, x_N) \in \mathbb{R}^N : |x_i| < R \} \) with \( 0 < R < 1 \) consider the function

\[
g_3(x) = \left| x_1 \log(|x_1|) \right|^{\frac{p}{N}} , \quad x_1 \neq 0.
\]

(6.5)

Using (3.0.21) one can verify that \( g_3 \in L\left( \frac{N}{p}, q \right) \) for \( q > \frac{N}{p} \) and hence \( g_3 \in \mathcal{F}^N_p \). However, \( g_3 \) does not satisfy (6.4), indeed along the curve \( x_2 = (x_1)^{\frac{1}{N}} \), the limit of \( |x|^p g_3(x) \) is infinity as \( x \) tends to 0 and this limit is zero as \( x \) tends to 0 along the \( x_1 \) axis.

6.1.2 Counterexamples

Now we derive a Pohozaev type identity for \( p \)-Laplacian, analogous to (4.10). More precisely, for the solutions of (6.1) we prove the following identity:

\[
\int_{\mathbb{R}^N} \{ x \cdot \nabla a(x) + p a(x) \} |u|^p = 0,
\]

(6.6)

under certain regularity assumptions on \( a \) and \( u \).

The Pohozaev identity is known for proving the nonexistence of solutions for certain class of partial differential equations on certain type of domains. Here we prove an identity in \( \mathbb{R}^N \), similar to Pohozaev identity. Further, we use this identity to prove the nonexistence of eigenvalues for (6.1) for certain classes of weight functions. In contrast to the linear case, we prove (6.6) under additional assumptions on \( a \) and we use a weak form of divergence theorem of Cuesta and Takac [28].

Theorem 6.1.3. Let \( a \in C^\alpha_{0\text{loc}}(\mathbb{R}^N) \) for some \( \alpha \in (0, 1) \) and let \( u \in \mathcal{D}^{1,p}_0(\mathbb{R}^N) \). Further, assume that \( a(x)|u|^p, x \cdot \nabla a(x)u^p \in L^1(\mathbb{R}^N) \). If \( u \) solves

\[
- \Delta_p u = a(x)|u|^{p-2} u
\]

(6.7)
in the weak sense, then the following identity holds:

$$\int_{\mathbb{R}^N} \{ x \cdot \nabla a(x) + p a(x) \} |u|^p = 0.$$  

**Proof.** First we prove a point wise identity valid in the complement of zero set of \(\nabla u\). For each \(\eta > 0\) let \(\Omega_\eta := \{ x \in \mathbb{R}^N : |\nabla u| > \eta \}\). Note that \(a \in L^q_{loc}(\mathbb{R}^N)\) with \(q > \frac{N}{p}\). Thus by Serrin’s local regularity results ([70]) available for quasilinear operators, the solutions of (6.7) are in \(C^2_{loc}(\mathbb{R}^N)\). Now Since \(\text{div}(|\nabla u|^{p-2} \nabla u)\) is an uniformly elliptic operator on \(\Omega_\eta\) and \(a(x)|u|^{p-2} u \in C^2_{loc}(\mathbb{R}^N)\), using the standard elliptic regularity theory, we get \(u \in C^2_{loc}(\Omega_\eta)\), see [40]. Thus one has the following point wise identity in \(\Omega_\eta\): 

$$- \Delta_p u = a(x)|u|^{p-2} u \quad \text{a.e in } \Omega_\eta, \ \eta > 0. \quad (6.8)$$

First, we choose a cut-off function \(\zeta \in C^\infty_c(\mathbb{R})\) such that

(i) \(0 \leq \zeta \leq 1\), \quad (ii) \(\zeta(r) = 1, \ 0 \leq r \leq 1\), \quad (iii) \(\zeta(r) = 0, \ r \geq 2\).

and for each \(n \in \mathbb{N}\) we define

$$\psi_n(x) = \zeta \left( \frac{|x|^2}{n^2} \right).$$

Then there exists \(c > 0\) independent of \(n\) such that

$$|\psi_n(x)|, |x||\nabla \psi_n(x)| \leq c, \ \forall x \in \mathbb{R}^N, \ \forall n \in \mathbb{N}. \quad (6.9)$$

Now we multiply equation (6.8) by \(\{ x \cdot \nabla u \} \psi_n\) to obtain the following point wise identity

$$- \Delta_p u \{ x \cdot \nabla u \} \psi_n = a(x)|u|^{p-2} u \{ x \cdot \nabla u \} \psi_n \quad \text{a.e in } \Omega_\eta, \ \eta > 0. \quad (6.10)$$

It is easy to verify the following point wise identities valid in \(\Omega_\eta\), for each \(\eta > 0\):

$$\text{div} \{|\nabla u|^{p-2} \nabla u \{ x \cdot \nabla u \} \psi_n\} = \Delta_p u \{ x \cdot \nabla u \} \psi_n - |\nabla u|^{p-2} \nabla u \cdot \nabla (\{ x \cdot \nabla u \} \psi_n). \quad (6.11)$$

$$|\nabla u|^{p-2} \nabla u \cdot \nabla (\{ x \cdot \nabla u \} \psi_n) = |\nabla u|^{p-2} \nabla u \cdot \nabla \{ x \cdot \nabla u \} \psi_n + \{ x \cdot \nabla u \} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi_n. \quad (6.12)$$
\[ |\nabla u|^p \nabla u \cdot \nabla \{ x \cdot \nabla u \} = |\nabla u|^p + |\nabla u|^{p-2} \sum_j x_j \sum_i \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_i \partial x_j} \]
\[ = |\nabla u|^p + |\nabla u|^{p-2} x \cdot (\nabla^2 u) \nabla u. \] 
\label{6.14}

\[ \text{div} (|\nabla u|^p) = |\nabla u|^p + x \cdot \nabla (|\nabla u|^p) \]
\[ = N |\nabla u|^p + p |\nabla u|^{p-2} x \cdot (\nabla^2 u) \nabla u. \] 
\label{6.15}

From (6.14) and (6.15) we get the following identity in \( \Omega_\eta \):
\[ |\nabla u|^{p-2} \nabla u \cdot \nabla \{ x \cdot \nabla u \} = \left( 1 - \frac{N}{p} \right) |\nabla u|^p + \frac{1}{p} \text{div}(x|\nabla u|^p). \] 
\label{6.16}

By combining all the above identities we obtain the following
\[ \Delta_p u \{ x \cdot \nabla \} \psi_n(x) = \text{div} \left\{ |\nabla u|^{p-2} \nabla u \{ x \cdot \nabla u \} \psi_n - \frac{1}{p} (x|\nabla u|^p) \psi_n \right\} \]
\[ - \left( 1 - \frac{N}{p} \right) |\nabla u|^p \psi_n - \{ x \cdot \nabla u \} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi_n \text{ a.e. in } \Omega_\eta. \]

We use (6.10) in the above identity to obtain the following identity a.e. in \( \Omega_\eta \):
\[ \text{div} \left\{ |\nabla u|^{p-2} \nabla u \{ x \cdot \nabla u \} \psi_n - \frac{1}{p} (x|\nabla u|^p) \psi_n \right\} = - a(x)|u|^{p-2} u \{ x \cdot \nabla u \} \psi_n \]
\[ + \left( 1 - \frac{N}{p} \right) |\nabla u|^p \psi_n + \{ x \cdot \nabla u \} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi_n. \]

For the convenience we use the following notations:
\[ G_n = |\nabla u|^{p-2} \nabla u \{ x \cdot \nabla u \} \psi_n - \frac{1}{p} (x|\nabla u|^p) \psi_n. \]
\[ H_n = - a(x)|u|^{p-2} u \{ x \cdot \nabla u \} \psi_n + \left( 1 - \frac{N}{p} \right) |\nabla u|^p \psi_n \]
\[ + \{ x \cdot \nabla u \} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi_n. \] 
\label{6.17}

