In this chapter, we are concerned with the existence of a nontrivial solution branch for the semilinear elliptic partial differential equations of the following type:

$$-\Delta u = \lambda f(x, u), \quad u \in D^{1,2}_0(\Omega).$$  \label{5.1}$$

The existence of a solution branch for the above equation is too general to be answered affirmatively, unless we make further assumptions on the structure of $f(x, u)$. The functions $f$ that we are considering here, can be put into two different classes, depending on whether $f(x, 0)$ is zero or not. These two classes leads to two different kinds of behaviour of the solution branches of (5.1), namely the presence of the trivial branch of (5.1) when $f(x, 0) = 0$ and its absence when $f(x, 0) \neq 0$. Moreover, in the first case we obtain solutions only for sufficiently large $\lambda$ and in the other case we obtain a solution branch for small $\lambda$. However, in both the cases, the existence of nontrivial solution branches are obtained via two different methods, both of them rely on the implicit function theorem in its core.

The rest of this chapter is divided into two sections, each of them studies the existence of a nontrivial solution branch of (5.1) with a different set of assumptions on the structure of $f(x, u)$ that fixes a suitable functional framework for (5.1). In Section 1, we prove the existence of a nontrivial solution branch for (5.1) using the Rabinowitz bifurcation theorem. In Section 2, we obtain the existence of positive solution branch for (5.1) by a direct application of the implicit function theorem, in a suitable functional
On solution branches of semilinear elliptic equations

framework. In that section we improve the regularity of the solution branch using a Moser type iteration. Further, some examples of weights satisfying our conditions are given in Section 2.

5.1 Existence of a solution branch using the bifurcation theory

In this section, as an application of Theorem 4.3.15, we study the existence of a non-trivial solution branch for the following semilinear equation:

\[- \Delta u = \lambda g u + \lambda h r(u), \quad u \in D_0^{1,2}(\Omega),\]  

(5.2)

where \( \lambda \) is a real parameter and \( \Omega \subset \mathbb{R}^N \) with \( N \geq 3 \). The domain \( \Omega \) is assumed to be connected; however no assumptions on the the boundedness of \( \Omega \) are made. The weights \( g \) and \( h \) are assumed to be in certain weak Lebesgue spaces. More precisely, we assume the following:

\[
(H1) \left\{ \begin{array}{l}
          r \in \mathcal{C}(\mathbb{R}), \quad |r(s)| \leq C|s|^{\gamma-1}, \text{ for } \gamma \in [1, 2^*) \text{ and } C > 0, \\
          \lim_{|s| \to 0} \frac{|r(s)|}{|s|} = 0, \text{ if } 1 \leq \gamma \leq 2.
        \end{array} \right.
\]

\[
(H2) \left\{ \begin{array}{l}
          g \in \mathcal{F}_N, \quad g^+ \neq 0, \\
          h \in \left\{ \begin{array}{ll}
                     \mathcal{F}_{\frac{\gamma}{2}} \text{ if } \gamma \geq 2, & \text{where } \frac{1}{\gamma} + \frac{2}{\gamma} = 1, \\
                     \mathcal{F}_{\frac{\gamma}{2}} \text{ if } 1 \leq \gamma < 2.
            \end{array} \right.
        \end{array} \right.
\]

A typical nonlinearity is the subcritical power, i.e. \( r(u) = |u|^\gamma - 1, \gamma \in [1, 2^*) \). Note that \((\lambda, 0)\) is a trivial solution branch of (5.2), since \( r(0) = 0 \). Here, we are interested in certain sufficient conditions that ensure the existence of a nontrivial solution branch for (5.2), bifurcating from the branch of zero solutions. Under the assumptions (H1) and (H2) we obtained a global bifurcation results that generalize the results of [60]. The results that we present in this section are published in [10].

For \( g, h \) and \( r \) satisfying \((H1)\) and \((H2)\), let \( S \) denote the set of non-trivial solutions of (5.2), i.e.,

\[
S = \{ (\lambda, u) \in \mathbb{R} \times D_0^{1,2}(\Omega) : (\lambda, u) \text{ solves (5.2), } u \neq 0 \}.
\]
Further, set
\[ \sigma(g) = \{ \lambda : \lambda \text{ is an eigenvalue of } (4.1) \}. \]

From Remark (4.4.4), it is clear that \( \sigma(g) \) is non-empty. Now we state our global bifurcation result:

**Theorem 5.1.1.** Assume (H1), (H2) and \( g^+ \neq 0 \). Then, there exists a connected branch \( C^+ \) in \( \mathcal{S}^{\mathbb{R} \times \mathcal{D}_0^{1,2}(\Omega)} \) bifurcating from \((\lambda_1, 0)\). Moreover,

(i) either \( C^+ \) is unbounded in \( \mathbb{R} \times \mathcal{D}_0^{1,2}(\Omega) \),

(ii) or \( (\lambda, 0) \in C^+ \) with \( \lambda_1 \neq \lambda \in \sigma(g) \).

First, we formulate our problem in a suitable functional framework. Note that the map \( (\mathcal{L})^1 : [L(2^*,2)]' \to \mathcal{D}_0^{1,2}(\Omega) \) is continuous, see Remark 2.3.6. Hence by defining

\[
L : \mathcal{D}_0^{1,2}(\Omega) \to \mathcal{D}_0^{1,2}(\Omega), \quad H : \mathbb{R} \times \mathcal{D}_0^{1,2}(\Omega) \to \mathcal{D}_0^{1,2}(\Omega),
\]

\[
L(u) = (-\Delta)^{-1}(gu), \quad H(\lambda, u) = (-\Delta)^{-1}(\lambda hr(u)),
\]

the problem (5.2) is equivalent to solving

\[
u = \lambda L(u) + H(\lambda, u), \quad u \in \mathcal{D}_0^{1,2}(\Omega).
\]

Under hypotheses (H1)-(H2), we show that \( H \) and \( L \) satisfy all the requirements of the following global bifurcation theorem of Rabinowitz:

**Theorem 5.1.2.** (Rabinowitz, [66]) Given a Banach space \((B, \| \cdot \|)\), consider a mapping

\[
G : \mathbb{R} \times B \to B, \quad G(\lambda, u) = \lambda L(u) + H(\lambda, u),
\]

where \( L : B \to B \) is a compact linear operator and \( H(\lambda, \cdot) : B \to B \) is a continuous compact mapping satisfying \( \lim_{\|u\| \to 0} \frac{\|H(\lambda, u)\|}{\|u\|} = 0 \). Let

\[
r(L) := \{ \mu \in \mathbb{R} : \mu^{-1} \text{ is an eigenvalue of } L \text{ with odd multiplicity} \},
\]

\[
\mathcal{S} := \{ (\lambda, u) \in \mathbb{R} \times B : (\lambda, u) \text{ is solution of } u = G(\lambda, u), u \neq 0 \}.
\]

Then, given \( \mu \in r(L) \), \( \mathcal{S} \) has a connected branch \( C_\mu \) bifurcating from \((\mu, 0)\).
On solution branches of semilinear elliptic equations

and

(i) either $C_\mu$ is unbounded in $\mathbb{R} \times B$,
(ii) or, $(\mu, 0) \in C_\mu$ with $\mu \neq \bar{\mu} \in r(L)$.

First we prove the following compactness result, which is a nonlinear version of Lemma 4.3.3.

**Lemma 5.1.3.** Let $r \in C^0(\mathbb{R})$ satisfying $|r(s)| \leq C|s|^{\alpha-1}$ for some $\alpha \in [2, 2^*)$ and $w \in \mathcal{F}_\alpha$ with $\frac{1}{\alpha} + \frac{\alpha}{2} = 1$. Then the operator

$$N : \mathcal{D}_0^{1,2}(\Omega) \to [L(2^*, 2)]', \quad N(u) := wr(u), \quad (5.7)$$

is compact.

