Chapter 4

Weighted eigenvalue problems for the Laplacian

In this chapter we consider the weighted eigenvalue problem for the Laplacian. More precisely, for a given connected domain $\Omega$ in $\mathbb{R}^N$ with $N > 2$, we study the sufficient conditions on a weight function $g$ for the existence of $\lambda \in \mathbb{R}$ and a weak solution $u \in \mathcal{D}_0^{1,2}(\Omega) \setminus \{0\}$ for the following problem:

$$-\Delta u = \lambda g u \text{ in } \Omega.$$  \hfill (4.1)

Recall that $u \in \mathcal{D}_0^{1,2}(\Omega)$ is said to be a weak solution of (4.1), if

$$\int_{\Omega} \nabla u \cdot \nabla v = \lambda \int_{\Omega} g uv, \quad \forall v \in \mathcal{D}_0^{1,2}(\Omega).$$  \hfill (4.2)

There are various sufficient conditions on the weight function $g$ available in the literature for the existence of a positive principal eigenvalue for (4.1). For example, see [61, 21, 22, 4] and the references therein. All these sufficient conditions required either the weight $g$ or its positive part $g^+$ to be in the Lebesgue space $L^{\frac{N}{2}}(\Omega)$. A sufficient condition beyond the Lebesgue spaces was obtained in [73] by Szulkin and Willem, by considering the problem (4.1) for a weight $g$ such that $g^+ \in L^{\frac{N}{2}}(\Omega)$ or having a faster decay than $|x|^{-2}$ at infinity and at any point in the domain.

The existence of a positive principal eigenvalue for (4.1) is closely related to the existence of a minimizer for the following Rayleigh quotient

$$R(u) := \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} g u^2},$$  \hfill (4.3)
with the domain of definition
\[ D^+(g) := \left\{ u \in D_0^{1,2}(\Omega) : \int_{\Omega} g u^2 > 0 \right\}. \]
Let
\[ \lambda_1 = \inf_{u \in D^+(g)} R(u). \tag{4.4} \]
Note that, if \( \lambda_1 > 0 \), then the following inequality holds:
\[ \int_{\Omega} g u^2 \leq \frac{1}{\lambda_1} \int_{\Omega} |\nabla u|^2, \quad \forall u \in D_0^{1,2}(\Omega). \tag{4.5} \]
In addition, if \( \lambda_1 \) is attained for some \( u \in D^+(g) \), then under certain integrability assumptions on \( g \), one may be able to derive Eq. (4.2) as the Euler-Lagrange equation for the minimizer. In order to enlarge the class of weight functions considered in the inequality (4.5) beyond the Lebesgue spaces, Visciglia [77] considered positive weights in \( L(\frac{N}{2}, \infty) \). Following this direction, Ramaswamy and Lucia in [60] proved the existence of a positive principal eigenvalue for (4.1), when \( g \) is such that
\[ g \in \bigcup_{1 \leq q < \infty} L \left( \frac{N}{2}, q \right), \quad g^+ \neq 0. \tag{4.6} \]
They have also shown that the above class of weights and those weights considered in [73] by Szulkin and Willem are independent. We unify all the previous works by proving the existence of a positive principal eigenvalue for (4.1), when \( g \) is such that
\[ g \in \bigcup_{1 \leq q < \infty} L \left( \frac{N}{2}, q \right), \quad g^+ \neq 0. \]

This chapter is organized as follows. In Section 1, we give examples of weights for which (4.1) admits a positive principal eigenvalue and relate our sufficient condition with various sufficient conditions available in the literature. In order to show that (4.1) does not admit an eigenvalue certain classes of weights, we derive a Pohozaev type identity in Section 2. The existence and the other qualitative properties of the first eigenvalue are discussed in Section 3. The existence of infinitely many eigenvalues of (4.1) is studied in Section 4. Further extensions and miscellaneous remarks are given in Section 5.
4.1 Examples of good weight functions

Indeed, in Chapter 3 we had given many examples of weights that are in $\mathcal{F}_T^N$. Thus by our main theorem of this chapter, every function $g$ such that $g^+ \in \mathcal{F}_T^N$ admits a minimizer for $J$ on $\mathcal{M}$. In particular Proposition 3.0.13 shows that

$$\bigcup_{1 \leq q < \infty} L\left(\frac{N}{2}, q\right) \subset \mathcal{F}_T^N. $$

This shows that weights considered in [4, 22, 21, 60, 61] are in $\mathcal{F}_T^N$ and hence our result subsumes all their results.

Another class of admissible weight functions that admit a minimizer for $J$ on $\mathcal{M}$ is provided by the work of Szulkin and Willem in [73]. More specifically they consider the functions $g$ defined by the following conditions:

$$g \in L^1_{\text{loc}}(\Omega), \quad g^+ = g_1 + g_2 \neq 0, \quad g_1 \in L^N_T(\Omega),$$

$$\lim_{|x| \to \infty, x \in \Omega} |x|^2 g_2(x) = 0, \quad \lim_{x \to a, x \in \Omega} |x - a|^2 g_2(x) = 0 \quad \forall a \in \Omega. $$

Lemma 4.1.1. Let $g$ be a measurable function satisfying condition (4.8). Then $g^+$ belongs to the space $\mathcal{F}_T^N$.

Proof. Clearly $g_1 \in \mathcal{F}_T^N$, since $L^N_T(\Omega) \subset \mathcal{F}_T^N$ (see Proposition 3.0.13). Further, by Theorem 3.0.24, $g_2 \in \mathcal{F}_T^N$. Hence the result. □

The above lemma shows that the weights that we consider here, include those that are in [73]. We emphasize that conditions (4.7) and (4.8) do not exhaust the space $\mathcal{F}_T^N$. This is illustrated by the following example:

Example 4.1.2. There are functions lying in $\mathcal{F}_T^N$ which fail to satisfy both (4.7) and (4.8). Let $\Omega = \{(x_1, \ldots, x_n) : |x_i| < R\}$ with $R = 2^{-\frac{N}{N-1}}$, and consider

$$g_1(x) = \begin{cases} 
|x_1|^{-\frac{2}{N}} \log (|\log(|x_1|)|)^{-1} & \text{if } x \in \Omega \text{ and } x_1 \neq 0, \\
0 & \text{otherwise.}
\end{cases}$$