In the new notations, the point wise identity (6.17) reads as below
\[ \text{div } G_n = H_n, \text{ a.e. in } \Omega_\eta. \] 
\label{6.18}
Next we show that (6.18) holds in \( \mathbb{R}^N \), in the sense of distributions. i.e.,

\[
\int_{\mathbb{R}^N} \text{div} (G_n) \phi = \int_{\mathbb{R}^N} H_n \phi, \quad \forall \phi \in \mathcal{C}^\infty_c(\mathbb{R}^N).
\]

For a given \( \varepsilon > 0 \) and \( \phi \in \mathcal{C}^\infty_c(\mathbb{R}^N) \), it is enough to prove that

\[
\left| \int_{\mathbb{R}^N} \text{div} (G_n) \phi - \int_{\mathbb{R}^N} H_n \phi \right| < \varepsilon.
\]

Because of (6.18), the above inequality will follow, provided there exists \( \eta > 0 \) such that

\[
\left| \int_{\Omega_\eta} \text{div} (G_n) \phi - \int_{\Omega_\eta} H_n \phi \right| < \varepsilon.
\]

Indeed, we prove that each of the integrals in the above inequality goes to zero as \( \eta \to 0 \). First, we estimate \( \int_{\Omega^c_\eta} \text{div} (G_n) \phi \). Note that support of \( G_n \) is compact and the boundary \( \partial \Omega_\eta \) is of class \( \mathcal{C}^1 \), since \( u \in \mathcal{C}^2(\Omega_\delta) \) for \( \delta < \eta \). Thus by the integration by parts, we get

\[
\int_{\Omega^c_\eta} \text{div} (G_n) \phi = \int_{\partial \Omega_\eta} G_n(y) \phi(y) \cdot \gamma(y) ds(y) - \int_{\Omega^c_\eta} G_n \cdot \nabla \phi.
\]

From the definition of \( G_n \), see (6.17), we obtain

\[
|G_n(x)| \leq \eta^p |x||\psi_n(x)| + \frac{1}{p} \eta^p |x||\psi_n| \leq 2\sqrt{2} n \eta^p, \quad \forall x \in \mathbb{R}^N.
\]

Thus

\[
\left| \int_{\Omega^c_\eta} \text{div} (G_n) \phi \right| \leq 2\sqrt{2} n \eta^p (||\phi||_1 + ||\nabla \phi||_1). \tag{6.19}
\]

Now we estimate \( H_n \). We have

\[
H_n = -a(x)|u|^{p-2}u \{ x \cdot \nabla u \} \psi_n + \left( 1 - \frac{N}{p} \right) |\nabla u|^p \psi_n + |\nabla u|^{p-2} \nabla u \cdot \{ x \cdot \nabla u \} \nabla \psi_n. \tag{6.20}
\]
Note that from (6.9), \(|x|\|\nabla \psi_n(x)\| < C\). Thus

\[
|H_n(x)| \leq \sqrt{2n\eta} |a(x)||u|^{p-1} |\psi_n(x)| + \left(1 - \frac{N}{p}\right) \eta^p + C\eta^p
\]

\[
\leq \tilde{C}\eta \left(1 + \|a(x)|u|^{p-1}\psi_n\|_1\right).
\] (6.21)

Therefore,

\[
\left|\int_{\Omega_n^c} H_n \phi\right| \leq \tilde{C}\eta \left(1 + \|a(x)|u|^{p-1}\psi_n(x)\|_1\right) \|\phi\|_1.
\] (6.22)

It is easy to verify that \(a(x)|u|^{p-1} \in L^1_{\text{loc}}(\mathbb{R}^N)\), since \(a \in C^1_{\text{loc}}(\mathbb{R}^N)\) and \(a(x)|u|^p \in L^1(\mathbb{R}^N)\). Thus from (6.19) and (6.22), we obtain the following inequality,

\[
\left|\int_{\Omega_n^c} \text{div}(G_n) \phi - \int_{\Omega_n^c} H_n \phi\right| < \varepsilon.
\] (6.23)

for sufficiently small \(\eta > 0\). Therefore we conclude that

\[
\text{div} G_n = H_n,
\]

in the sense of distributions. Note that by the regularity results of Tolksdorf [75], the solutions of (6.7) are \(C^1_{\text{loc}}(\mathbb{R}^N)\). Thus \(G_n\) is continuous in \(B_{\sqrt{2n}}\).

Further, \(G_n\) vanishes on the boundary of \(B_{\sqrt{2n}}\). Now we use the weak divergence theorem due to Cuesta and Takac [28] to obtain

\[
\int_{B_{\sqrt{2n}}} H_n(x)dx = 0.
\] (6.24)

Next we find a point wise identity for \(a(x)|u|^{p-2}u \{x \cdot \nabla u\} \psi_n\). Let \(F(u) = \frac{|u|^p}{p}\). Thus we obtain the following identity, for a.e. in \(\mathbb{R}^N\):

\[
\text{div} \{x a(x) F(u) \psi_n(x)\} = N a(x) F(u) \psi_n(x) + x \cdot \nabla a(x) F(u) \psi_n(x)
+ a(x)|u|^{p-2} u \{x \cdot \nabla u\} \psi_n(x) + a(x) F(u) x \cdot \nabla \psi_n(x).
\]
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Therefore

\[
H_n(x) = N a(x) F(u) \psi_n + x \cdot \nabla a(x) F(u) \psi_n - \text{div} \{ x a(x) F(u) \psi_n \} \\
+ \left(1 - \frac{N}{p}\right) |\nabla u|^p \psi_n + \{ x \cdot \nabla u \} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi_n.
\]

(6.25)

In particular the above identity holds in the sense of distributions. Now again by applying weak divergence theorem of [28], by noting that \( \psi_n \) vanishes on the boundary we obtain:

\[
\int_{B_{\sqrt{2}n}} \left\{ N a(x) F(u) + x \cdot \nabla a(x) F(u) + \left(1 - \frac{N}{p}\right) |\nabla u|^p \right\} \psi_n(x) \\
+ \int_{B_{\sqrt{2}n}} \{ x \cdot \nabla u \} |\nabla u|^{p-2} \nabla u + a(x) F(u) x \cdot \nabla \psi_n(x) = 0. \quad (6.26)
\]

Note that each term in the above integrals is integrable in entire \( \mathbb{R}^N \). Now by letting \( n \) tend to infinity, using dominated convergence theorem, we obtain

\[
\int_{\mathbb{R}^N} N a(x) F(u) + x \cdot \nabla a(x) F(u) + \left(1 - \frac{N}{p}\right) |\nabla u|^p = 0. \quad (6.27)
\]

Further, from (6.7), we have

\[
\int_{\mathbb{R}^N} |\nabla u|^p = \int_{\mathbb{R}^N} a(x) |u|^p. \quad (6.28)
\]

Now by substituting the above identity in (6.27) we obtain the required identity

\[
\int_{\mathbb{R}^N} \{ x \cdot \nabla a(x) + p a(x) \} |u|^p = 0.
\]

Remark 6.1.4. Let \( a \) and \( u \) be as in the above theorem. If \( x \cdot \nabla a(x) + p a(x) \) has a definite sign, then we have the nonexistence of solution for (6.7). In particular, we have the nonexistence of an eigenvalue for (6.1), when \( x \cdot \nabla g(x) + p g(x) \) has a definite sign.