**Proof.** Let $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,2}(\Omega)$. First we show that a subsequence of $\{N(u_n)\}$ converges to $N(u)$ in $[L(2^*, 2)]'$. For $\phi \in C_c^\infty(\Omega)$ and for $v \in L(2^*, 2)$ we write:

$$|N(u_n)(v) - N(u)(v)| \leq \int_{\Omega} |\phi(r(u_n) - r(u))||v| + \int_{\Omega} |w - \phi||r(u_n) - r(u)||v|. \quad (5.8)$$

Using the growth condition on $r$, we have $r(u_n) - r(u) \in L^{\frac{2^*}{\alpha - 1}}(\Omega)$, since $\mathcal{D}_0^{1,2}(\Omega) \hookrightarrow L^{\frac{2^*}{\alpha - 1}}(\Omega)$. Now we estimate the first integral in the right hand side of (5.8) using the Hölder inequality in Lebesgue spaces.

$$\int_{\Omega} |\phi(r(u_n) - r(u))||v| \leq \left\| \phi (r(u_n) - r(u)) \right\|_{[2^*]'} \left\| v \right\|_{2^*}. \quad (5.9)$$

Using our growth assumption on $r$, we have

$$|\phi(r(u_n) - r(u))|^{[2^*]'} \leq C_2|\phi| \left( |u_n|^{(\alpha-1)[2^*]'} + |u|^{(\alpha-1)[2^*]'} \right). \quad (5.10)$$

Note that $(\alpha-1)[2^*]' < 2^*$ and $\mathcal{D}_0^{1,2}(\Omega) \hookrightarrow L_q^q(\Omega)$ is compact for $q \in [1, 2^*)$. Thus the right hand side of (5.10) converges in $L^{\frac{2^*}{\alpha - 1}}(\Omega)$ up to a subsequence, say $\{u_{n_k}\}$ of $\{u_n\}$. Further, since $r$ is continuous, $r(u_{n_k}) \to r(u)$ a.e. in $\Omega$. Now by applying the generalized dominated convergence theorem (Theorem 17 Chapter 4 of [67]), we have $\phi r(u_{n_k}) \to \phi r(u)$ in $L^{2^*}(\Omega)$. Therefore from (5.9), noting that $\|v\|_{2^*} \leq C_0 \|v\|_{(2^*, 2)}$, we deduce the existence of
k_0 \in \mathbb{N} \text{ such that}
\int_\Omega |\phi||r(u_{nk}) - r(u)||v| < \varepsilon \|v\|_{(2^*,2)}, \ \forall \ k \geq k_0. \quad (5.11)

The second integral in (5.8) can be estimated using the growth assumption on \(r\), Proposition 2.2.7 and using the Lorentz-Sobolev embedding as below:

\[
\int_\Omega |w - \phi||r(u_n) - r(u)||v| \leq C \int_\Omega |w - \phi||u_n|^{\alpha-1} + |u|^{\alpha-1}|v|
\leq C_1 \|w - \phi\|_{(\overline{\alpha},\infty)}\|u_n|^{\alpha-1} + C_1 |u|^{\alpha-1}\|\frac{1}{(\alpha-1,2)}\||v|_{(2^*,2)}.
\]

Note that,
\[
\|u_n|^{\alpha-1} + |u|^{\alpha-1}\|_{(\frac{2^*}{\alpha-1},2)} \leq C_2 \left(\|u_n\|_{(2^*,2)}^{\alpha-1} + \|u\|_{(2^*,2)}^{\alpha-1}\right), \ \forall \ n.
\]

Therefore
\[
m = \sup_n \left\{\|u_n|^{\alpha-1} + |u|^{\alpha-1}\|_{(\frac{2^*}{\alpha-1},2)}\right\} < \infty.
\]

Since \(w \in \mathcal{F}_p\), we choose \(w_\varepsilon \in C^\infty_c(\Omega)\), such that
\[
\|w - w_\varepsilon\|_{(\overline{\alpha},\infty)} \leq \frac{\varepsilon}{C_1 m}.
\]

Now by taking \(\phi = w_\varepsilon\) in (5.12), we obtain
\[
\int_\Omega |w - \phi||r(u_n) - r(u)||v| \leq \varepsilon \|v\|_{(2^*,2)}. \quad (5.12)
\]

Putting together (5.8), (5.12) and (5.12) we conclude
\[
\int_\Omega |w||r(u_{nk}) - r(u)||v| < \widetilde{C} \varepsilon \|v\|_{(2^*,2)}, \quad \forall \ k \geq k_0,
\]

where \(\widetilde{C}\) is a constant. This shows that \(\|N(u_{nk}) - N(u)\|_{[2^*,2]'} \to 0\). Thus every subsequence of \(N(u_n)\) has a convergent subsequence and it converges to \(N(u)\). Thus conclude that \(N(u_n)\) converges to \(N(u)\) in \([L(2^*,2)]\). This completes the proof.

Using the above lemma one can deduce the following proposition,

**Proposition 5.1.4.** Assume \((H1)-(H2)\) hold. Then the mappings \(L\) and \(H\)
defined by (5.3) are compact. Furthermore,

$$\lim_{\|u\| \to 0} \frac{\|H(\lambda, u)\|_{\mathcal{D}^{1,2}_0(\Omega)}}{\|u\|_{\mathcal{D}^{1,2}_0(\Omega)}} = 0.$$  \hfill (5.13)

**Proof.** Using (H1) and the fact that \(r(s)\) is bounded on \([\delta, 1]\), for any \(\delta > 0\), we see that

$$|r(s)| \leq \begin{cases} C_0 |s| & \text{if } \gamma \in [1, 2], \\ C |s|^{\gamma - 1} & \text{if } \gamma > 2. \end{cases} \quad (5.14)$$

Let us consider the following maps defined from \(\mathcal{D}^{1,2}_0(\Omega)\) to \([L(2^*, 2)]'\)

$$\tilde{L}(u) = g(x)u, \quad \tilde{H}(u) = h(x)r(u).$$

From Lemma 5.1.3 we have \(\tilde{L}\) and \(\tilde{H}\) are compact. Thus by Remark 2.3.6 we deduce that \(L\) and \(H\) are continuous and compact. Next we prove that

$$\lim_{\|u\| \to 0} \frac{\|\tilde{H}(u)\|}{\|u\|} = 0.$$ \hfill (5.15)

First we assume that \(\gamma > 2\). Thus by using the Hölder inequality and the Lorentz-Sobolev embedding we obtain the following:

$$\|h r(u)\|_{([2^*], 2)} \leq C \|h|u|^{\gamma - 1}\|_{([2^*], 2)}$$

$$\leq C_1 \|h\|_{(\gamma, \infty)} \|u|^{\gamma - 1}\|_{([2^*], 2)}$$

$$\leq C_2 \|h\|_{(\gamma, \infty)} \|u\|_{[2^*, 2]}^{\gamma - 1}$$

$$\leq C_3 \|h\|_{(\gamma, \infty)} \|\nabla u\|_2^{\gamma - 1},$$

where all the constants appearing above are independent of \(u\). Therefore

$$\frac{\|\tilde{H}(u)\|_{([2^*], 2)}}{\|u\|_{\mathcal{D}^{1,2}_0(\Omega)}} \leq C_3 \|h\|_{(\gamma, \infty)} \|u\|_{\mathcal{D}^{1,2}_0(\Omega)}^{\gamma - 2}.$$  \hfill (5.15)

Thus (5.15) holds when \(\gamma > 2\).