Clearly $g_1$ does not satisfy (4.8), since it fails to satisfy the conclusions of Lemma 3.0.23. To show that $g_1$ does not satisfy (4.7), we first note that the
distribution function of $g_1$ is given by:

$$\alpha_{g_1}(s) = 2^R N^{N-1} |x_1(s)|,$$

where $x_1(s)$ is the first coordinate of $x \in \Omega$, such that $s = g_1(x)$. Now for $t \in (0, (2R)^N)$,

$$g_1^*(t) = \inf \{ s : \alpha_{g_1}(s) \leq t \} = \inf \{ s : 2^R N^{N-1} |x_1(s)| \leq t \}$$

$$= \inf \left\{ |x_1(s)|^{-\frac{N}{N-1}} \left[ \log \left( |\log |x_1(s)|| \right) \right]^{-1} : 2^R N^{N-1} |x_1(s)| \leq t \right\}$$

$$= \left( \frac{t}{2^R N^{N-1}} \right)^{-\frac{N}{N-1}} \left[ \log \left( \frac{t}{2^R N^{N-1}} \right) \right]^{-1}$$

$$= t^{-\frac{N}{N-1}} \log (|\log t|)^{-1}.$$

Observe that our choice of $R$ gives that $2^R N^{N-1} = 1$ and $(2R)^N < 1$. Since $|\Omega| = (2R)^N$, $g_1^*(t) = 0$ for $t \in [(2R)^N, \infty)$. Thus

$$g_1^*(t) = \begin{cases} 
  t^{-\frac{N}{N-1}} \log (|\log t|)^{-1}, & 0 < t < (2R)^N, \\
  0, & t \geq (2R)^N.
\end{cases}$$

A straightforward computation shows

$$\int_0^\infty \left\{ t^{\frac{N}{N-1}} g_1^*(t) \right\}^q \frac{dt}{t} = \int_0^{(2R)^N} \frac{1}{|\log |\log t||^q} \frac{dt}{t} = \int_0^\infty \frac{1}{|\log y|^q} dy.$$

Thus $g_1 \notin L(\frac{N}{2}, q)$ for any $q \in [1, \infty)$, since the last integral is divergent. However

$$\lim_{t \to 0} t^{-\frac{N}{N-1}} g_1^*(t) = \lim_{t \to 0} \frac{1}{|\log (|\log t|)|^{-1}} = 0 = \lim_{t \to \infty} t^{-\frac{N}{N-1}} g_1^*(t).$$

Now using characterization (3.4) of $F_{\frac{N}{2}}$, we conclude that $g_1 \in F_{\frac{N}{2}}$.

### 4.2 A POHOZAEV TYPE IDENTITY

In order to show that for certain classes of weights (4.1) does not admit an eigenvalue, we derive a Pohozaev type identity. We consider an elliptic partial differential equation of the form $-\Delta u = a u$, where $a$ is a weight function satisfying certain integrability conditions. We derive a necessary condition, namely a Pohozaev type identity, for the existence of a solution. Our Pohozaev type identity is inspired by Proposition 4.5 of [74].
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Theorem 4.2.1. Let \( u \in D^{1,2}_0(\mathbb{R}^N) \cap H^2_{\text{loc}}(\mathbb{R}^N) \) and \( a \in L^1_{\text{loc}}(\mathbb{R}^N) \) such that \( a(x)u^2, x \cdot \nabla a(x)u^2 \in L^1(\mathbb{R}^N) \). If \( u \) solves
\[
- \Delta u = a(x)u
\] (4.9)
in the weak sense, then the following identity holds:
\[
\int_{\mathbb{R}^N} \{ x \cdot \nabla a(x) + 2a(x) \} u^2 = 0. \tag{4.10}
\]

Proof. Since \( u \in H^2_{\text{loc}}(\mathbb{R}^N) \) solves Eq. (4.9), we must have
\[
- \Delta u = a(x)u, \text{ a.e. in } \mathbb{R}^N. \tag{4.11}
\]
First, we choose a cut-off function \( \phi \in C^\infty_c(\mathbb{R}) \) such that
\[
(i) \ 0 \leq \phi \leq 1, \quad (ii) \ \phi(r) = 1, \ 0 \leq r \leq 1, \quad (iii) \ \phi(r) = 0, \ r \geq 2.
\]
Now for each \( n \in \mathbb{N} \), we define
\[
\psi_n(x) = \phi \left( \frac{|x|^2}{n^2} \right).
\]
It is easy to see that there exists a constant \( c > 0 \), independent of \( n \), such that
\[
|\psi_n(x)|, |x \cdot \nabla \psi_n(x)| \leq c, \quad \forall x \in \mathbb{R}^N, \ \forall n \in \mathbb{N}.
\]
By multiplying Eq. (4.11) by \( \{ x \cdot \nabla \} \psi_n(x) \) we obtain the following point wise identity:
\[
- \Delta u \{ x \cdot \nabla \} \psi_n(x) = a(x)u \{ x \cdot \nabla \} \psi_n(x), \text{ a.e. } x \in \mathbb{R}^N. \tag{4.12}
\]
Now, straightforward calculations show that the following point wise identities hold:
\[
\begin{align*}
\text{div}( \nabla u \{ x \cdot \nabla \} \psi_n(x) ) & = \Delta u \{ x \cdot \nabla \} \psi_n(x) + \nabla u \cdot \nabla ( \{ x \cdot \nabla \} \psi_n(x) ), \\
\nabla u \cdot \nabla ( \{ x \cdot \nabla \} \psi_n(x) ) & = \nabla u \cdot \nabla \{ x \cdot \nabla \} \psi_n(x) + \{ x \cdot \nabla \} \nabla u \cdot \nabla \psi_n(x), \\
\n\nabla u \cdot \nabla \{ x \cdot \nabla \} & = |\nabla u|^2 + \nabla u \cdot ( \nabla^2 u ) x, \\
\text{div}(x|\nabla u|^2) & = N |\nabla u|^2 + x \cdot \nabla(|\nabla u|^2), \\
x \cdot \nabla(|\nabla u|^2) & = 2 \nabla u \cdot ( \nabla^2 u ) x.
\end{align*}
\] (4.13)
From the above identities we obtain the following:

\[
\Delta u \{x \cdot \nabla u\} \psi_n(x) = \text{div} \left( \nabla u \{x \cdot \nabla u\} \psi_n(x) - \frac{1}{2} (x |\nabla u|^2) \psi_n(x) \right) \\
- \left( 1 - \frac{N}{2} \right) |\nabla u|^2 \psi_n(x) - \nabla u \cdot \{x \cdot \nabla u\} \nabla \psi_n(x). \tag{4.14}
\]