Example 6.1.5. For example, when \( a(x) = \frac{1}{(1+|x|^2)^2} \), one can verify that

\[
x \cdot \nabla a(x) + p a(x) > 0.
\]
Thus no solution for (6.7) and hence (6.1) does not admit an eigenvalue, when \( g(x) = \frac{1}{(1+|x|^2)^2} \).

**Remark 6.1.6.** We emphasize that the above identity is only a necessary condition and we may not be able to make any conclusion, when the quantity \( x \cdot \nabla a(x) + p a(x) \) is zero. For example, for the Hardy potential \( a(x) = \frac{1}{|x|^p} \), we have

\[
x \cdot \nabla a(x) + p a(x) = 0.
\]

Thus our identity does not prove the nonexistence of an eigenvalue for (6.1).

Later we show that (6.1) does not admit a positive principal eigenvalue for the weight \( g(x) = \frac{1}{|x|^p} \).

### 6.2 Existence of the first eigenvalue and its properties

#### 6.2.1 Existence of an eigenvalue

As in the case Laplacian, here also we use a variational technique for proving the existence of an eigenvalue for (6.1). First let us recall the following definitions:

\[
R_p(u) = \frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} g|u|^p},
\]

\[
D^+_p(g) = \left\{ u \in D_{0}^{1,p}(\Omega) : \int_{\Omega} g|u|^p > 0 \right\},
\]

\[
M_p = \left\{ u \in D_{0}^{1,p}(\Omega) : \int_{\Omega} g|u|^p = 1 \right\},
\]

\[
J_p(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p.
\]

From the definition of \( J_p \), it is obvious that \( J_p \) is coercive and weakly lower semi-continuous on \( D_{0}^{1,p}(\Omega) \). Now, in addition if \( M_p \) is weakly closed, then standard theorem in functional analysis gives the existence of a minimizer for \( J_p \) on \( M_p \). However, \( M_p \) is far from being weakly closed. The weak closedness of \( M_p \) is related to the compactness of the following nonlinear functional:

\[
G_p(u) = \frac{1}{p} \int_{\Omega} g|u|^p.
\]

Due to our weak assumptions on \( g \), the map \( G_p \) may not be even continuous. However, for our objective, weak limits of all weakly convergent
sequences are not required to be in $\mathcal{M}_p$; it is sufficient that only the weak limits of all sequence minimizing $J_p$ over $\mathcal{M}_p$ lie in $\mathcal{M}_p$. We show this is indeed true, under our assumptions on $g^+$. First we prove the following preparatory lemma:

**Lemma 6.2.1.** Let $g^+ \in F_N^p \setminus \{0\}$ and let

$$G^+_p(u) = \frac{1}{p} \int_\Omega g^+ |u|^p.$$ 

Then $G^+_p$ is compact.

**Proof.** Let $\{u_n\}$ converge weakly to $u$ in $\mathcal{D}_0^{1,p}(\Omega)$. First we show that a subsequence of $\{G^+_p(u_n)\}$ converges to $G^+_p(u)$. Let $\phi \in C_0^\infty(\Omega)$ be arbitrary. Now we write:

$$p(G^+_p(u_n) - G^+_p(u)) = \int_\Omega \phi (|u_n|^p - |u|^p) + \int_\Omega (g^+ - \phi) (|u_n|^p - |u|^p). \quad (6.29)$$

First we estimate the second integral using Lorentz Sobolev embedding and Hölder inequality as below

$$\int_\Omega |(g^+ - \phi)| \left(|u_n|^p - |u|^p\right) \leq C \|g^+ - \phi\|_{(N,p,\infty)} \left(\|u_n\|^p_{(p^*,p)} + \|u\|^p_{(p^*,p)}\right).$$

where $C$ is a constant which depends only on $N$ and $p$. Clearly $u_n$ is a bounded sequence in $L(p^*,p)$. Let

$$m := \sup_n \left\{\|u_n\|^p_{(p^*,p)} + \|u\|^p_{(p^*,p)}\right\}.$$

Now using the definition of the space $F_N^p$, for a given $\varepsilon > 0$, we choose $g_\varepsilon \in C_0^\infty(\Omega)$ so that

$$\|g^+ - g_\varepsilon\|_{(N,p,\infty)} < \frac{\varepsilon}{m C}.$$ 

Thus by taking $\phi = g_\varepsilon$, we obtain

$$\int_\Omega |(g^+ - g_\varepsilon)| \left(|u_n|^p - |u|^p\right) < \varepsilon. \quad (6.30)$$

Since $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p_{loc}(\Omega)$ compactly, the first integral in (6.29) converges to
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zero with $\phi = g_\varepsilon$. Thus there exists $n_0 \in \mathbb{N}$, so that

$$\int_{\Omega} g_\varepsilon (|u_n|^p - |u|^p) < \varepsilon, \quad \forall n > n_0.$$  

Now using (6.30), we get

$$|G_p^+(u_n) - G_p^+(u)| < C \varepsilon, \quad \forall n > n_0,$$  

(6.31)

where $C$ is a constant independent of $n$. Thus we conclude that the sequence $G_p^+(u_n)$ converges to $G_p^+(u)$.  

Now we are in a position to prove the existence of a minimizer for $J_p$ on $\mathcal{M}_p$.  

**Theorem 6.2.2.** Let $\Omega$ be a domain in $\mathbb{R}^N$ with $p \in (1, N)$ and let $g \in L^1_{\text{loc}}(\Omega)$. If $g^+ \in L^1_p(\Omega \setminus \{0\})$, then $J_p$ admits a minimizer on $\mathcal{M}_p$.

**Proof.** Since $g \in L^1_{\text{loc}}(\Omega)$ and $g^+ \neq 0$, there exists $\varphi \in C^\infty_c(\Omega)$ such that $\int_{\Omega} g|\varphi|^p > 0$ (see Lemma 6.3.9) and hence $\mathcal{M}_p \neq \emptyset$. Let $\{u_n\}$ be a minimizing sequence of $J_p$ on $\mathcal{M}_p$. Thus

$$\lim_{n \to \infty} J_p(u_n) = \lambda_1 = \inf_{u \in \mathcal{M}_p} J_p(u).$$

By the coercivity of $J_p$, $\{u_n\}$ is bounded and hence using the reflexivity of $D_0^{1,p}(\Omega)$ we obtain a subsequence of $\{u_n\}$ that converges weakly to some $u \in D_0^{1,p}(\Omega)$. We denote the subsequence by $\{u_n\}$ itself. Now using the compactness of $G_p^+$, we get

$$\lim_{n \to \infty} \int_{\Omega} g^+|u_n|^p = \int_{\Omega} g^+|u|^p.$$  

(6.32)

Now as $u_n \in \mathcal{M}_p$ we write,

$$\int_{\Omega} g^-|u_n|^p = \int_{\Omega} g^+|u_n|^p - 1.$$  

Since the embedding $D_0^{1,p}(\Omega) \hookrightarrow L^p_{\text{loc}}(\Omega)$ is compact, up to a subsequence $u_n \to u$ a.e in $\Omega$. Hence using (6.32) and the Fatou’s lemma, we obtain

$$\int_{\Omega} g^-|u|^p \leq \int_{\Omega} g^+|u|^p - 1.$$
this shows that \( \int_{\Omega} g|u|^p \geq 1 \). Set \( \tilde{u} := \frac{u}{(\int_{\Omega} g|u|^p)^{1/p}} \). Now the weak lower semi-continuity of \( J_p \) yields the following:

\[
\lambda_1 \leq J_p(\tilde{u}) = \frac{J_p(u)}{\int_{\Omega} g|u|^p} \leq J_p(u) \leq \liminf_n J_p(u_n) = \lambda_1
\]

Thus the equality must hold at each step and hence \( \int_{\Omega} g|u|^p = 1 \), which shows that \( u \in M_p \) and \( J_p(u) = \lambda_1 \). \( \square \)

**Remark 6.2.3.** It is easy to see that there is a one to one correspondence between the minimizers \( J_p \) on \( M_p \) and the minimizers of \( R_p \) on \( D_1^+(g) \).