Next we prove the property (5.15) for \(\gamma \in [1, 2]\). Let us fix \(\varepsilon > 0\). Note that \(r(s)/|s|^{2^*-1}\) is continuous and hence bounded on \([\delta, 1]\), for any \(\delta > 0\). Thus using
5.1 Existence of a solution branch using the bifurcation theory

(5.14), we get $s_0, C_1 > 0$ depending only on $\varepsilon$ such that

$$|r(s)| \leq \varepsilon, \forall |s| < s_0 \quad \text{and} \quad |r(s)| \leq C_1 |s|^{2^* - 1}, \quad \forall |s| \geq s_0. \quad (5.16)$$

For each $u \in D_0^1(\Omega)$, let

$$E := \{ x \in \Omega : |u(x)| < s_0 \} \quad \text{and} \quad F := \{ x \in \Omega : |u(x)| \geq s_0 \}.$$ 

Using the triangular inequality, we obtain

$$\| h r(u) \chi_E \|_{(2^*, 2)} \leq \| h r(u) \chi_E \|_{(2^*, 2)} + \| h r(u) \chi_F \|_{(2^*, 2)}. \quad (5.17)$$

Let us estimate each term in the right hand side of (5.17) using Hölder inequality and Lorentz-Sobolev embedding. The first term is handled as follows:

$$\| h r(u) \chi_E \|_{(2^*, 2)} \leq C_2 \| h \|_{(\frac{N}{2}, \infty)} \| r_1(u) \chi_E \|_{(2^*, 2)} \leq C_2 \varepsilon \| h \|_{(\frac{N}{2}, \infty)} \| u \|_{(2^*, 2)}. \quad (5.18)$$

To estimate $\| h r(u) \chi_F \|_{(2^*, 2)}$, we write $h = \phi + (h - \phi)$ with $\phi \in C^\infty_c(\Omega)$. Now using (5.14) and (5.16) we have the following estimates:

$$\| h r(u) \chi_F \|_{(2^*, 2)} \leq C_1 \| \phi \|_{(2^*, 2)} + C_2 \| (h - \phi) \|_{(2^*, 2)} \leq C_3 \| \phi \|_{\infty} \| u \|_{(2^*, 2)} + C_4 \| h - \phi \|_{(\frac{N}{2}, \infty)} \| u \|_{(2^*, 2)}. \quad (5.19)$$

Since $h \in F_{\frac{N}{2}}$, when $\gamma \in [1, 2]$, we choose $h_\varepsilon \in C^\infty_c(\Omega)$ such that

$$\| h - h_\varepsilon \|_{(\frac{N}{2}, \infty)} < \varepsilon.$$ 

Now by taking $\phi = h_\varepsilon$ in (5.19) we deduce that

$$\| h r(u) \chi_F \|_{(2^*, 2)} \leq C_5 \left\{ \| \nabla u \|_{2}^{2^* - 1} + \varepsilon \| \nabla u \|_{2} \right\}. \quad (5.20)$$

From (5.17),(5.19) and (5.20), we conclude that (5.15) holds for $\gamma \in [1, 2]$. \qed

Now we are in a position to prove the existence of a global branch of solutions for problem (5.2):

**Proof of Theorem 5.1.1:** Theorem 4.3.15 shows that $\lambda_1$ is an eigenvalue of multiplicity one. This fact with Lemma 5.1.4 shows that all the conditions of Theorem 5.1.2 are satisfied and hence the proof follows. \qed
5.2 A positive solution branch using the implicit function theorem

In this section, we consider the following type of semilinear elliptic equation:

\[- \Delta u = \lambda a f(u) \text{ in } D_{0}^{1,2}(\mathbb{R}^N), \]

where $\lambda$ is a real parameter and $N \geq 3$. Further, we assume that $f$ is in $C^1(\mathbb{R})$ with $f(0) \neq 0$ and $a$ may change sign.

Here we are looking for sufficient conditions on the weight function $a$ and on the nonlinearity $f$ for the existence of a positive solution branch for (5.21) in $D_{0}^{1,2}(\mathbb{R}^N)$. Note that, $(\lambda, 0)$ is no longer a solution branch for the above equation as in the case considered in the previous section, since $f(0) \neq 0$. Thus none of the methods, which assume the existence of a trivial solution branch, is applicable here. However, under certain assumptions on the weight function $a$ and on the nonlinearity $f$, we apply the implicit function theorem to obtain a nontrivial solution branch for (5.21). We make the following assumptions:

(A1) $f \in C^1(\mathbb{R})$ such that $f(0) \neq 0$ and there exists $s_0 > 0$ such that

$$|f'(s)| \leq C |s|^\gamma - 2, \forall |s| \geq s_0 \text{ with } \gamma \in [2, 2^*).$$

(A2) $a \in L^{\frac{2N}{N+2}}(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ with $r > \{\frac{\gamma}{2}, 2\}$, where $\frac{\gamma}{2}$ is the conjugate exponent of $\{\frac{\gamma}{2}\}$. 

The main novelty of our hypotheses is that the weight function $a$ need not be smooth and the function $f$ not necessarily bounded, but we demand that $a$ lies in certain Lebesgue spaces and $f$ is $C^1(\mathbb{R})$ with subcritical growth at infinity. The results that we present in this section have appeared in [11].

First we prove the existence of a solution branch of (5.21) for small $\lambda$, under the assumptions (A1) and (A2). The positivity of the solution branch is obtained under further assumptions on $a$. For $\varepsilon \geq 0$, we consider the following perturbed linearised problem:

\[- \Delta v = f(0)(a - \varepsilon a^-) \text{ in } \mathbb{R}^N. \]

Using a simple interpolation we see that $a \in L^q(\mathbb{R}^N)$ for $q \in \left[ \frac{2N}{N+2}, r \right]$. In particular we have $a \in L^2(\mathbb{R}^N)$. Now using the Newtonian potential, the existence of a weak solution for (5.22) in $H^1(\mathbb{R}^N)$ is well known, see Theorem 9.9 of [40]. Further, from (A2), we have $a \in L^r(\mathbb{R}^N)$ with $r > \frac{N}{2}$,
since $\gamma \geq \frac{N}{2}$. Thus the weak solutions of (5.22) are continuous, see Corollary 9.18 of [40]. Let $v_\varepsilon$ be a weak solution of (5.22) that is continuous. Thus for each $\varepsilon \geq 0$, $v_\varepsilon \in D^{1,2}_0(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and satisfies equation (5.22) in the weak sense, i.e.,

$$\int_{\mathbb{R}^N} \nabla v_\varepsilon \cdot \nabla u = \int_{\mathbb{R}^N} f(0)(a - \varepsilon a^-)u \quad \forall u \in D^{1,2}_0(\mathbb{R}^N).$$

We obtain the positivity of the solution branch of (5.21) under one of the following assumptions:

(A3) $v_0$ is positive in $\mathbb{R}^N$ and $f(0) a \geq 0$ a.e. near infinity.

(A4) $v_\varepsilon$ is positive in $\mathbb{R}^N$, for some $\varepsilon > 0$.

The existence of the solution branch of (5.21) rely on the implicit function theorem as in the work of Brown and Afrouzi [3]. Further we improve the integrability of solutions of (5.21) using a Moser type iteration as in [33], where Drábek proved the integrability of the positive solutions of quasilinear partial differential equation involving p-Laplacian in $\mathbb{R}^N$. The key difference and the difficulty in our case is the presence of $f(0)$ (which is zero in [33]), that produces an additional term in the iteration and inhomogeneity in the inequalities. We show that the regularity assumptions on $a$ are sufficient to tackle this difficulty. Moreover, we improve the integrability for a general solution, not just for the positive solutions of (5.21). Using the improved integrability of solutions, we prove the decay at infinity and the Hölder continuity of solutions of (5.21) using certain classical results available in the literature.

5.2.1 Existence of a solution branch

In [3], authors considered only bounded domains and smooth weight function $a$. They have used the fact that the map $\Delta$ is a linear homeomorphism between appropriate Hölder spaces. Since we are considering entire $\mathbb{R}^N$ and $a$ is only in certain Lebesgue spaces, we look for a functional framework so that $\Delta$ can be identified as a linear homeomorphism and $a f(u)$ as a $C^1$ map between appropriate function spaces. We set $X = D^{1,2}_0(\mathbb{R}^N)$ and identify $-\Delta$ as a mapping between $X$ and its dual $X'$, with the obvious action

$$\langle -\Delta u, v \rangle_{(X',X)} = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v.$$
From Corollary 2.3.6, we know that $-\Delta$ is a linear isometry from $X$ onto $X'$. Now we prove that $a \, f(u)$ is a $C^1$ map on $X$.