Next we derive an identity for the right hand side of (4.12). Let \( F(u) = \frac{u^2}{r^2} \).
One can easily verify the following identities:

\[
\text{div} (x a(x) F(u) \psi_n(x)) = N a(x) F(u) \psi_n(x) + x \cdot \nabla \{a(x) F(u) \psi_n(x)\}.
\]

\[
x \cdot \nabla \{a(x) F(u) \psi_n(x)\} = x \cdot \nabla a(x) F(u) \psi_n(x) + a(x) u \{x \cdot \nabla u\} \psi_n(x) \\
+ a(x) F(u) x \cdot \nabla \psi_n(x).
\]

From the above identities we get

\[
a(x) u \{x \cdot \nabla u\} \psi_n(x) = \text{div} \left( x a(x) F(u) \psi_n(x) \right) - N a(x) F(u) \psi_n(x) \\
- x \cdot \nabla a(x) F(u) \psi_n(x) - a(x) F(u) x \cdot \nabla \psi_n(x). \tag{4.15}
\]

Now using (4.12), (4.14) and (4.15) we obtain the following identity:

\[
\text{div} \left\{ \left( \nabla u \{x \cdot \nabla u\} + x a(x) F(u) - \frac{1}{2} (x |\nabla u|^2) \right) \psi_n(x) \right\} \\
= \left( 1 - \frac{N}{2} \right) |\nabla u|^2 \psi_n(x) + \{N a(x) F(u) + x \cdot \nabla a(x) F(u)\} \psi_n(x) \\
+ \{(x \cdot \nabla u) \nabla u + a(x) F(u) x\} \cdot \nabla \psi_n(x). \tag{4.16}
\]

Since \( \psi_n(x) = 0 \) in the complement of the ball \( B_{\sqrt{2}n} \), the classical divergence theorem on the ball \( B_{\sqrt{2}n} \) yields the following:

\[
\int_{\mathbb{R}^N} \left\{ \left( 1 - \frac{N}{2} \right) |\nabla u|^2 + N a(x) F(u) + x \cdot \nabla a(x) F(u) \right\} \psi_n(x) dx \\
= -\int_{\mathbb{R}^N} \{(x \cdot \nabla u) \nabla u + a(x) F(u) x\} \cdot \nabla \psi_n(x) dx. \tag{4.17}
\]

Note that, the integrands of each of the above integrals are bounded by functions that are in \( L^1(\mathbb{R}^N) \). Thus by applying the dominated convergence theorem to pass through the limit, noting that that \( \psi_n(x) \to 1 \) and \( \nabla \psi_n(x) \to 0 \).
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as \( n \to \infty \), we obtain

\[
\int_{\mathbb{R}^N} \left( 1 - \frac{N}{2} \right) \left| \nabla u \right|^2 + \left\{ N a(x) + x \cdot \nabla a(x) \right\} \frac{u^2}{2} = 0. \tag{4.18}
\]

Further, multiplying the equation (4.11) by \( u \) yields

\[
\int_{\mathbb{R}^N} \left| \nabla u \right|^2 \, dx = \int_{\mathbb{R}^N} a(x) u^2. \tag{4.19}
\]

Now by substituting (4.19) in (4.18) we get the required identity.

\[
\int_{\mathbb{R}^N} \left\{ x \cdot \nabla a(x) + 2 a(x) \right\} u^2 = 0.
\]

\( \square \)

The following corollary is an immediate consequence of the above theorem:

**Corollary 4.2.2.** Let \( a \) be as in the above theorem. If \( x \cdot \nabla a(x) + 2 a(x) \) has a definite sign, then we have the nonexistence of solution for (4.9) in \( \mathcal{D}^{1,2}_0(\mathbb{R}^N) \cap H^2_{\text{loc}}(\mathbb{R}^N) \).

**Example 4.2.3.** For the weight \( a(x) = \frac{1}{1+|x|^2} \) one can see that

\[
x \cdot \nabla a(x) + 2 a(x) > 0.
\]

Thus we have the nonexistence of solution for (4.9) in \( u \in \mathcal{D}^{1,2}_0(\mathbb{R}^N) \cap H^2_{\text{loc}}(\mathbb{R}^N) \). More importantly, in this case we have the nonexistence of solution of (4.9) in \( \mathcal{D}^{1,2}_0(\mathbb{R}^N) \) itself, since any \( \mathcal{D}^{1,2}_0(\mathbb{R}^N) \) solution of (4.9) is in \( H^2_{\text{loc}}(\mathbb{R}^N) \). This follows from the Calderon-Zygmund regularity theorem, since \( a \in L^\infty(\mathbb{R}^N) \) and \( u \in L^2_{\text{loc}}(\mathbb{R}^N) \).

**Remark 4.2.4.** It is easy to see that \( \frac{1}{1+|x|^2} \in L(\frac{N}{2}, \infty) \). Thus, by using the Hölder inequality (Theorem 2.2.7) and the Lorentz-Sobolev embedding (Theorem 2.3.10) we obtain the following Hardy inequality:

\[
\int_{\mathbb{R}^N} \left| \nabla u \right|^2 \leq \frac{1}{\lambda_1} \int_{\mathbb{R}^N} \frac{1}{1+|x|^2} u^2, \text{ for } u \in \mathcal{D}^{1,2}_0(\mathbb{R}^N),
\]

where

\[
\lambda_1 = \inf \left\{ \int_{\mathbb{R}^N} \left| \nabla u \right|^2 : u \in \mathcal{D}^{1,2}_0(\mathbb{R}^N), \int_{\mathbb{R}^N} \frac{u^2}{1+|x|^2} = 1 \right\}.
\]
Note that $\frac{1}{x_1}$ is the best constant in the above inequality. Thus we conclude that the best constant is not attained for any $u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N)$, otherwise a minimizer would be a weak solution of (4.1).