Note that \( R_p \) is not regular enough to conclude that \( u \) is an eigenfunction of (6.2) corresponding to \( \lambda_1 \), using the critical point theory.

**Proposition 6.2.4.** Let \( g \) be as in the above theorem and let \( u \) be a minimizer of \( J_p \) on \( M_p \). Then \( u \) is an eigenfunction of (6.1) corresponding to \( \lambda_1 \).

**Proof.** For each \( \phi \in C_c^\infty(\Omega) \), using the dominated convergence theorem one can verify that \( R_p \) admits the directional derivative along \( \phi \). Since \( u \) is a minimizer of \( J_p \) on \( D_1^+(g) \) we get

\[
\frac{d}{dt} R_p(u + t\phi)|_{t=0} = 0.
\]

Therefore

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi = \lambda_1 \int_{\Omega} g |u|^{p-2} u \phi, \quad \forall \phi \in C_c^\infty(\Omega).
\]

Now using the density of \( C_c^\infty(\Omega) \) in \( D_0^{1,p}(\Omega) \) we obtain

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v = \lambda_1 \int_{\Omega} g |u|^{p-2} u v, \quad \forall v \in D_0^{1,p}(\Omega).
\]

This shows that \( u \) is an eigenfunction of (6.1) corresponding to \( \lambda_1 \). \( \square \)

### 6.2.2 Sign and the uniqueness of minimizer

First we prove that eigenfunctions corresponding to \( \lambda_1 \) are of constant sign. Due to the lack of regularity of first eigenfunctions, we can’t use the classical strong maximum principle to prove that the first eigenfunctions are of constant sign. For \( p = 2 \), we used the strong maximum principle due to
Brezis and Ponce [19] to show that first eigenfunctions are of constant sign (see Chapter 4). In [50] a similar strong maximum principle has been proved for more general quasilinear operators. From Proposition 3.2 of [50] one can obtain the following lemma.

**Lemma 6.2.5 (Strong Maximum principle for $\Delta_p$).** Let $u \in D_0^{1,p}(\Omega)$ and $V \in L_{loc}^1(\Omega)$ be such that $u, V \geq 0$ a.e in $\Omega$. If $Vu^{p-1} \in L_{loc}^1(\Omega)$ and $u$ satisfies the following differential inequality (in the sense of distributions)

$$-\Delta_p u + V(x)u^{p-1} \geq 0 \quad \text{in } \Omega.$$

Then either $u \equiv 0$ or $u > 0$ a.e.

Now from the above lemma we have the following result.

**Lemma 6.2.6.** Let $g$ be as in Theorem 6.2.2. Then each eigenfunction corresponding to $\lambda_1$ is of constant sign.

**Proof.** It is clear that the eigenfunctions corresponding to $\lambda_1$ are the minimizers of $R_p$ on $D_0^+(g)$. Let $u$ be a minimizer of $R_p$ on $D_0^+(g)$. Since $u \neq 0$ either $u^+$ or $u^-$ is non zero. Without loss of generality we may assume that $u^+ \neq 0$. Now by taking $u^+$ as a test function in (6.2), we see that $u^+$ also minimizes $R_p$ on $D_0^+(g)$. Thus by Proposition 6.2.4, we see that $u^+$ also solves (6.1) in the weak sense,

$$-\Delta_p u^+ - \lambda_1 g(u^+)^{p-1} = 0, \quad \text{in } \Omega.$$

In particular we have the following differential inequality in the sense of distributions,

$$-\Delta_p u^+ + \lambda_1 g^- (u^+)^{p-1} = \lambda_1 g^+(u^+)^{p-1} \geq 0, \quad \text{in } \Omega.$$

It is clear that $g^-$ and $u^+$ satisfy all the assumptions of Lemma 6.2.5, if we show that $g^-(u^+)p \in L_{loc}^1(\Omega)$. Since $g|u|^p \in L^1(\Omega)$, we have $(g^-)^{\frac{1}{q}}(u^+)^{p-1} \in L^q(\Omega)$, where $q$ is the conjugate exponent of $p$. Further, $(g^-)^{\frac{1}{p}} \in L_{loc}^p(\Omega)$, since $g \in L_{loc}^1(\Omega)$. Let us write

$$g^-(u^+)^{p-1} = (g^-)^{\frac{1}{p}}(g^-)^{\frac{1}{q}}(u^+)^{p-1}.$$

Now we use Hölder inequality to conclude that $g^-(u^+)^{p-1} \in L_{loc}^1(\Omega)$. Now in view of Lemma 6.2.5, we obtain $u^+ > 0$ a.e. and hence $u = u^+$. Moreover, the zero set of $u$ is of measure of zero. □
6.2 Existence of the first eigenvalue and its properties

From the above lemma, it is clear that \( \lambda_1 \) is a principal eigenvalue of (6.1). Next we prove the uniqueness of the positive principal eigenvalue of (6.1), using the Picone’s identity for the p-Laplacian. In [6], Picone’s identity is proved for \( C^1 \) functions. However it is not difficult to obtain a similar identity for less regular functions.

**Lemma 6.2.7.** (Picone’s identity) Let \( u \geq 0, v > 0 \) a.e. and let \( |\nabla v|, |\nabla u| \) exist as measurable functions. Then the following identity holds a.e.

\[
|\nabla u|^p + (p - 1) \frac{u^p}{v^p} |\nabla v|^p - \frac{u^{p-1}}{v^p-1} |\nabla v|^{p-2} \nabla v = |\nabla u|^p - \nabla \left( \frac{u^p}{v^{p-1}} \right) \cdot |\nabla v|^{p-2} \nabla v.
\]

Furthermore, the left and the right hand sides of the above identity are non-negative.

Now we prove the uniqueness of the positive principal eigenvalue of (6.1).

**Lemma 6.2.8.** Let \( g \in L(\frac{N}{p}, \infty) \) and let \( \lambda > 0 \) be a positive principal eigenvalue of (6.1). Then

\[
\lambda = \lambda_1 = \inf \left\{ \int_{\Omega} |\nabla u|^p : u \in \mathcal{M}_p \right\}.
\]

**Proof.** Let \( v \in \mathcal{D}_0^{1,p}(\Omega) \) be a positive eigenfunction of (6.1) corresponding to \( \lambda \). Let \( u \in \mathcal{M}_p \). Thus there exists a sequence \( \{\phi_n\} \) in \( C^\infty(\Omega) \) such that \( \|u - \phi_n\|_{\mathcal{D}_0^{1,p}(\Omega)} \to 0 \) and \( \int_{\Omega} g|u|^p = 1 \). Note that \( \frac{|\phi_n|^p}{v+\varepsilon} \in \mathcal{D}_0^{1,p}(\Omega) \) (see Proposition 2.3.7). Now by applying the Picone’s identity for \( |\phi_n| \) and \( v + \varepsilon \), we obtain the following inequality:

\[
0 \leq \int_{\Omega} |\nabla |\phi_n||^p - \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla \left( \frac{|\phi_n|^p}{(v+\varepsilon)^{p-1}} \right). \tag{6.33}
\]

Since \( v \) is an eigenfunction of (6.1) corresponding to \( \lambda \), we have

\[
\int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla \left( \frac{|\phi_n|^p}{(v+\varepsilon)^{p-1}} \right) = \lambda \int_{\Omega} g v^{p-1} \frac{|\phi_n|^p}{(v+\varepsilon)^{p-1}}. \tag{6.34}
\]