**Lemma 5.2.1.** Let $a$ and $f$ satisfy conditions (A1) and (A2). Then the map $F : L^{2^*}(\mathbb{R}^N) \to [L^{2^*}(\mathbb{R}^N)]'$ defined by

$$F(u) = a \, f(u)$$

is $C^1$. Further $D F(u) : L^{2^*}(\mathbb{R}^N) \to [L^{2^*}(\mathbb{R}^N)]'$ is given by

$$D F(u) = a \, f'(u).$$

**Proof.** Using (A1) and the mean value theorem one can easily deduce that

$$|f(s)| < C(1 + |s|^\gamma - 1), \quad |f'(s)| < C(1 + |s|^{\gamma - 2}), \forall s \in \mathbb{R}, \quad (5.24)$$

for some positive constant $C$ depending on $f$. Using the interpolation on Lebesgue spaces and (A2) we get $a \in L^q(\mathbb{R}^N)$ for $q \in \left(\frac{2N}{N+2}, r\right]$. Note that $2^* = \frac{2N}{N+2}$. Now for $u \in L^{2^*}(\mathbb{R}^N)$, using (5.24) and H"{o}lder inequality we get $a \, f(u) \in [L^{2^*}(\mathbb{R}^N)]'$. Thus the map $F(u)$ is well defined from $L^{2^*}(\mathbb{R}^N)$ to $[L^{2^*}(\mathbb{R}^N)]'$.

Next, we show that $F$ is continuous. Let $u_n, u \in L^{2^*}(\mathbb{R}^N)$ and $u_n \to u$ in $L^{2^*}(\mathbb{R}^N)$. Thus up to a subsequence $u_n \to u$ a.e. in $\mathbb{R}^N$ and since $f$ is continuous, $f(u_n) \to f(u)$ a.e in $\mathbb{R}^N$.

$$|a \, f(u_n) - a \, f(u)| \leq |a| \{|f(u_n)| + |f(u)|\} \leq 2C|a| + C|a| \{ |u_n|^\gamma - 1 + |u|^\gamma - 1 \},$$

using (5.24). Now as $|u_n|^\gamma - 1 \to |u|^\gamma - 1$ in $L^{2^*\gamma^{-1}}$ and $a \in L^\gamma$ using H"{o}lder inequality we get $|a||u_n|^\gamma - 1 \to |a||u|^\gamma - 1$ in $[L^{2^*}(\mathbb{R}^N)]'$. Thus the right hand side of the above inequality converges to $2C|a| \{ 1 + |u|^\gamma - 1 \}$ in $[L^{2^*}(\mathbb{R}^N)]'$ and hence by the generalized dominated convergence theorem $\{a \, f(u_n) - a \, f(u)\} \to 0$ in $[L^{2^*}(\mathbb{R}^N)]'$. By a similar calculation one can show that $F$ is $C^1$ and its derivative is given by $D F(u) = a \, f'(u)$.

**Theorem 5.2.2.** Let $a$, $f$ satisfy conditions (A1)-(A2). Then there exist a $\varepsilon_0 > 0$ and a $C^1$ map $u : (-\varepsilon_0, \varepsilon_0) \to X$ such that for $\lambda \in (-\varepsilon_0, \varepsilon_0)$, $u(\lambda)$ satisfies

$$-\Delta u(\lambda) = \lambda \, a \, f(u(\lambda)) \quad (5.25)$$
5.2 A positive solution branch using the implicit function theorem

in the weak sense. Further the derivative \( u'(0) := \frac{d}{d\lambda} u(\lambda)|_{\lambda=0} \) satisfies

\[
- \Delta(u'(0)) = f(0) a
\]

(5.26)
in the weak sense.

Proof. By Sobolev embedding, \( X \) is embedded in \( L^2^\ast(\mathbb{R}^N) \) and hence the map \( F : \mathbb{R} \times X \to X' \) defined by

\[
F(\lambda, u) = -\Delta u - \lambda a f(u)
\]
is well defined. Now as \( \tilde{F} \) is \( C^1 \) and \( -\Delta \) is a linear isometry, \( F \) is \( C^1 \) and its partial derivatives are given by

\[
\frac{\partial F}{\partial u}(\lambda, u) = -\Delta - \lambda a f'(u), \quad \frac{\partial F}{\partial \lambda}(\lambda, u) = -a f(u).
\]

Observe that \( F(0, 0) = 0 \) and \( \frac{\partial F}{\partial \lambda}(0, 0) = -\Delta \), which is invertible. Hence by applying the implicit function theorem to \( F(\lambda, u) = 0 \) at \( (\lambda, u) = (0, 0) \), we get a neighbourhood \( (-\varepsilon_0, \varepsilon_0) \) of 0 and a unique map \( u : (-\varepsilon_0, \varepsilon_0) \to X \) such that \( u(0) = 0 \) and

\[
F(\lambda, u(\lambda)) = 0, \quad \forall \lambda \in (-\varepsilon_0, \varepsilon_0).
\]

Further the map \( u \) is \( C^1 \). It is clear that \( u(\lambda) \) satisfies the Eq. (5.25) in the weak sense. Let us differentiate the above equation with respect \( \lambda \) and evaluate at \((0, 0)\):

\[
\frac{\partial F}{\partial \lambda}(0, 0) + \frac{\partial F}{\partial u}(0, 0) \circ u'(0) = 0.
\]

Now by substituting for \( \frac{\partial F}{\partial \lambda}(0, 0) \) and \( \frac{\partial F}{\partial u}(0, 0) \) we get \( -\Delta(u'(0)) = f(0) a \).

Henceforth, for the convenience we denote \( u(\lambda) \) by \( u_\lambda \).

Remark 5.2.3. From the uniqueness of the weak solution of (5.23), we see that \( u(0) = v_0 \). Thus

\[
\lim_{\lambda \to 0} \left\| \frac{u_\lambda}{\lambda} - v_0 \right\|_{W^{1,2}(\mathbb{R}^N)} = 0.
\]
5.2.2 Regularity of the solution branch

In this section we improve the regularity of solutions using Moser type iterations as in [33].

**Lemma 5.2.4.** Let $a, f$ satisfy (A1) and (A2). Let $u \in D^{1,2}_0(\mathbb{R}^N)$ satisfy

$$-\Delta u = \lambda a f(u)$$

in the weak sense; then $u \in L^s$ for every $s \in [2^*, \infty)$.

**Proof.** For a positive function $w \in D^{1,2}_0(\mathbb{R}^N)$ and $k \in \mathbb{N}$, we define

$$w_k = \min\{w, k\}.$$

Now for $\alpha > 0$, let

$$v_k = [(u^+)_k]^{2\alpha+1} - [(u^-)_k]^{2\alpha+1}.$$

From Corollary 2.3.9, we have $v_k \in D^{1,2}_0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Thus from the equation we have the following

$$\int \nabla u \cdot \nabla v_k = \lambda \int a f(u) v_k. \quad (5.27)$$

First we find a lower bound for the left hand side of (5.27)

$$\int \nabla u \cdot \nabla v_k = \int \nabla (u^+)_k \cdot \nabla [(u^+)_k]^{2\alpha+1} + \nabla (u^-)_k \cdot \nabla [(u^-)_k]^{2\alpha+1}$$

$$= (2\alpha + 1) \int |\nabla (u^+)_k|^2 [(u^+)_k]^{2\alpha} + |\nabla (u^-)_k|^2 [(u^-)_k]^{2\alpha}$$

$$= \frac{(2\alpha + 1)}{(\alpha + 1)^2} \int |\nabla [(u^+)_k]^{\alpha+1}|^2 + |\nabla [(u^-)_k]^{\alpha+1}|^2$$

$$= \frac{(2\alpha + 1)}{(\alpha + 1)^2} \int |\nabla [u_k]^{\alpha+1}|^2$$

$$\geq \frac{1}{C_s} \frac{(2\alpha + 1)}{(\alpha + 1)^2} \left( \int |u_k|^{2^*(\alpha+1)} \right)^{\frac{2}{2^*}}. \quad (5.28)$$