In order to apply Theorem 4.2.1 for proving the nonexistence of an eigenvalue for (4.1), we must show that any solution of (4.1) is in $H^2_{\text{loc}}(\mathbb{R}^N)$. However one can obtain (4.10), even for the solutions of (4.9) that are in $H^2_{\text{loc}}(\mathbb{R}^N \setminus \{x_1, \ldots, x_n\})$. For example, we have the following corollary:

**Corollary 4.2.5.** Let $u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N) \cap H^2_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ and $a \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that $a(x)u^2$, $x \cdot \nabla a(x)u^2 \in L^1(\mathbb{R}^N)$. If $u$ solves

$$- \Delta u = a(x)u$$

in the weak sense, then the following identity holds:

$$\int_{\mathbb{R}^N} \{ x \cdot \nabla a(x) + 2a(x) \} u^2 = 0. \quad (4.21)$$

**Proof.** One can imitate the proof of Theorem 4.2.1, by choosing the following cut-off functions:

$$\psi_n(x) = (1 - \phi(n^2|x|^2))\phi \left( \frac{|x|^2}{n^2} \right).$$

The following example is considered in [73, 74].

**Example 4.2.6.** For the weight $a(x) = \frac{1}{|x|^2(1+|x|^2)}$ one can verify that

$$x \cdot \nabla a(x) + 2a(x) < 0$$

and that any $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$ solution of (4.9) is at least in $H^2_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$. Thus from the above corollary we have the nonexistence of the solution for (4.9) in $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$. In particular, we have the nonexistence of an eigenvalue for (4.1), when $g(x) = \frac{1}{|x|^2(1+|x|^2)}$.

**Remark 4.2.7.** Note that for the Hardy potential, $a(x) = \frac{1}{|x|^2}$, we get

$$x \cdot \nabla a(x) + 2a(x) = 0.$$ 

Thus our identity holds. However, using the scale invariance of equation (4.1), when $g(x) = \frac{1}{|x|^2}$, one can obtain the nonexistence of a principal
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4.3.1 Existence of an eigenvalue

In this section we prove the existence of an eigenvalue for (4.1) using a direct variational principle. For \( g \in L^1_{\text{loc}}(\Omega) \), first of all recall the definitions of \( R, D^+(g) \) and let us define the following:

\[
M := \left\{ u \in D^1_0(\Omega) : \int_{\Omega} gu^2 = 1 \right\},
\]

\[
J(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2.
\]

Now we consider the following minimization problems:

\[
(P) \quad \text{Minimize } R \text{ on } D^+(g),
\]

\[
(P) \quad \text{Minimize } J \text{ on } M.
\]

Observe that due to the homogeneity of the Rayleigh quotient \( R \), problems \((P)\) and \((\bar{P})\) are equivalent. Furthermore, if problem \((P)\) admits a minimizer \( u \), then we show that \( u \) is an eigenfunction of (4.1), corresponding to the eigenvalue \( R(u) \). Recall that

\[
\lambda_1 = \inf_{u \in M} J(u).
\]

Note that, if \((P)\) admits a minimizer, then \( \lambda_1 \) must be nonzero. From the definition of \( \lambda_1 \), it is clear that \( \lambda_1 \geq 0 \). Thus, for the existence of a minimizer for \( J \) on \( M \), we must have \( \lambda_1 > 0 \). In the following lemma we show that \( \lambda_1 \) is positive, when \( g \in L^1_{\text{loc}}(\Omega) \) and \( g^+ \in F_N^* \). In fact, we obtain a positive lower bound for \( \lambda_1 \) in terms of \( \|g^+\|_{(\frac{N}{p}, \infty)} \).

**Lemma 4.3.1.** If \( g \in L^1_{\text{loc}}(\Omega) \) and \( g^+ \in F_N^* \), then \( \lambda_1 > 0 \).

**Proof.** Let \( u \in M \). Thus by Lorentz-Sobolev embedding (Theorem 2.3.10), \( u \in L(2^*, 2) \). Now using property (2.10) we get \( u^2 \in L(\frac{2^*}{1}, 1) \). Further, \( g^+ \in L(\frac{N}{2}, \infty) \), which is the dual space of \( L(\frac{2^*}{1}, 1) \). Thus by using Hölder
inequality (see (2.9)) we obtain:

\[
\int_{\Omega} g^+ u^2 \leq C \| g^+ \|_{\left( \frac{N}{2}, \infty \right)} \| u^2 \|_{\left( \frac{2^*}{2}, 1 \right)},
\]

where \( C \) is a constant independent of \( u \). Moreover using (2.10) and Lorentz-Sobolev embedding we get

\[
\| u^2 \|_{\left( \frac{2^*}{2}, 1 \right)} \leq \| u \|_{(2^*,2)}^2 \leq C_s \int_{\Omega} |\nabla u|^2,
\]

where \( C_s \) is the best constant that appears in Lorentz-Sobolev embedding. Note that \( \int_{\Omega} g u^2 \leq \int_{\Omega} g^+ u^2 \). Since \( u \in M \), we have \( \int_{\Omega} g u^2 = 1 \). Now using (4.22) and (4.23) we obtain the following inequality:

\[
1 \leq C \| g^+ \|_{\left( \frac{N}{2}, \infty \right)} \int_{\Omega} |\nabla u|^2.
\]

Therefore

\[
\frac{1}{CC_s \| g^+ \|_{\left( \frac{N}{2}, \infty \right)}} \leq \lambda_1.
\]

\[\Box\]

From the definition of the space \( D_0^{1,2}(\Omega) \), it is obvious that \( J \) is coercive and weakly lower semi-continuous. Now using some standard results in functional analysis one may conclude the existence of a minimizer for \( J \) on \( M \), provided \( M \) is weakly closed. But, this is far from being satisfied, due to the weak assumption on \( g^- \). Observe that the map \( G \) defined as

\[
G(u) := \frac{1}{2} \int_{\Omega} g u^2
\]

may not be even continuous and hence the set \( M \) is not even be closed in \( D_0^{1,2}(\Omega) \). Nevertheless, we are not interested in the existence of weak limits of every weakly convergent sequence in \( M \), rather we are concerned particularly with the existence of a weak limit of the minimizing sequence of \( J \) on \( M \). This may hold true, even when \( M \) is not weakly closed.

In the sequel, for the ease of exposition, we will say that a map \( T: X \to Y \) between Banach spaces \( X \) and \( Y \) is completely continuous if \( x_n \to x \) (weakly) in \( X \) implies that \( T(x_n) \to T(x) \) (strongly in \( Y \)).