Now from (6.33) and (6.34) we obtain the following:

\[
0 \leq \int_{\Omega} |\nabla |\phi_n||^p - \lambda \int_{\Omega} g v^{p-1} \frac{|\phi_n|^p}{(v+\varepsilon)^{p-1}}. \tag{6.35}
\]
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By letting $\varepsilon \to 0$, the dominated convergence theorem yields:

$$0 \leq \int_{\Omega} |\nabla \phi_n|^p - \lambda \int_{\Omega} g \phi_n^p.$$  \hspace{1cm} (6.36)

Note that $\int_{\Omega} |\nabla \phi_n|^p = \int_{\Omega} |\nabla \phi_n|^p$. Now using Corollary 2.2.8 and the Fatou’s lemma we let $n \to \infty$ to obtain the following inequality:

$$0 \leq \int_{\Omega} |\nabla u|^p - \lambda \int_{\Omega} g u^p.$$  \hspace{1cm} (6.37)

Therefore

$$\lambda \leq \int_{\Omega} |\nabla u|^p, \quad \forall u \in \mathcal{M}_p.$$  \hspace{1cm} (6.38)

This completes the proof. \hfill \square

**Remark 6.2.9.** As a consequence of the above lemma, we see that the eigenfunctions corresponding to other eigenvalues of (6.1) must change sign.

**Remark 6.2.10.** The continuity argument that we used for proving the simplicity of the first eigenvalue of Laplacian (see Proposition 4.3.14) is not applicable here, since the sum of two eigenfunctions corresponding to an eigenvalue of $p$-Laplacian is no longer an eigenfunction. In [59], Lucia and Prashanth obtained the simplicity of the first eigenvalue of (6.1), if it exit, even when $g \in L^r(\Omega), r > 1$ and $\Omega$ is connected. Later in [50], Kawohl, Lucia and Prashanth extended this result to $g$ which are only in $L^1_{1\text{loc}}(\Omega)$. For a proof of simplicity of $\lambda_1$ we refer to Theorem 1.3 of [50].

From Theorem 6.2.2, Lemma 6.2.4, Remark 6.2.9 and Theorem 1.3 of [50] we have the following result:

**Theorem 6.2.11.** Let $\Omega$ be an open connected subset of $\mathbb{R}^N$. Let $p \in (1, N)$ and let $g \in L^1_{1\text{loc}}(\Omega)$ such that $g^+ \in \mathcal{F}_N^p \setminus \{0\}$. Then

$$\lambda_1 = \inf \{J_p(u) : u \in \mathcal{M}_p\}$$  \hspace{1cm} (6.39)

is the unique positive principal eigenvalue of (6.1). Furthermore, each eigenfunction corresponding to $\lambda_1$ is of constant sign and $\lambda_1$ is simple.

### 6.2.3 Radial solution

Now we give sufficient conditions for the radial symmetry of eigenfunctions, if exists, corresponding to the eigenvalue $\lambda_1$ of (6.1). Here we assume that
6.2 Existence of the first eigenvalue and its properties

In [17], Bhattacharya proved the radial symmetry of the first eigenfunctions of (6.1), when \( g \equiv 1 \) and \( \Omega \) is ball. Here we prove that all positive eigenfunctions corresponding to \( \lambda_1 \) are radial and radially decreasing, provided \( g \) is nonnegative, radial and radially decreasing. Thus our result is a two fold generalization of results of Bhattacharya, as we allow more general weight functions and the domain can be \( \mathbb{R}^N \). Our result uses certain rearrangement inequalities. We emphasize that here we are not assuming any conditions to ensure that \( \lambda_1 \) is an eigenvalue.

Theorem 6.2.12. Let \( \Omega \) be a ball centred at origin or \( \mathbb{R}^N \). Let \( g \) be nonnegative, radial and radially decreasing measurable function. If \( \lambda_1 \) is an eigenvalue of (6.1), then any positive eigenfunction corresponding to \( \lambda_1 \) is radial and radially decreasing.

Proof. Let \( u \) be a positive eigenfunction of (6.1) corresponding to \( \lambda_1 \). Let \( u_* \) and \( g_* \) be the symmetric decreasing rearrangement of \( u \) and \( g \) respectively. Since \( g \) is nonnegative, radial and radially decreasing, we use property (a) of Proposition 2.1.11 to conclude that \( g = g_* \) a.e. Further, as \( u \) is positive by property (c) of Proposition 2.1.11 we obtain \( (u^p)_* = (u_*)^p \) a.e. Now by the Hardy-Littlewood inequality (see Theorem 2.1.13),

\[
\int_{\Omega} g u^p \leq \int_{\Omega} g_*(u^p)_* = \int_{\Omega} g(u_*)^p.
\]

Further due to Polya-Szego we have the following inequality:

\[
\int_{\Omega} |\nabla u_*|^p \leq \int_{\Omega} |\nabla u|^p.
\]

Thus

\[
\frac{1}{\Omega} g(u_*)^p \int_{\Omega} |\nabla u_*|^p \leq \frac{1}{\Omega} g(u)^p \int_{\Omega} |\nabla u|^p.
\]  

(6.40)

Since \( u \) is a minimizer of \( R_p \) on \( \mathcal{D}_p^+ (g) \), equality holds in (6.40) and hence \( u_* \) also minimizes \( R_p \) on \( \mathcal{D}_p^+ (g) \). Now as \( \lambda_1 \) is simple, we get \( u_* = \alpha u \) a.e. for some \( \alpha > 0 \). This shows that \( u \) is radial and radially decreasing.

Next we prove that the Hardy potential, \( \frac{1}{|x|^p} \) does not admit a positive principal eigenvalue for (6.1).

Proposition 6.2.13. Let \( g(x) = \frac{1}{|x|^p} \), \( x \in \mathbb{R}^N \). Then (6.1) does not admit a positive principal eigenvalue.
Proof. By the above lemma, it is enough to show that \( \lambda_1 \) is not an eigenvalue of (6.1), when \( g(x) = \frac{1}{|x|^p} \). If \( \lambda_1 \) is an eigenvalue of (6.1), then by Theorem 1.3 of [50] we must have \( \lambda_1 \) is simple. Further, if \( u \) is an eigenfunction of (6.1) corresponding \( \lambda_1 \), then using the scale invariance of (6.1), for each \( \alpha \in \mathbb{R} \), one can verify that
\[
v_{\alpha}(x) = u(\alpha x)
\]
is also an eigenfunction of (6.1), corresponding to \( \lambda_1 \). Now using the simplicity of \( \lambda_1 \) and the radial symmetry of \( u \), one can show that
\[
u(x) = |x|^{1-\frac{N}{p}} u(1).
\]
This is a contradiction as \( |x|^{1-\frac{N}{p}} \notin D_0^{1,p}(\Omega) \).

6.3 Existence of infinitely many eigenvalues

In this section we discuss the existence of infinitely many eigenvalues of (6.1). Our proof relies on the Ljusternik-Schnirelmann theory on a \( C^1 \) manifold due to Szulkin [72]. The classical Ljusternik-Schnirelmann minimax theorem uses a deformation homotopy which requires the set \( \mathcal{M}_p \) to be at least a \( C^{1,1} \) manifold (i.e, transition maps are \( C^1 \) and its derivatives are locally Lipschitz). However Szulkin, in [72] developed Ljusternik-Schnirelmann theorem on a \( C^1 \) manifold using the Ekeland variational principle. It is worth mentioning that due to the weaker assumptions on the weight \( g \), the set \( \mathcal{M}_p \) that we are considering does not even possess a manifold structure from the topology inherited from \( D_0^{1,p}(\Omega) \). However, we define a suitable topology on \( \mathcal{M}_p \) that makes \( \mathcal{M}_p \) a \( C^1 \) Banach manifold and use a result of [72] to get an infinite sequence of eigenvalues tending to infinity.