The equality sign in the first and fourth steps are due to the disjointness of the supports of $u^+_k$ and $u^-_k$. The last inequality comes from the Sobolev embedding, where $C_s$ is the best constant appearing in the corresponding Sobolev inequality. Next, we get an upper bound for the right hand side of
5.2 A positive solution branch using the implicit function theorem

(5.27) using the growth condition on $f$ and the integrability of $a$:

$$
|\lambda \int a f(u)v_k| \leq \lambda \int |a||f(u)||u|^{2\alpha+1} \\
\leq \lambda C \int |a|(1 + |u|^{\gamma-1})|u|^{2\alpha+1} \\
\leq \lambda C \int |a||u|^{2\alpha+1} + \lambda C \int |a||u|^{2\alpha+\gamma}. \quad (5.29)
$$

We estimate the second term in (5.29) using H"older inequalities and Sobolev embedding

$$
\int |a||u|^{2\alpha+\gamma} \leq \|a\|_r \|u|^{2\alpha+\gamma}\|_{r'}.
$$

Now

$$
\int |u|^{(2\alpha+\gamma) r'} = \int |u|^{(\gamma-2)r'}|u|^{2(\alpha+1)r'}.
$$

Let $p = \frac{2}{(\gamma-2)r'}$. It is given that $r > (\frac{2}{\gamma})'$. Thus $r' < (\frac{2}{\gamma}) < (\frac{2}{\gamma-2})$ and hence $p > 1$. Now by H"older inequality applied for $p,p'$ we get

$$
\int |u|^{(2\alpha+\gamma) r'} \leq \left(\int |u|^{2} \right)^{\frac{r'}{p}} \left(\int |u|^{2(\alpha+1) r'} \right)^{\frac{1}{r'}}.
$$

Let $q = 2r'p'$. Then $q = 2r' \frac{2}{2r'-(\gamma-2)r'}$. Now one can see that $q < 2^*$, using the fact that $r > (\frac{2}{\gamma})'$ and the following implications:

$$
q < 2^* \iff \frac{2r'}{2^*-(\gamma-2)r'} < 1 \iff 2r' < 2^* - (\gamma-2)r' \iff r' < \frac{2^*}{\gamma}.
$$

Now

$$
\left\| u \right\|^{(2\alpha+\gamma)}_{r'} \leq \left(\int |u|^{2} \right)^{\frac{1}{p'}} \left(\int |u|^{(\alpha+1)q} \right)^{\frac{2}{q}}.
$$

Therefore using (5.30)

$$
\int |a||u|^{2\alpha+\gamma} \leq C_2 \left(\int |u|^{(\alpha+1)q} \right)^{\frac{2}{q}} = C_2 \|u\|^{2(\alpha+1)}_{(\alpha+1)q},
$$

where $C_2$ is a constant independent of $\alpha, k$.

Next, we estimate the first integral in (5.29). Let $s_\alpha = (\frac{\alpha+1}{2\alpha+1})q$. Observe
that \(\frac{\alpha + 1}{2\alpha + 1} > \frac{1}{2}\) and hence \(s_\alpha > \frac{q}{2} = r' p' \geq r'\). Let \(\tilde{\alpha}\) be such that \((\tilde{\alpha} + 1)q = 2^*\), then for any \(\alpha \geq \tilde{\alpha}\), \((\frac{\alpha + 1}{2\alpha + 1})q \leq (\frac{\tilde{\alpha} + 1}{2\tilde{\alpha} + 1})q < 2^*\). Thus for \(\alpha \geq \tilde{\alpha}\), we have \(s_\alpha \in (r', 2^*)\) and hence \(s'_\alpha \in (2^*, r)\).

\[
\int |a||u|^{2\alpha + 1} \leq \|a\|_{s'_\alpha} \|u\|_{(1+\alpha)q}^{2\alpha + 1}.
\]

Thus we get an upper bound for the right hand side of (5.27) provided \(\alpha \geq \tilde{\alpha}\).

\[
|\lambda \int a \, f(u)v_k| \leq \lambda C_3 \left\{ \|a\|_{s'_\alpha} \|u\|_{(1+\alpha)q}^{2\alpha + 1} + \|u\|_{(1+\alpha)q}^{2\alpha + 1} \right\}, \tag{5.31}
\]

where \(C_3\) is a constant independent of \(\alpha\). Now from (5.28) and (5.31)

\[
\frac{1}{C_s} \left(\frac{2\alpha + 1}{(\alpha + 1)^2}\right)^{\frac{2^*}{2\alpha + 1}} \leq \lambda C_3 \left\{ \|a\|_{s'_\alpha} \|u\|_{(1+\alpha)q}^{2\alpha + 1} + \|u\|_{(1+\alpha)q}^{2\alpha + 1} \right\}.
\]

Thus

\[
\left(\int |u_k|^{2\alpha + 1} \right)^{-\frac{(\alpha + 1)q}{2\alpha + 1}} \leq C_s \left(\frac{2\alpha + 1}{(\alpha + 1)^2}\right)^{\frac{2^*}{2\alpha + 1}} \lambda C_3 \left\{ \|a\|_{s'_\alpha} \|u\|_{(1+\alpha)q}^{2\alpha + 1} + \|u\|_{(1+\alpha)q}^{2\alpha + 1} \right\}.
\]

Therefore

\[
\|u_k\|_{(1+\alpha)2^*} \leq \lambda C_4(\alpha) \left\{ \|a\|_{s'_\alpha} \|u\|_{(1+\alpha)q}^{\frac{2\alpha + 1}{2\alpha + 1}} + \|u\|_{(1+\alpha)q}^{2\alpha + 1} \right\}.
\]

Now by the monotone convergence theorem we get

\[
\|u\|_{(1+\alpha)2^*} \leq \lambda C_4(\alpha) \left\{ \|a\|_{s'_\alpha} \|u\|_{(1+\alpha)q}^{\frac{2\alpha + 1}{2\alpha + 1}} + \|u\|_{(1+\alpha)q}^{2\alpha + 1} \right\}. \tag{5.32}
\]

Let \(\alpha_1\) be such that \((1 + \alpha_1)q = 2^*\). Note that for this \(\alpha_1\) all the terms in the right hand side of (5.32) are finite and hence \(u \in L^{(\alpha_1+1)2^*}(\mathbb{R}^N)\). Now set \(\beta = \frac{2^*}{q}\) and hence \(u \in L^{2^*,\beta}\). In the next iteration, we choose \(\alpha = \alpha_2\) in (5.32) so that \((1 + \alpha_2)q = (1 + \alpha_1)2^*\) and \(u \in L^{2^*,\beta^2}\). Indeed, for each \(n \in \mathbb{N}\) we choose \(\alpha_n\) so that \((1 + \alpha_n)q = (1 + \alpha_{n-1})2^*\). Thus we get \(u \in L^{2^*,\beta^n}\). Since \(\beta > 1\), using the interpolations on Lebesgue spaces we get \(u \in L^p\) for all \(p \in [2^*, \infty)\).

As an immediate corollary we have the following
5.2 A positive solution branch using the implicit function theorem

Corollary 5.2.5. Let $u_\lambda$ be as in Theorem 5.2.2, then $u_\lambda \in L^p$, for $2^* \leq p < \infty$. Moreover $\|u_\lambda\|_p \leq C_p$, $\forall \lambda \in (-\varepsilon_0, \varepsilon_0)$.

Proof. Since $u_\lambda$ solves $-\Delta u_\lambda = \lambda a_f(u_\lambda)$, for $\lambda \in (-\varepsilon_0, \varepsilon_0)$ from Lemma 5.2.4 we obtain $u_\lambda \in L^p(\mathbb{R}^N)$, $\forall p \in \left[2^*, \infty\right)$. Note that right hand side of (5.2.4) is uniformly bounded for $\lambda \in (-\varepsilon_0, \varepsilon_0)$.