\textbf{Remark 4.3.2.} Indeed, a completely continuous map is continuous. In addition, if the Banach space \( X \) is reflexive, then the complete continuity of
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T is equivalent to the compactness of T.

Since all the solution spaces that we come across in this thesis are reflexive Banach spaces, hereafter we use the word compact instead of completely continuous.

Next we prove the following result.

**Lemma 4.3.3.** Let $g^+ \in \mathcal{F}_N \setminus \{0\}$. Let

$$G^+(u) = \frac{1}{2} \int_\Omega g^+ u^2.$$

Then $G^+ : \mathcal{D}^{1,2}_0(\Omega) \rightarrow \mathbb{R}$ is compact.

**Proof.** Let $u_n \rightharpoonup u$ in $\mathcal{D}^{1,2}_0(\Omega)$. We show that $\{G^+(u_n)\}$ converges to $G^+(u)$. For $\phi \in C_c^\infty(\Omega)$, we have

$$2(G^+(u_n) - G^+(u)) = \int_\Omega \phi(u_n^2 - u^2) + \int_\Omega (g^+ - \phi)(u_n^2 - u^2). \quad (4.25)$$

We estimate the second integral using Lorentz Sobolev embedding and Hölder inequality as below

$$\int_\Omega |g^+ - \phi||u_n^2 - u^2| \leq C \|g^+ - \phi\|_{(\frac{N}{2},\infty)} \left\{ \|u_n\|^2_{(2^*,2)} + \|u\|^2_{(2^*,2)} \right\} \quad (4.26)$$

where $C$ is a constant independent of $\phi$. Since the sequence $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}_0(\Omega)$, $\{u_n\}$ is also bounded in $L(2^*,2)$ by Theorem 2.3.10. Let

$$m := \sup_n \left\{ \|u_n\|^2_{(2^*,2)} + \|u\|^2_{(2^*,2)} \right\}.$$

Now using the definition of the space $\mathcal{F}_N$, for a given $\varepsilon > 0$, we choose $g_\varepsilon \in C_c^\infty(\Omega)$ so that

$$\|g^+ - g_\varepsilon\|_{(\frac{N}{2},\infty)} < \frac{\varepsilon}{mC}.$$

Thus by taking $\phi = g_\varepsilon$ in (4.26), we obtain

$$\int_\Omega |(g^+ - g_\varepsilon)| |(u_n^2 - u^2)| < \varepsilon. \quad (4.27)$$

Since $\mathcal{D}^{1,2}_0(\Omega) \hookrightarrow L^2_{loc}(\Omega)$ compactly, the first integral in the right hand side of (4.25) with $\phi = g_\varepsilon$ can be made arbitrary small for large $n \in \mathbb{N}$. Hence
we can choose \( n_0 \in \mathbb{N} \), so that
\[
\int_{\Omega} g_n (u_n^2 - u^2) < \varepsilon, \quad \forall n > n_0. \tag{4.28}
\]
Now using (4.27), (4.28) together with (4.25) we conclude that \( \{G^+(u_n)\} \) converges to \( G^+(u) \).

**Remark 4.3.4.** If we assume that \( g \in F^2_N \setminus \{0\} \), i.e. both \( g^+ \) and \( g^- \) are in \( F^2_N \setminus \{0\} \), then the map \( G : D^{1,2}_0(\Omega) \to \mathbb{R} \) is also compact.

Now we are in a position to prove the existence of a minimizer for \( J \) on \( \mathcal{M} \).

**Theorem 4.3.5.** Let \( g \in L^{1}_{\text{loc}}(\Omega) \) and let \( g^+ \in F^2_N \setminus \{0\} \). Then \( J \) admits a minimizer on \( \mathcal{M} \).

**Proof.** Since \( g \in L^{1}_{\text{loc}}(\Omega) \) and \( g^+ \neq 0 \), there exists \( \varphi \in C^\infty_c(\Omega) \) such that \( \int_{\Omega} g^2 \varphi^2 > 0 \) (see for example, Proposition 4.2 of [50]) and hence \( \mathcal{M} \neq \emptyset \). Let \( \{u_n\} \) be a minimizing sequence of \( J \) on \( \mathcal{M} \), i.e.
\[
\lim_{n \to \infty} J(u_n) = \lambda_1 = \inf_{u \in \mathcal{M}} J(u).
\]
Now by the coercivity of \( J \), \( \{u_n\} \) is bounded in \( D^{1,2}_0(\Omega) \). Hence using the reflexivity of \( D^{1,2}_0(\Omega) \) we obtain a subsequence of \( \{u_n\} \) that converges weakly to some \( u \in D^{1,2}_0(\Omega) \). Let us denote the subsequence by \( \{u_n\} \) itself. Since the map \( G^+ \) is compact, we get
\[
\lim_{n \to \infty} \int_{\Omega} g^+ u_n^2 = \int_{\Omega} g^+ u^2. \tag{4.29}
\]
Now as \( u_n \in \mathcal{M} \) we write,
\[
\int_{\Omega} g^- u_n^2 = \int_{\Omega} g^+ u_n^2 - 1.
\]
Since the embedding \( D^{1,2}_0(\Omega) \hookrightarrow L^{2}_{\text{loc}}(\Omega) \) is compact, up to a subsequence \( u_n \to u \) a.e in \( \Omega \). Now we apply the Fatou’s lemma and let \( n \) goes to infinity in the above equation to obtain
\[
\int_{\Omega} g^- u^2 \leq \int_{\Omega} g^+ u^2 - 1.
\]
This shows that \( \int_{\Omega} gu^2 \geq 1 \). Setting \( \tilde{u} := \frac{u}{(\int_{\Omega} gu^2)^{1/2}} \), the weak lower semi-continuity of \( J \) yields the following,

\[
\lambda_1 \leq J(\tilde{u}) = \frac{J(u)}{\int_{\Omega} gu^2} \leq J(u) \leq \liminf_n J(u_n) = \lambda_1.
\]

Thus equality must hold at each step and hence \( \int_{\Omega} gu^2 = 1 \), which shows that \( u \in \mathcal{M} \) and \( J(u) = \lambda_1 \).

\[\square\]

**Remark 4.3.6.** Note that, when \( g \in \mathcal{F}_\mathcal{N} \setminus \{0\} \) the map \( G : \mathcal{D}_0^{1,2}(\Omega) \to \mathbb{R} \) is compact. Hence

\[
\lim_{n \to \infty} \int_{\Omega} g u_n^2 = \int_{\Omega} g u^2
\]

for the minimizing subsequence \( \{u_n\} \) obtained in the above proof. Since \( u_n \in \mathcal{M} \), we have \( \int_{\Omega} g u_n^2 = 1 \) and hence \( \int_{\Omega} g u^2 = 1 \). Thus we conclude that \( u \in \mathcal{M} \) and \( J(u) = \lambda_1 \).