First we give the following definitions: Let \( \mathcal{M} \) be a \( C^1 \) manifold and \( f \in C^1(\mathcal{M};\mathbb{R}) \). Denote the differential of \( f \) at \( u \) by \( df(u) \), an element of \( (T_u\mathcal{M})^* \), the cotangent space of \( \mathcal{M} \) at \( u \) (see section 27.4 of [30] for definition and properties ).

Let \( A \) be a closed and symmetric (i.e, \( -A = A \)) subset of \( \mathcal{M} \). The Krasnoselskii genus \( \gamma(A) \) is defined to be the smallest integer \( k \) for which there exists a non-vanishing odd continuous mapping from \( A \) to \( \mathbb{R}^k \). If there exists no such map for any \( k \), then we define \( \gamma(A) = \infty \) and we set \( \gamma(\emptyset) = 0 \). For more details and properties of genus we refer to Chapter 7 of [66].

We say that a map \( f \in C^1(\mathcal{M};\mathbb{R}) \) satisfies the Palais-smale (P.S. for short ) condition on \( \mathcal{M} \), if a sequence \( \{u_n\} \subset \mathcal{M} \) is such that \( f(u_n) \to \lambda \)
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and \( df(u_n) \to 0 \) then \( \{u_n\} \) possesses a convergent subsequence.

From Corollary 4.1 of [72] one can deduce the following theorem:

**Theorem 6.3.1 (Szulkin’s Theorem).** Let \( \mathcal{M} \) be a closed symmetric \( \mathcal{C}^1 \)
submanifold of a real Banach space \( X \) and \( 0 \notin \mathcal{M} \). Let \( f \in \mathcal{C}^1(\mathcal{M};R) \) be
an even function which satisfies P.S. condition on \( \mathcal{M} \) and bounded below. Define

\[
c_j := \inf_{A \in \Gamma_j} \sup_{x \in A} f(x),
\]

where \( \Gamma_j = \{ A \subset \mathcal{M} : A \text{ is compact and symmetric about origin, } \gamma(A) \geq j \} \).

If for a given \( j \), \( c_j = c_{j+1} = \ldots = c_{j+p} \equiv c \), then \( \gamma(K_c) \geq p + 1 \), where
\( K_c = \{ x \in \mathcal{M} : f(x) = c, df(x) = 0 \} \).

Note that the set, \( \mathcal{M}_p = \left\{ u \in D_0^{1,p}(\Omega) : \int_\Omega g|u|^p = 1 \right\} \), may not even
possess a manifold structure from the topology of \( D_0^{1,p}(\Omega) \), due to the weak assumptions on \( g^- \). However we show that \( \mathcal{M}_p \) admits a \( \mathcal{C}^1 \) Banach manifold structure from a subspace of \( D_0^{1,p}(\Omega) \).

For \( g^- \in L_{1,loc}^{1}(\Omega) \), we define

\[
\|u\|_X^p := \int_\Omega |\nabla u|^p + \int_\Omega g^- |u|^p.
\]

\[
X := \{ u \in D_0^{1,p}(\Omega) : \|u\|_X < \infty \}.
\]

Then one can easily verify the following:

- \( X \) is a Banach space with the norm \( \| \cdot \|_X \) and \( X \) is reflexive.
- Since \( g^- \) is locally integrable, \( \mathcal{C}^\infty_c(\Omega) \) is contained in \( X \).
- Let \( g \in L_1^{loc}(\Omega) \) and \( g^+ \in \mathcal{F}_{\frac{p}{p-1}} \). Then \( \mathcal{D}_p^+(g) \) is contained in \( X \). This can be seen as below:

\[
\int_\Omega g^- |u|^p < \int_\Omega g^+ |u|^p \leq C \|g^+\|_{L^p(\Omega)} \|u\|_{\mathcal{D}_p^{1,p}(\Omega)}^p < \infty,
\]

where \( C \) is the constant involving the constants that are appearing in the Lorentz-Sobolev embedding and the Hölder inequality. Note that the first inequality follows as \( \int_\Omega g^- |u|^p > 0 \), for \( u \in \mathcal{D}_p^{1,p}(g) \).

- \( X \) is continuously embedded into \( \mathcal{D}_0^{1,p}(\Omega) \). Thus \( X \) embedded continuously into the Lorentz space \( L(p^\ast, p) \) and embedded compactly into \( L_{1,loc}^p(\Omega) \).
We denote the dual space of $X$ by $X'$ and the duality action by $\langle \cdot , \cdot \rangle$.

Using the definition of the norm one can easily see that, the map $G_p^-$, defined by

$$G_p^-(u) := \frac{1}{p} \int_{\Omega} g^- |u|^p,$$

is continuous on $X$. Further, using the dominated convergence theorem one can verify that $G_p^-$ is continuously differentiable on $X$ and its derivative is given by

$$\langle G_p^-'(u), v \rangle = \int_{\Omega} g^- |u|^{p-2} u v.$$

Similarly using the Sobolev embedding and the Hölder inequality one can easily verify that $G_p^+$ is $C^1$ in $D_0^{1,p}(-\Omega)$ and in particular on $X$. The derivative of $G_p^+$ is given by

$$\langle G_p^+(u), v \rangle = \int_{\Omega} g^+ |u|^{p-2} u v.$$

Note that for $u \in M_p$, $\langle G_p'(u), u \rangle = p$ and hence the map $G_p'(u) \neq 0$. Recall that, $c \in \mathbb{R}$ is called a regular value of $G_p$, if $G_p'(u) \neq 0$ for all $u$ such that $G_p(u) = c$. Thus we have the following lemma:

**Lemma 6.3.2.** Let $g$ be as in theorem 6.2.11. Then the map $G_p$ is in $C^1(X; \mathbb{R})$ and $G_p' : X \to X'$ is given by

$$\langle G_p'(u), v \rangle = \int_{\Omega} g|u|^{p-2} u v.$$

Further, 1 is a regular value of $G_p$.

**Remark 6.3.3.** In view of Example 27.2 of [30], the above lemma shows that $M_p$ is a $C^1$ Banach submanifold of $X$. Note that $M_p$ is symmetric about the origin as the map $G_p$ is even.

Next we show that $J_p$ satisfies all the conditions to apply Theorem 6.3.1.

**Lemma 6.3.4.** $J_p$ is a $C^1$ functional on $M_p$ and the derivative of $J_p$ is given by

$$\langle J_p'(u), v \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v.$$

The proof is straightforward.
6.3 Existence of infinitely many eigenvalues

Remark 6.3.5. Using Proposition 6.4.35 of [35], one can deduce that

\[ \|dJ_p(u)\| = \min_{\lambda \in \mathbb{R}} \left\| J'_p(u) - \lambda G'_p(u) \right\|. \]  

(6.42)

Thus \( dJ_p(u_n) \to 0 \) if and only if there exists a sequence \( \{\lambda_n\} \) of real numbers such that \( J'_p(u_n) - \lambda_n G'_p(u_n) \to 0 \).

In the next lemma we prove the compactness of the map \( G^+_p \), that we use for showing that the map \( J_p \) satisfies P.S. condition on \( M_p \).

Lemma 6.3.6. The map \( G^+_p : X \to X' \) is compact.