The next lemma gives a representation formula for the solutions of Poisson equation.

Lemma 5.2.6. Let $g \in L^{2^*}(\mathbb{R}^N)$ and $u \in D^{1,2}_0(\mathbb{R}^N)$ be such that $-\Delta u = g$. Then

$$ u(x) = (\Gamma * g)(x) = \int_{\mathbb{R}^N} \Gamma(x - y)g(y)dy, $$

where $\Gamma$ is the fundamental solution of $-\Delta$ on $\mathbb{R}^N$.

Proof. Since $g \in L^{2^*}(\mathbb{R}^N)$, by the Calderon-Zygmund $L^p$ regularity theory, the Newtonian potential $\Gamma * g \in W^{2,2^*}(\mathbb{R}^N)$ and

$$ -\Delta(\Gamma * g) = g \text{ a.e. in } \mathbb{R}^N. $$

In particular $\Delta(u - \Gamma * g) = 0$ in the sense of distributions. Thus $u - \Gamma * g$ is a harmonic function in $L^{2^*}(\mathbb{R}^N)$. Therefore we conclude that $u = \Gamma * g$ is the unique solution of $-\Delta u = g$ in $D^{1,2}_0(\mathbb{R}^N)$. \qed

Now we have the following representation formula for the solution $u_\lambda$ of (5.21).

Proposition 5.2.7. Let $u_\lambda$ be as in Theorem 5.2.2. Then

$$ u_\lambda(x) = \int_{\mathbb{R}^N} \Gamma(x - y)a(y)f(u_\lambda(y))dy. $$

Moreover $u_\lambda \in W^{2,p}(\mathbb{R}^N)$ for all $p \in [(2^*)', r)$ and

$$ \|u_\lambda\|_{W^{2,p}} \leq \tilde{C}_p, \forall \lambda \in (-\varepsilon_0, \varepsilon_0). \quad (5.34) $$

Proof. From Corollary 5.2.5, we get $|u_\lambda|^{\gamma - 1} \in L^s$ for $s \in \left(\frac{2^*}{\gamma}, \infty\right)$ and by the H"older inequality $|a| |u_\lambda|^{\gamma - 1} \in L^q$ with $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{s}$. Now using hypotheses
(A1) and (A2), one can easily see that $a f(u) \in L^q$ for $q \in [(2^*)', r]$ and also $2^* \in [(2^*)', r]$. Since $-\Delta u = \lambda a f(u)$, from Lemma 5.2.6, we obtain

$$u_\lambda(x) = \lambda \int_{\mathbb{R}^N} f(u(y)) a(y) \Gamma(x - y) dy.$$  

Further, using the Calderon-Zygmund $L^p$ regularity theory for the Laplacian, we get $u_\lambda \in W^{2,p}(\mathbb{R}^N)$ for all $p \in [(2^*)', r)$. Now as the map $f \rightarrow f \ast \Gamma$ is continuous from $L^p(\mathbb{R}^N)$ into $W^{2,p}(\mathbb{R}^N)$, using (5.33) we obtain a uniform bound for $\|u_\lambda\|_{W^{2,p}}$.

Using Morrey’s inequality, we obtain the following Hölder regularity of solutions $u_\lambda$.

**Proposition 5.2.8.** Let $u_\lambda$ be as in Theorem 5.2.2, then $u_\lambda \in C^\alpha(\mathbb{R}^N)$. Moreover

$$\|u_\lambda\|_{C^\alpha} \leq C_{\alpha}, \forall \lambda \in (-\varepsilon_0, \varepsilon_0). \quad (5.35)$$

**Proof.** From Morrey’s inequality, for $p > \frac{N}{2}$, we know that $W^{2,p}(\mathbb{R}^N) \hookrightarrow C^\alpha(\mathbb{R}^N)$ with $\alpha = \text{fractional part of } \{2 - \frac{N}{p}\}$. Since $r > \frac{N}{2}$, using the above proposition, we conclude that $u_\lambda \in C^\alpha(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$. The uniform bound for $\|u_\lambda\|_\alpha$ follows as the embedding of $W^{2,p}(\mathbb{R}^N)$ is continuous and $u_\lambda$ is uniformly bounded in $W^{2,p}(\mathbb{R}^N)$. \qed

To get the decay of the solutions, we first recall the following version of Theorem 8.17 of [40]:

**Lemma 5.2.9.** Let $u \in D_{0,1}^{1,2}(\mathbb{R}^N)$ satisfy $-\Delta u = g$ in the weak sense. If $g \in L^q(\mathbb{R}^N)$ with $q > \frac{N}{2}$ then for every $y \in \mathbb{R}^N$ and $R > 0$,

$$\sup_{x \in B_R(y)} |u(x)| \leq C(R, 2^*) \left\{ \|u\|_{L^{2^*}(B_{2R}(y))} + \|g\|_{L^q(B_{2R}(y))} \right\}. \quad (5.36)$$

In the next proposition we prove the decay of $u_\lambda$ at infinity.

**Proposition 5.2.10.** Let $u_\lambda$ be as in the Theorem 5.2.2. Then $u_\lambda(x) \rightarrow 0$ as $|x| \rightarrow \infty$ uniformly for $\lambda \in (-\varepsilon_0, \varepsilon_0)$.

**Proof.** Observe that $r > \frac{N}{2}$ and $a f(u_\lambda) \in L^q(\mathbb{R}^N)$ with $q > \frac{N}{2}$. Now by Lemma 5.2.9 we have the following

$$\sup_{x \in B_R(y)} |u_\lambda(x)| \leq C(R, 2^*) \left\{ \|u_\lambda\|_{L^{2^*}(B_{2R}(y))} + \|a f(u_\lambda)\|_{L^q(B_{2R}(y))} \right\}.$$
5.2 A positive solution branch using the implicit function theorem

Note that the right hand side of above inequality goes to 0 as $|y| \to \infty$. Moreover the convergence is uniform with respect $\lambda$ as the estimates in the proof of Lemma 5.2.4 can be made independent of $\lambda$ by choosing an upper bound for $\lambda$. \hfill \Box

5.2.3 Positivity of the solution branch

In this section we prove the positivity of the solution curve. Here we obtain the positivity of $u_\lambda$ under the additional assumptions (A3) or (A4). The regularity of the solution curve at 0, in $C^\alpha$ topology, is one of the essential ingredients the of our proof. More precisely, we make use of the differentiability of the map $u : (-\varepsilon, \varepsilon) \to C^\alpha(\mathbb{R}^N)$ at 0. The $C^1$ regularity of the map $u : (-\varepsilon, \varepsilon) \to D^{1,2}_0(\mathbb{R}^N)$, is a part of the implicit function theorem. However, it is not clear whether the map $u : (\varepsilon, \varepsilon) \to C^\alpha(\mathbb{R}^N)$ is $C^1$. Here, using a similar technique as in (5.2.4), we show that the map $u : (\varepsilon, \varepsilon) \to L^p(\mathbb{R}^N)$ is continuously differentiable at 0, for every $p \geq 2^*$. From Remark 5.2.3, we know that $u'(0) = v_0$, where $v_0$ is the continuous weak solution of (5.23) for $\varepsilon = 0$. Further,

$$\lim_{\lambda \to 0} \left\| \frac{u_\lambda}{\lambda} - v_0 \right\|_{D^{1,2}_0(\mathbb{R}^N)} = 0.$$  

**Lemma 5.2.11.** Let $u_\lambda$ be as in Theorem (5.2.2). Then for each $p \in [2^*, \infty)$ the map $u : (\varepsilon, \varepsilon) \to L^p(\mathbb{R}^N)$ is continuously differentiable at 0. i.e

$$\lim_{\lambda \to 0} \left\| \frac{u_\lambda}{\lambda} - v_0 \right\|_p = 0.$$

**Proof.** First observe that $(\frac{u_\lambda}{\lambda} - v)$ satisfies the following:

$$-\Delta \left( \frac{u_\lambda}{\lambda} - v_0 \right) = a \left( f(u_\lambda) - f(0) \right). \tag{5.37}$$