**Proposition 4.3.7.** Let \( g \in L^1_{\text{loc}}(\Omega) \) and \( g^+ \in \mathcal{F}_\mathcal{N} \setminus \{0\} \). Let \( u \) be a minimizer of \( R \) on \( \mathcal{D}^+(g) \). Then \( u \) is an eigenfunction corresponding to the eigenvalue \( \lambda_1 \) of (4.1).

**Proof.** Notice that \( u \in \mathcal{D}^+(g) \) implies that \( u \neq 0 \). For each \( \phi \in \mathcal{C}^\infty_c(\Omega) \), using dominated convergence theorem one can verify that \( R \) admits directional derivative along \( \phi \). Now since \( u \) is a minimizer of \( J \) on \( \mathcal{D}^+(g) \) we get

\[
\frac{d}{dt} R(u + t\phi)|_{t=0} = 0.
\]

Therefore

\[
\int_{\Omega} \nabla u \cdot \nabla \phi = \lambda_1 \int_{\Omega} g u \phi, \quad \forall \phi \in \mathcal{C}^\infty_c(\Omega).
\]

Now we use the density of \( \mathcal{C}^\infty_c(\Omega) \) in \( \mathcal{D}_0^{1,2}(\Omega) \) to conclude that

\[
\int_{\Omega} \nabla u \cdot \nabla v = \lambda_1 \int_{\Omega} g u v, \quad \forall v \in \mathcal{D}_0^{1,2}(\Omega).
\]

(4.30) \[\square\]

**Remark 4.3.8.** If we assume that \( g \in L^1_{\text{loc}}(\Omega) \) and \( g^- \in \mathcal{F}_\mathcal{N} \setminus \{0\} \), then by the same analysis for the weight \( -g \) we obtain a maximum negative eigenvalue \( \mu_1 \). In particular, when \( g \in \mathcal{F}_\mathcal{N} \) with \( g^+, g^- \neq 0 \), then we obtain a positive and a negative eigenvalue for (4.1).
4.3.2 Sign of the eigenfunctions

In this section we discuss the sign of the minimizers that we obtained in Proposition 4.3.5. First we study the principal nature of $\lambda_1$. Since $R(u) = R(|u|)$, if $u$ is an eigenfunction corresponding to $\lambda_1$ then $|u|$ is also an eigenfunction. However this does not mean that $\lambda_1$ is a principal eigenvalue, unless the zero set of $u$ is of measure zero. Note that, the sign of a measurable function is well defined only when its zero set is of measure zero. Thus, if $u$ and $g$ are regular enough, one may use classical maximum principle to deduce that $|u|$ is positive except for a set of zero measure. However, when these functions have less regularity one may use a weaker version of the strong maximum principle due to Ancona[8] or Brezis and Ponce [19]. Here we use the following result of Brezis and Ponce, see Corollary 4 of [19]:

Theorem 4.3.9. (Strong Maximum Principle) Let $O \subset \mathbb{R}^N$ be a non-empty open connected bounded set and $a \in L^1_{\text{loc}}(O)$ with $a \geq 0$. Assume that $au \in L^1_{\text{loc}}(O)$ and $\Delta u$ is a Radon measure on $O$. If

$$-\Delta u + au \geq 0 \text{ in } D'(O)$$

and $u = 0$ on a subset of $O$ with positive measure, then $u = 0$ a.e. in $O$.

Now using the above theorem we prove that the eigenfunctions corresponding to $\lambda_1$ are of constant sign.

Proposition 4.3.10. Let $g$ be as in Theorem 4.3.5. Then every eigenfunctions corresponding to $\lambda_1$ is of constant sign.

Proof. Let $u$ be an eigenfunction corresponding to $\lambda_1$. Clearly $u^+$ or $u^-$ is nonzero. Without loss of generality assume that $u^+ \neq 0$. Now by taking $v = u^+$ in (4.2) we see that $u^+$ minimizes $R$ on $D^+(g)$. Thus by Proposition 4.3.7, $u^+$ solves (4.1) in the weak sense,

$$-\Delta u^+ - \lambda_1 gu^+ = 0, \text{ in } \Omega. \quad (4.31)$$

Thus we have the following differential inequality in the sense of distribution,

$$-\Delta u^+ + \lambda_1 g^- u^+ = \lambda_1 g^+ u^+ \geq 0, \text{ in } \Omega.$$ 

Note that equation (4.31) shows that $\Delta u^+ \in L^1_{\text{loc}}(\Omega)$ and in particular a
4.3 Existence and some properties of the first eigenvalue

Radon measure on $\Omega$. Furthermore,
\[(g^-)^{\frac{1}{2}} \in L^2_{loc}(\Omega), \text{ since } g^- \in L^1_{loc}(\Omega),\]
\[(g^-)^{\frac{1}{2}}(u^+) \in L^2(\Omega), \text{ since } u^+ \text{ is a solution } g(u^+)^2 \in L^1(\Omega).\]

Now let us write
g^+ - u^+ = (g^-)^{\frac{1}{2}}(g^-)^{\frac{1}{2}}u^+
and then use Hölder inequality to obtain $g^+-u^+ \in L^1_{loc}(\Omega)$. Now in view of
Theorem 4.3.9, we obtain $u^+ > 0$ a.e.. Thus $u = u^+$ and moreover, the zero
set of $u$ is of measure zero. $\square$

4.3.3 The uniqueness of the positive principal eigenvalue

Here we prove the uniqueness of the positive principal eigenvalue using a
Picone’s identity.

Theorem 4.3.11. If $\lambda > 0$ is any eigenvalue of (4.1) with an eigenfunction
which does not change sign, then $\lambda = \lambda_1$.