Proof. Let \( u_n \to u \) in \( X \) and \( v \in X \). Let \( q \) be the conjugate exponent of \( p \). Now using the Lorentz-Sobolev embedding and the Hölder inequality available for the Lorentz spaces, one can verify the following:

\[
(g^+)^{\frac{1}{q}} (|u_n|^{p-2} u_n - |u|^{p-2} u) \in L \left( \frac{p^*}{p-1}, \frac{p}{p-1} \right),
\]

\[ (g^+)^{\frac{1}{p}} |v| \in L(p, p), \]

\[ \| (g^+)^{\frac{1}{p}} v \|_p \leq C \| g^+ \|_{L(\frac{q}{p}, \infty)} \| v \|_{(p^*, p)}, \]

where \( C \) is a constant that depends only on \( p, N \). Now by using usual Hölder inequality we get,

\[
| (G'_p(u_n) - G'_p(u), v) | \leq \int_{\Omega} g^+ \left( |u_n|^{p-2} u_n - |u|^{p-2} u \right) |v|.
\]

\[
\leq \left( \int_{\Omega} g^+ \left( |u_n|^{p-2} u_n - |u|^{p-2} u \right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left( \int_{\Omega} g^+ |v|^p \right)^{\frac{1}{p}}.
\]

\[
\leq \| g^+ \|_{L(\frac{q}{p}, \infty)} \| v \|_{(p^*, p)} \left( \int_{\Omega} g^+ \left( |u_n|^{p-2} u_n - |u|^{p-2} u \right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}.
\]

Thus

\[
\| G'_p(u_n) - G'_p(u) \| \leq \| g^+ \|_{L(\frac{q}{p}, \infty)} \left( \int_{\Omega} g^+ \left( |u_n|^{p-2} u_n - |u|^{p-2} u \right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}.
\]
Now it is enough to show that
\[
\left( \int_{\Omega} g^+ \left( |u_n|^{p-2}u_n - |u|^{p-2}u \right) \right)^{\frac{p-1}{p}} \to 0, \quad \text{as } n \to \infty.
\]

Let \( \varepsilon > 0 \) and \( g_\varepsilon \in C^\infty_c(\Omega) \) be arbitrary.

\[
\int_{\Omega} g^+ \left( |u_n|^{p-2}u_n - |u|^{p-2}u \right) \frac{p}{p-1} = \int_{\Omega} g_\varepsilon \left( |u_n|^{p-2}u_n - |u|^{p-2}u \right) \frac{p}{p-1} + \int_{\Omega} (g^+ - g_\varepsilon) \left( |u_n|^{p-2}u_n - |u|^{p-2}u \right) \frac{p}{p-1}
\]

(6.43)

First we estimate the second integral. Note that \( |(u_n|^{p-2}u_n - |u|^{p-2}u| \) is bounded in \( L^\left( \frac{p}{p-1} \right) \). Let

\[
m = \sup_n \left\| \left( |u_n|^{p-2}u_n - |u|^{p-2}u \right) \right\|_{L^\left( \frac{p}{p-1} \right)}.
\]

\[
\int_{\Omega} (g^+ - g_\varepsilon) \left( |(u_n|^{p-2}u_n - |u|^{p-2}u\right) \frac{p}{p-1} \leq C \cdot m \cdot \left\| (g^+ - g_\varepsilon) \right\|_{L^\left( \frac{p}{p-1} \right)} \leq C \cdot m \cdot \left\| (g^+ - g_\varepsilon) \right\|_{L^\left( \frac{p}{p-1} \right)},
\]

where the constant \( C \) includes all the constants that appear in the Hölder inequality and the Lorentz-Sobolev embedding. Now since \( g^+ \in F^\frac{N}{p} \), from the definition of \( F^\frac{N}{p} \), we can choose \( g_\varepsilon \in C^\infty_c(\Omega) \) such that

\[
m \cdot \left\| (g^+ - g_\varepsilon) \right\|_{L^\left( \frac{p}{p-1} \right)} < \frac{\varepsilon}{2C}.
\]

Thus we can make the second integral in (6.43) smaller than \( \frac{\varepsilon}{2} \) for a suitable choice of \( g_\varepsilon \). Since \( X \) is embedded compactly into \( L^\left( \frac{p}{p-1} \right) \), the first integral converges to zero up to a subsequence \( \{u_{nk}\} \) of \( \{u_n\} \). Hence we get \( k_0 \in \mathbb{N} \) so that

\[
\int_{\Omega} g^+ \left( |u_{nk}|^{p-2}u_{nk} - |u|^{p-2}u \right) \frac{p}{p-1} < \varepsilon, \quad \forall k > k_0.
\]

Now the uniqueness of limit of subsequence helps us to conclude, as in Lemma 6.2.1, that \( \left( \int_{\Omega} g^+ \left( |u_n|^{p-2}u_n - |u|^{p-2}u \right) \right)^{\frac{p-1}{p}} \to 0 \) as \( n \to \infty \).

Hence the proof. 

\( \square \)
6.3 Existence of infinitely many eigenvalues

**Definition 6.3.7.** For $\lambda \in \mathbb{R}^+$, we define $A_\lambda : X \to X'$ as

$$A_\lambda = J'_p + \lambda G_p^-.$$  

In the next proposition we show that the map $J_p$ indeed satisfies P.S. condition on the $M_p$.

**Proposition 6.3.8.** $J_p$ satisfies P.S. condition on $M_p$.

**Proof.** Let $\{u_n\}$ be a sequence in $M_p$, such that $J_p(u_n) \to \lambda$ and $dJ_p(u_n) \to 0$. Thus there exists a sequence $\{\lambda_n\}$ such that

$$J'_p(u_n) - \lambda_n G_p(u_n) \to 0$$  

as $n \to \infty$. (6.44)

Since $J_p(u_n)$ is bounded, using the estimate (6.41), we see that $\{G_p^-(u_n)\}$ is bounded. Thus the sequence $\{u_n\}$ is bounded in $X$ and hence by the reflexivity we may assume that $u_n \to u$, by passing to a subsequence. Since $G_p^+$ is compact, we get $G_p^+(u_n) \to G_p^+(u)$. Now by Fatou’s lemma

$$\int_{\Omega} g^- |u|^p \leq \liminf \int_{\Omega} g^+ |u_n|^p - 1 = \int_{\Omega} g^+ |u|^p - 1.$$  

(6.45)

Thus $\int_{\Omega} g|u|^p \geq 1$ and hence $u \neq 0$. Further, $\lambda_n \to \lambda$ as $n \to \infty$, since

$$p(J_p(u_n) - \lambda_n) = \left< J'_p(u_n) - \lambda_n G_p(u_n), u_n \right> \to 0.$$  

Now we write (6.44) as

$$A_\lambda(u_n) - \lambda_n G_p^+(u_n) \to 0.$$  

Since $\lambda_n \to \lambda$, we obtains $A_\lambda(u_n) - A_\lambda(u_n) \to 0$. Now the compactness of $G_p^+$ yields the strong convergence of $A_\lambda(u_n)$ and hence $\langle A_\lambda(u_n), u_n - u \rangle \to 0$. Since $u_n \to u$, using Lemma 4.3 of [73] one obtain $u_n \to u$. \hfill $\square$

Next we show that $\Gamma_n \neq \emptyset$, using the same argument as in Lemma 4.4.1

**Lemma 6.3.9.** For each $n \in \mathbb{N}$, the set $\Gamma_n \neq \emptyset$.

**Proof.** The idea is to construct odd continuous maps from $S^{n-1} \to M$, for each $n \in \mathbb{N}$. Let $\Omega^+ = \{x : g^+(x) > 0\}$. Since $|\Omega^+| > 0$, using the Lebesgue-Besicovitch differentiation theorem, one can choose $n$ points $x_1, x_2, \ldots, x_n$ in
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\[ \Omega^+ \] such that

\[ \lim_{r \to 0} \frac{1}{|B_r(x_i)|} \int_{B_r(x_i)} g(y) dy = g(x_i) > 0. \]