Now we set $w = (\frac{u_\lambda}{\lambda} - v_0)$. For $k, \alpha > 0$, let

$$v_k = \left[ (w^+)_k \right]^{2\alpha+1} - \left[ (w^-)_k \right]^{2\alpha+1}.$$  

Multiply Eq: (5.37) by $v_k$ to obtain the following:

$$\int \nabla w \cdot \nabla v_k = \int a \left( f(u_\lambda) - f(0) \right) v_k. \tag{5.38}$$

Using the calculations similar to those that yield (5.28), we obtain a lower
bound for the left hand side of (5.38) as below:

\[ \int \nabla w \cdot \nabla v_k \geq \frac{1}{C_s} \frac{(2\alpha + 1)}{(\alpha + 1)^2} \left( \int (|w_k|^{\alpha+1})^{2^*} \right)^\frac{2}{2^*}, \]  

(5.39)

where \( C_s \) is the best constant appearing in the corresponding Sobolev inequality. Next we estimate the right hand side of 5.38. Using the growth condition on \( f \) and the mean value theorem, we obtain the following:

\[ \int a (f(u) - f(0))v_k \leq \int |a||u\|^\gamma v_k. \]  

(5.40)

Since \( |v_k| \leq |w|^{2\alpha+1} \) and \( |u\|^{\gamma-1} \leq C, \forall \lambda \in (-\varepsilon, \varepsilon) \), see (5.35), from the above inequality, we deduce the following

\[ \int a (f(u) - f(0))v_k \leq C \int |a||w|^{2\alpha+1}. \]  

(5.41)

Note that the right hand side of (5.41) is similar to the first integral in the right hand side of (5.29). Here also we make use of the integrability assumptions on \( a \) to estimate the above integral. As in the proof of Theorem 5.2.4, we set \( q = \frac{2^* - 2}{2^* - (\gamma-2)^r} \) and \( s_\alpha = (\frac{\alpha+1}{2\alpha+1}) q \). Let \( \tilde{\alpha} \) be such that \( (\tilde{\alpha} + 1)q = 2^* \). For any \( \alpha \geq \tilde{\alpha} \), one can verify that \( s_\alpha \in (2^*, r) \) and hence \( a \in L^{s_\alpha} (\mathbb{R}^N) \). Now from the Hölder inequality we get,

\[ \int |a||w|^{2\alpha+1} \leq \|a\|_{s_\alpha'} \|w\|_{(1+\alpha)q}^{(2\alpha+1)}. \]  

(5.42)

By combining (5.39), (5.41), (5.42) and using the monotone convergence theorem we obtain:

\[ \|w\|_{2^*(\alpha+1)}^{2(\alpha+1)} \leq C \|a\|_{s_\alpha'} \|w\|_{(1+\alpha)q}^{(2\alpha+1)}. \]  

(5.43)

Therefore

\[ \left\| \frac{u_\lambda}{\lambda} - v_0 \right\|_{2^*(\alpha+1)}^{2(\alpha+1)} \leq C \|a\|_{s_\alpha'} \left\| \frac{u_\lambda}{\lambda} - v_0 \right\|_{(1+\alpha)q}^{(2\alpha+1)}. \]  

(5.44)

Now, if we choose \( \alpha_1 = \tilde{\alpha} \), then \((1+\alpha)q = 2^* \) and hence when \( \lambda \to 0 \), the right hand side of the above inequality converges to zero. Thus for \( \beta = \frac{2^*}{q} \), we conclude that the map \( u : (\varepsilon_0, \varepsilon_0) \to L^{2^*\beta} \) is differentiable at 0. In the next step, we choose \( \alpha = \alpha_2 \) in the above inequality so that \((1+\alpha_2)q = (1+\alpha_1)2^* \) and hence we deduce the differentiability of the map
5.2 A positive solution branch using the implicit function theorem

$u$ at 0, in $L^{2^*\beta^2}$ topology. Since $\beta > 1$ and by interpolation on Lebesgue spaces, for all $p \in [2^*, \infty)$, we get $u$ is differentiable at 0, in $L^p$ topology.

For large $p$, the embedding of $W^{2,p}(\mathbb{R}^N)$ in $C^\alpha(\mathbb{R}^N)$ is continuous and linear. Thus the map $u : (-\varepsilon_0, \varepsilon_0) \to C^\alpha(\mathbb{R}^N)$ is differentiable at 0. In particular we have the following result.

**Corollary 5.2.12.** Let $u_\lambda$ be as in Theorem 5.2.2. Then

$$\lim_{\lambda \to 0} \frac{\|u_\lambda - v_0\|_\infty}{\lambda} = 0. \quad (5.45)$$

Now we prove the solution branch $u_\lambda$ is positive for small positive $\lambda$.

**Theorem 5.2.13.** Let $u_\lambda$ be as in Theorem 5.2.2. Let $v_0 \in D_0^{1,2}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ be such that

$$-\Delta v_0 = f(0) a. \quad (5.46)$$

If $v_0$ is positive in $\mathbb{R}^N$ and $f(0)a \geq 0$ a.e. near infinity, then there exists $\lambda_0 > 0$, such that $u_\lambda > 0$, for $0 < \lambda < \lambda_0$.

**Proof.** The idea is to use the positivity of $v_0$ to obtain the positivity of the solution branch in a ball for small positive $\lambda$ and then use the definite sign of the weight at infinity to get the same in the complement. Without loss of generality we assume that $f(0) > 0$. Thus by the continuity of $f$, there exists $\delta > 0$ such that $f(s) > 0$ for $|s| < \delta$. Let us fix $R > 0$ sufficiently large, so that $a \geq 0$ and $|u_\lambda(x)| < \delta$ in $B_R^c$ for $|\lambda| < \varepsilon_0$. This is possible since $u_\lambda(x) \to 0$ as $|x| \to \infty$ uniformly with respect to $\lambda$ (thanks to Proposition 5.2.10). Hence

$$af(u_\lambda(x)) \geq 0 \text{ in } B_R^c.$$

Further, from (5.45),

$$\lim_{\lambda \to 0} \frac{\|u_\lambda - v_0\|_\infty}{\lambda} = 0. \quad (5.47)$$

Set $m = \min_{B_R} v_0$. Note that $m$ is positive, since $v_0$ is continuous and $v_0 > 0$. Further, by (5.47) there exists a $\lambda_0$ such that $0 < \lambda_0 < \varepsilon_0$ and

$$\left| \frac{u_\lambda}{\lambda}(x) - v_0(x) \right| < \frac{m}{2}, \forall x \in \overline{B_R}, \forall \lambda \in (-\lambda_0, \lambda_0).$$

Thus $\frac{u_\lambda}{\lambda}(x) > 0$ for $x \in \overline{B_R}$ and hence $u_\lambda(x) > 0$ for $0 < \lambda < \lambda_0$ in $\overline{B_R}$.