Proof. Let $v$ be a positive eigenfunction corresponding to the eigenvalue $\lambda$. Let $u \in \mathcal{M}$. Thus there exists a sequence $\{\phi_n\}$ such that $\phi_n \in C_c^\infty(\Omega)$ and as $n \to \infty$,
\[\|u - \phi_n\|_{D^{1,2}_0(\Omega)} \to 0.\]

Further, $\frac{\phi_n^2}{(v + \epsilon)} \in D^{1,2}_0(\Omega)$ (see Proposition 2.3.7) is a valid test function in
(4.2). Thus we obtain
\[\int_\Omega |\nabla \phi_n|^2 - \lambda \int_\Omega g v \frac{\phi_n^2}{(v + \epsilon)} = \int_\Omega |\nabla \phi_n|^2 - \lambda \int_\Omega g v \frac{\phi_n^2}{(v + \epsilon)}. \tag{4.32}\]

A straight calculation shows that the following “Picone’s identity” holds:
\[|\nabla \phi_n|^2 - \nabla v \cdot \nabla \left( \frac{\phi_n^2}{(v + \epsilon)} \right) = \left| \nabla \phi_n - \left( \frac{\phi_n}{v + \epsilon} \right) \nabla v \right|^2. \tag{4.33}\]

By plugging (4.33) in (4.32) we deduce that
\[0 \leq \int_\Omega |\nabla \phi_n|^2 - \lambda \int_\Omega g v \frac{\phi_n^2}{(v + \epsilon)}.\]
Using dominated convergence theorem, we let \( \varepsilon \to 0 \), to obtain

\[
0 \leq \int_{\Omega} |\nabla \phi_n|^2 - \lambda \int_{\Omega} g\phi_n^2.
\]

Now by letting \( n \to \infty \) and by the continuity of the map \( G^+ \) (see Corollary 2.2.8) and by Fatou’s lemma, we obtain

\[
0 \leq \int_{\Omega} |\nabla u|^2 - \lambda \int_{\Omega} gu^2 = \int_{\Omega} |\nabla u|^2 - \lambda.
\]

Therefore

\[
\lambda \leq \int_{\Omega} |\nabla u|^2, \quad \forall u \in \mathcal{M}.
\]

Hence by the definition of \( \lambda_1 \), we conclude that \( \lambda = \lambda_1 \).

**Remark 4.3.12.** In particular the above theorem shows that all the eigenfunctions corresponding to an eigenvalue \( \lambda > \lambda_1 \) must change sign.

### 4.3.4 The simplicity of the first eigenvalue

Using the connectedness of \( \Omega \), we show that \( \lambda_1 \) is simple. For the proof, we adapt a connectedness argument as in [29, 61]. To show that any two eigenfunctions corresponding to \( \lambda_1 \) are linearly dependent, we prove the following lemma first.

**Lemma 4.3.13.** Let \( u \) and \( v \) be two measurable functions on \( \Omega \) and let \( u > 0 \) a.e. Then there exists \( \varepsilon_0 > 0 \) such that for all \( 0 < \varepsilon < \varepsilon_0 \), \( u - \varepsilon v > 0 \) on a set \( E_\varepsilon \) of positive measure.

**Proof.** If not, we can find \( \varepsilon_n \downarrow 0 \) and sets \( E_n \), each of measure zero, such that \( u - \varepsilon_n v \leq 0 \) on \( E_n^c \). Since \( E = \cup E_n \) also has measure zero and \( u - \varepsilon_n v \leq 0 \) for all \( n \) on \( E^c \), we deduce that \( u \leq 0 \) a.e, a contradiction.

**Proposition 4.3.14.** Let \( g \) and \( \lambda_1 \) be as in Theorem 4.3.5. Then \( \lambda_1 \) is simple.

**Proof.** Let \( u_1, u_2 \) be two nonnegative eigenfunctions corresponding to \( \lambda_1 \). Define

\[
A^+ := \{ \alpha \in \mathbb{R} : u_1 + \alpha u_2 > 0 \text{ a.e. } \},
\]

\[
A^- := \{ \alpha \in \mathbb{R} : u_1 + \alpha u_2 < 0 \text{ a.e. } \}.
\]

(i) Since \( 0 \in A^+ \), \( A^+ \neq \emptyset \).
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(ii) Set $u = u_2$ and $v = u_1$ in the above lemma and choose $\varepsilon > 0$ such that $u_2 - \varepsilon u_1 > 0$ on a set $E_\varepsilon$ of positive measure. Then by Proposition 4.3.10, $u_2 - \varepsilon u_1 > 0$ a.e. Then $u_1 - \frac{1}{\varepsilon} u_2 < 0$ a.e. and so $A^- \neq \emptyset$.

(iii) Let $\alpha \in A^+$. Then $u_1 + \alpha u_2 > 0$ a.e. Set $u = u_1 + \alpha u_2$ and $v = u_2$ in the above lemma to obtain an $\varepsilon > 0$ such that $u_1 + \alpha u_2 - \varepsilon u_2 > 0$ on a set $E_\varepsilon$ of positive measure. Since $u_1 + (\alpha - \varepsilon) u_2$ is an eigenfunction of (4.1), by Proposition 4.3.10, $u_1 + (\alpha - \varepsilon) u_2 > 0$ a.e. Thus for every $\beta > \alpha - \varepsilon$, $u_1 + \beta u_2 > 0$ a.e. Thus $A^+$ is open.

(iv) Let $\alpha \in A^-$. Then $-(u_1 + \alpha u_2) > 0$ a.e. Set $u = -(u_1 + \alpha u_2)$ and $v = u_2$ in the above lemma to deduce, as in (iii), that $A^-$ is also open.

Now from the connectedness of $\mathbb{R}$, we deduce that $\mathbb{R} \setminus (A^+ \cup A^-) \neq \emptyset$. Hence there exists $\alpha_0$ such that $u_1 + \alpha_0 u_2$ vanishes on a set of positive measure and therefore must vanish a.e. in $\Omega$ (by Proposition 4.3.10). This shows that $u_1$ and $u_2$ are linearly dependent. Thus we conclude that the first eigenfunction is unique up to a constant multiple.

Now we state the main result of this chapter. In view of Theorem 4.3.5, Proposition 4.3.10, Theorem 4.3.11 and Proposition 4.3.14 we have the following theorem:

**Theorem 4.3.15.** Let $N \geq 3$ and let $g \in L^1_{loc}(\Omega)$ such that $g^+ \in \mathcal{F}_N^1 \setminus \{0\}$. Then

$$\lambda_1 = \inf \{ J(u) : u \in M \}$$

is the unique positive principal eigenvalue of (4.1). Furthermore, each eigenfunction corresponding to $\lambda_1$ is of constant sign and $\lambda_1$ is simple.