Thus there exists $R > 0$, such that $B_R(x_i) \cap B_R(x_j) = \emptyset$ and

\[ \int_{B_r(x_i)} g(y) dy > 0, \text{ for } 0 < r < R. \]

Now one can choose $r$ such that $0 < r < R$ and

\[ \int_{B_R(x_i) \setminus B_r(x_i)} |g(y)| dy < \int_{B_r(x_i)} g(y) dy. \] (6.46)

Let $u_i \in C^\infty_c(B_R(x_i))$ such that $0 \leq u_i(x) \leq 1$ and $u_i \equiv 1$ on $B_r(x_i)$. Now using (6.46) we have the following

\[ \int_{B_r(x_i)} g|u_i|^p = \int_{B_r(x_i)} g + \int_{B_R(x_i) \setminus B_r(x_i)} g|u_i|^p \]

\[ \geq \int_{B_r(x_i)} g - \int_{B_R(x_i) \setminus B_r(x_i)} |g| > 0. \]

Thus we get $v_i = \frac{u_i}{(\int_{\Omega} g|u_i|^p)^{\frac{1}{p}}} \in M$. Note that the support of $v_i$’s are disjoint.

Now for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\sum |\alpha_i|^p = 1$, we have $\sum \alpha_i u_i \in C^\infty_c(\Omega)$ and $\int_{\Omega} g \sum \alpha_i v_i |^p = 1$. It is easy to see that the map $\phi(\alpha) = \sum \alpha_i u_i$ is an odd continuous map from $S^{n-1}$ into $\mathcal{M}_p$. Thus $\phi(S^{n-1})$ is compact and symmetric about origin. Now from the definition of genus it follows that $\gamma(\phi(S^{n-1})) \geq \gamma(S^{n-1}) = n$. \hfill \Box

Now we are in a position to adapt the Ljusternik-Schnirelmann theorem available for $C^1$ manifold in our situation and prove the existence of infinitely many eigenvalues for (6.1). In fact we prove the following theorem:

**Theorem 6.3.10.** Let $p \in (1, N)$, $g \in L^1_{\text{loc}}(\Omega)$ and $g^+ \in \mathcal{F}_p \setminus \{0\}$. Then (6.1) admits a sequence of positive eigenvalues going to $\infty$.

**Proof.** From all the above $J_p$ and $\mathcal{M}_p$ satisfy all the requirements of Theorem 6.3.1, for each $j \in \mathbb{N}$ and so we have $\gamma(K_{c_j}) \geq 1$. Thus $K_{c_j} \neq \emptyset$ and hence there exists $u_j \in M$ such that $dJ_p(u_j) = 0$ and $J_p(u_j) = c_j$. Therefore $c_j$ is an eigenvalue of (6.1) and $u_j$ is an eigenfunction corresponding to $c_j$.

A proof for the unboundedness of the sequence $\{c_n\}$ is given in [45], see Theorem 2. For the sake of completeness we adapt the same idea in our situation. Recall that the space $X$ is separable (see 3.5 of [2]) and hence $X$
admits a biorthogonal system \( \{e_m, e_m^*\} \), (see Proposition 1.f.3 of [55] ) such that
\[
\{e_m, m \in \mathbb{N}\} = X, \quad e_m^* \in X', \quad \langle e_m^*, e_n \rangle = \delta_{n,m}.
\]
\[
\langle e_m^*, x \rangle = 0, \quad \forall m \Rightarrow x = 0.
\]
Let
\[
E_n = \text{span} \{e_1, e_2, \ldots, e_n\},
\]
and let
\[
E_{n-1}^\perp = \text{span} \{e_{n+1}, e_{n+2}, \ldots\}.
\]
Since \( E_{n-1}^\perp \) is of codimension \( n - 1 \), for any \( A \subseteq \Gamma_n \) we have \( A \cap E_{n-1}^\perp \neq \emptyset \) (see Proposition 7.8 of [66]).

Let
\[
\mu_n = \inf_{A \subseteq \Gamma_n} \sup_{A \cap E_{n-1}^\perp} J_p(u), \quad n = 1, 2, \ldots
\]

Now we show that \( \mu_n \to \infty \). If possible let \( \{\mu_n\} \) be bounded, then there exists \( u_n \in E_{n-1}^\perp \cap M \) such that \( \mu_n \leq J_p(u_n) < c \) for some constant \( c > 0 \). Since \( u_n \in M \), the estimate (6.41) shows that \( u_n \) is indeed bounded in \( X \). Thus \( u_n \to u \) for some \( u \in X \). Now by the choice of biorthogonal system, for each \( m, \langle e_m^*, u_n \rangle \to 0 \) as \( n \to \infty \). Thus \( u_n \to 0 \), in \( X \) and hence \( u = 0 \), a contradiction to \( \int g|u|^q \geq 1 \) (see the conclusion followed by the estimate (6.45)). Therefore \( \mu_n \to \infty \) and hence \( c_n \to \infty \) as \( \mu_n \leq c_n \).

\textbf{Remark 6.3.11.} If \( g \in \mathcal{F}_N^p \) and \( g^- \neq 0 \), then there exists a sequence \( \mu_n \) of negative eigenvalues of (6.1) tending to \( -\infty \).

### 6.4 Miscellaneous Remarks

In this section we study some extensions of Theorem 6.2.11.

One can study the existence of ground states for \( \Delta_p \) with more general subcritical nonlinearities on the right hand side of (6.1). Precisely, for given \( V, g \) locally integrable on a domain \( \Omega \subset \mathbb{R}^N \) with \( V \geq 0 \) but \( g \) allowed to change sign, one can look for the positive solutions in \( D_0^{1,p}(\Omega) \) for the following problem
\[
\begin{align*}
\Delta_p u + V|u|^{p-2}u &= \lambda g|u|^{q-2}u, \\
u|\partial\Omega &= 0,
\end{align*}
\]
where \( q \in [p, p^*) \) and \( 1 < p < N \).
Theorem 6.4.1. If \( g^+ \in \mathcal{F}_p \setminus \{0\} \) with \( \frac{1}{p} + \frac{2}{p^*} = 1 \), then (6.47) has a positive solution.

One has to just verify that \( G_p(u) = \int_\Omega g^+ |u|^q \) is compact, then by arguing as in the proposition 6.2.11, it is immediate that \( \int_\Omega \{ |\nabla u|^p + V |u|^p \} \) has a positive minimizer on \( \mathcal{M}_q = \{ u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g |u|^q = 1 \} \). Then by the homogeneity of the Rayleigh quotient \( R = \frac{\int_\Omega \{ |\nabla u|^p + V |u|^p \}}{(\int_\Omega g |u|^q)^\frac{q}{p}} \) corresponding to (6.47) we get a minimizer of \( R_p \) on \( \{ u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g |u|^q > 0 \} \). For the positivity of this minimizer one can use Lemma (6.2.5).

Remark 6.4.2. Let \( g \) be as in the above remark. Then the following generalized Hardy-Sobolev inequality holds

\[
\left( \int_\Omega g |u|^q \right)^\frac{2}{p} \leq \frac{1}{\lambda_1} \int_\Omega \{ |\nabla u|^p + V |u|^p \}, \quad \forall u \in \mathcal{D}_0^{1,p}(\Omega), \int_\Omega g |u|^q > 0
\]

(6.48)

where \( \lambda_1 \) is the minimum of \( \int_\Omega \{ |\nabla u|^p + V |u|^p \} \) on \( \mathcal{M}_q \). Further the best constant is attained. This extends the results of Visciglia [77] for \( p \neq 2 \).