Next, we show that $u_\lambda$ is positive also in $B_R^c$, for $0 < \lambda < \lambda_0$. Since $u_\lambda$ is a
weak solution of (5.25), we have the following
\[ \int_{\mathbb{R}^N} \nabla u_\lambda \cdot \nabla w = \lambda \int_{\mathbb{R}^N} a f(u_\lambda) w, \quad w \in D_0^{1,2}(\mathbb{R}^N). \]
In particular, for \( 0 < \lambda < \lambda_0 \), choose \( w = u_\lambda^- \). Observe that \( \text{supp}(u_\lambda^-) \subseteq B^c_R \).
Therefore
\[ -\int_{B^c_R} |\nabla u_\lambda^-|^2 = \lambda \int_{B^c_R} a f(u_\lambda^-) u_\lambda^- . \] (5.48)
Since the right hand side is nonnegative, we get \( |\nabla u_\lambda^-| = 0 \) and hence \( u_\lambda^- \) is a constant on \( B^c_R \). Since \( u_\lambda^- \in D_0^{1,2}(\mathbb{R}^N) \) we get \( u_\lambda^- \equiv 0 \) in \( B^c_R \). Observe that \( u_\lambda \) is continuous and superharmonic in \( B^c_R \). Since \( \inf_{B^c_R} u_\lambda = 0 \), by the strong minimum principle, applicable to the continuous version of superharmonic functions in \( L^1_{\text{loc}} \) (see, Theorem 9.4, [54]), we have either
\[ u_\lambda > 0 \quad \text{or} \quad u_\lambda \equiv 0, \quad \text{in} \ B^c_R, \]
but \( u_\lambda \equiv 0 \) leads to a contradiction to (5.25) as \( a f(0) \not\equiv 0 \) in \( B^c_R \). Thus we conclude that
\[ u_\lambda > 0 \quad \text{in} \ \mathbb{R}^N, \quad \text{for} \ 0 < \lambda < \lambda_0. \]
This completes the proof of the theorem. \( \square \)

In the next theorem we relax the sign restriction of \( a \) at infinity, by assuming the existence of a positive solution for the perturbed linearised problem (5.22).

**Theorem 5.2.14.** Let \( u_\lambda \) be as in Theorem 5.2.2. Let \( v_\varepsilon \in D_0^{1,2}(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N) \) be such that
\[ -\Delta v_\varepsilon = f(0) (a - \varepsilon a^-). \] (5.49)
If \( v_\varepsilon > 0 \) in \( \mathbb{R}^N \) for some \( \varepsilon > 0 \), then there exists \( \lambda_0 > 0 \) such that \( u_\lambda > 0 \), for \( 0 < \lambda < \lambda_0 \).

**Proof.** Here we adapt an idea of Cac et al. [24]. Note that from Proposition 5.2.7, we have the following representation formula for \( v_\varepsilon \):
\[ v_\varepsilon(x) = f(0) \int_{\mathbb{R}^N} \Gamma(x - y) (a^+(y) - (1 + \varepsilon) a^-(y)) dy. \]
Now using the continuity of \( f \) at 0, for any \( \mu > 0 \), there exists \( \delta > 0 \) such that
\[ (1 - \mu) f(0) < f(s) < (1 + \mu) f(0), \quad \forall s \in (-\delta, \delta). \] (5.50)
Since \( u_\lambda \to v \) uniformly in \( \mathbb{R}^N \) and \( v \) is bounded, we have \( u_\lambda \to 0 \) uniformly in \( \mathbb{R}^N \) as \( \lambda \to 0 \). Thus there exists \( \lambda_0 > 0 \) such that \( \|u_\lambda\|_\infty < \delta, 0 < \lambda < \lambda_0 \). Now from (5.50), for each \( \lambda \in (0, \lambda_0) \), we obtain

\[
(1 - \mu)f(0) < f(u_\lambda(y)) < (1 + \mu)f(0), \quad \forall y \in \mathbb{R}^N.
\] (5.51)

Further, from Proposition 5.2.7, we have

\[
u_\lambda(x) = \lambda \int_{\mathbb{R}^N} \Gamma(x - y)a(y)f(u_\lambda(y))dy.
\]

Therefore using (5.51), we obtain the following inequalities

\[
u_\lambda = \lambda \int_{\mathbb{R}^N} \Gamma(x - y)(a^+(y) - a^-(y))f(u_\lambda(y))dy
\]

\[
> \lambda f(0) \int_{\mathbb{R}^N} \Gamma(x - y) \left[(1 - \mu)a^+(y) - (1 + \mu)a^-(y)\right]dy
\]

\[
= \lambda(1 - \mu)f(0) \int_{\mathbb{R}^N} \Gamma(x - y) \left[a^+(y) - \frac{(1 + \mu)}{(1 - \mu)}a^-(y)\right]dy
\]

\[
= \lambda(1 - \mu)f(0) \int_{\mathbb{R}^N} \Gamma(x - y) \left[a(y) - \frac{2\mu}{1 - \mu}a^-(y)\right]dy. \quad (5.52)
\]

Now for the choice of \( \mu = \frac{\varepsilon}{2 + \varepsilon} \), we see that \( \frac{2\mu}{1 - \mu} = \varepsilon \). Thus from (5.52) we conclude that, for \( 0 < \lambda < \lambda_0 \)

\[
u_\lambda(x) > \frac{2\lambda}{2 + \varepsilon} v_\varepsilon(x), \quad \forall x \in \mathbb{R}^N.
\] (5.53)

This completes the proof, \( v_\varepsilon > 0 \).  \( \square \)

**Remark 5.2.15.** It is worth noting that under the same assumptions on \( f \) and \( a \), if the solution of the linearised problem \( v \) is positive, then we get a \( \lambda_0 > 0 \) such that \( u_\lambda < 0 \) a.e. in \( \mathbb{R}^N \) for \( -\lambda_0 < \lambda < 0 \).

The assumption \( f(0) \neq 0 \) is necessary for obtaining a solution for small \( \lambda \) as shown in the following lemma.

**Lemma 5.2.16.** Let \( |f(s)| \leq C|s| \) and let \( a \in L^\infty(N) \). Then (5.21) does not admit a nontrivial solution for small \( \lambda \).

**Proof.** Let \( u \) be a nontrivial weak solution of (5.21) with \( \lambda > 0 \). Then using
Hölder inequality and Sobolev embedding we have the following:

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \leq C \lambda \int_{\mathbb{R}^N} |a||u|^2 \leq C \lambda \|a\|_{\frac{2}{2'}} \|u\|_{\frac{2}{2'}}^2 \leq C_4 C \lambda \|a\|_{\frac{2}{2'}} \int_{\mathbb{R}^N} |\nabla u|^2,
\]

where \(c_s\) is the constant that appears in the Sobolev inequality. Thus \(\lambda\) must be greater than \(\frac{1}{C_4 \|a\|_{\frac{2}{2'}}}\) and hence there is no nontrivial solution for \(\lambda\) smaller than \(\frac{1}{C_4 \|a\|_{\frac{2}{2'}}}\). \(\square\)

Now we give some examples of nonlinear functions that satisfy our assumptions (A1) and (A2).

**Example 5.2.17.** Let \(f\) be one of the following functions

\[
1 + s; \cos(s); \frac{s + 1}{s^2 + 1}; \log(2 + s^2).
\]

Then \(f'\) is a bounded function with \(f(0) \neq 0\). Using the mean value theorem it is easy to see that

\[
|f(s)| \leq C(1 + |s|),
\]

for some \(C > 0\). In general, one can have a nonlinearity, like

\[
1 + |s|^{\gamma-1}; \ (\log(a + |s|^b))^{\gamma-1}, a, b > 1,
\]

where \(\gamma \in [2, 2^*)\) with \(f'\) having a growth as in (A1). Now when \(f'\) is bounded, one can choose \(a \in L^{\frac{2N}{2N-2}}(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)\) with \(r > \frac{N}{2}\); otherwise depending on \(\gamma\) choose \(a \in L^{\frac{2N}{2N-2}}(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)\) with \(r > \tilde{\gamma}\) in order to satisfy (A2).

**Remark 5.2.18.** In order to obtain a positive solution for (5.21), when \(f(0) > 0\), we may assume that \(a\) satisfies

\[
\int_{\mathbb{R}^N} \frac{a^+(x)}{|x - y|^{N-2}}dy > \int_{\mathbb{R}^N} \frac{a^-(x)}{|x - y|^{N-2}}dy, \ \forall x \in \mathbb{R}^N.
\]

for the existence of a positive solution for the linearised problem (5.21). Similarly for getting a positive solution for (5.22), we may assume

\[
\int_{\mathbb{R}^N} \frac{a^+(x)}{|x - y|^{N-2}}dy > \int_{\mathbb{R}^N} (1 + \varepsilon) \frac{a^-(x)}{|x - y|^{N-2}}dy, \ \forall x \in \mathbb{R}^N.
\]