4.4 Existence of an infinite sequence of eigenvalues

In this section, we prove the existence of an infinite sequence of eigenvalues of (4.1), when $g \in L^1_{loc}(\Omega)$ and $g^+ \in \mathcal{F}_N^1$. First of all we obtain an infinite orthogonal set in $\mathcal{M}$, using an idea from Proposition 4.2 of [50].

**Lemma 4.4.1.** Let $g \in L^1_{loc}(\Omega)$ and $g^+ \neq 0$. Then for each $n \in \mathbb{N}$, there exist $v_1, v_2, \ldots, v_n \in \mathcal{M}$ with disjoint supports.

**Proof.** Let $\Omega^+ = \{x : g^+(x) > 0\}$. Since $|\Omega^+| > 0$, using the Lebesgue-Besicovitch differentiation theorem, one can choose $n$ points $x_1, x_2, \ldots x_n$ in $\Omega^+$ such that

$$\lim_{r \to 0} \frac{1}{|B_r(x_i)|} \int_{B_r(x_i)} g(y)dy = g(x_i) > 0.$$
Thus there exists $R > 0$, such that $B_R(x_i) \cap B_R(x_j) = \emptyset$ and

$$
\frac{1}{|B_r(x_i)|} \int_{B_r(x_i)} g(y) dy > \frac{g(x_i)}{2}, \forall r \in (0, R).
$$

In particular there exists $m > 0$ such that $\int_{B_r(x_i)} g(y) dy > m, \forall r \in \left(\frac{R}{2}, R\right)$. Thus we can choose $r \in \left(\frac{R}{2}, R\right)$ so that

$$
\int_{B_r(x_i) \setminus B_r(x_i)} |g(y)| dy < \int_{B_r(x_i)} g(y) dy.
$$

Let $u_i \in C^\infty_c(B_R(x_i))$ be such that $0 \leq u_i(x) \leq 1$ and $u_i \equiv 1$ on $B_r(x_i)$. Now using (4.36) we have the following

$$
\int_{B_R(x_i)} gu_i^2 = \int_{B_r(x_i)} g + \int_{B_R(x_i) \setminus B_r(x_i)} gu_i^2 \geq \int_{B_r(x_i)} g - \int_{B_R(x_i) \setminus B_r(x_i)} |g| > 0
$$

Take $v_i = \frac{u_i}{(\int_{\Omega} gu_i^2)^{\frac{1}{2}}}$. Clearly $v_i \in \mathcal{M}$ and their supports are disjoint.

**Corollary 4.4.2.** Let $Y$ be a finite dimensional subspace of $\mathcal{D}_0^{1,2}(\Omega)$. Then $Y^\perp \cap \mathcal{M} \neq \emptyset$.

**Proof.** Let $\dim(Y) = m$. Thus by applying the above lemma for $m + 1$, we get $v_1, v_2, \ldots, v_{m+1} \in \mathcal{M}$ with disjoint supports. Let $Z = \text{span}\{Y, v_1, \ldots, v_{m+1}\}$. Since the set $\{v_1, v_2, \ldots, v_{m+1}\}$ is linearly independent, we have

$$
\dim(Z) \geq m + 1 > \dim(Y).
$$

Thus, there exists $v \in Z$, $v \perp Y$ and hence $v = \sum_{i=1}^{m+1} \alpha_i v_i$, for some $\alpha_i \in \mathbb{R}$, not all zeros. Furthermore,

$$
\int_{\Omega} g v^2 = \alpha_1^2 + \ldots + \alpha_{m+1}^2.
$$

Let $\bar{v} = \frac{v}{\sqrt{\alpha_1^2 + \ldots + \alpha_{m+1}^2}}$. Then $\bar{v} \in \mathcal{M}$ and $\bar{v} \perp Y$.

**Theorem 4.4.3.** Let $g \in L^1_{\text{loc}}(\Omega)$ and $g^+ \in \mathcal{F}_N \setminus \{0\}$. Then (4.1) admits a sequence of positive eigenvalues going to $\infty$.

**Proof.** For each $n \in \mathbb{N}$, using Theorem 4.3.5, we obtain $u_n \in \mathcal{M}$ such that

$$
\lambda_n = J(u_n) = \inf_{\{u \in \mathcal{M} \setminus \mathcal{M}_{n-1}\}} J(u),
$$
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where \( Y_0 = \{0\} \), \( Y_{n-1} = \text{span}\{u_1, u_2, \ldots, u_{n-1}\} \). From the above corollary, it is clear that \( \{u \in M : u \perp Y_{n-1}\} \neq \emptyset \), for each \( n \). Now by the same argument as in Proposition 4.3.7, we see that

\[
\int_{\Omega} \nabla u_n \cdot \nabla v = \lambda_n \int_{\Omega} g u_n v, \quad \forall v \in Y_{n-1}^\perp.
\]

Further, for \( i = 1, 2, \ldots, n-1 \)

\[
0 = \int_{\Omega} \nabla u_i \cdot \nabla u_n = \lambda_i \int_{\Omega} g u_i u_n.
\]

But for \( v \in Y_{n-1} \), \( v = \sum_{i=1}^{n-1} a_i u_i \) and hence \( \int_{\Omega} g u_n v = 0 \), for \( v \in Y_{n-1} \). Thus

\[
\int_{\Omega} \nabla u_n \cdot \nabla v = \lambda_n \int_{\Omega} g u_n v, \quad \forall v \in D_0^{1,2}(\Omega).
\]

Thus we obtain an infinite orthogonal set \( \{u_n\} \) of eigenfunctions of (4.1) in \( M \). Next we show that the sequence \( \{\lambda_n\} \) is unbounded. Now by setting \( v_n = \frac{u_n}{\sqrt{\lambda_n}} \), we see that \( \{v_n\} \) is an orthonormal sequence in \( D_0^{1,2}(\Omega) \) and hence \( v_n \to 0 \). Note that

\[
\lambda_n^{-1} = \int_{\Omega} g v_n^2 \leq \int_{\Omega} g^+ v_n^2.
\]

Now we use the compactness of \( G^+ \) to obtain

\[
0 \leq \lim_{n \to \infty} \lambda_n^{-1} \leq \lim_{n \to \infty} \int_{\Omega} g v_n^2 = 0.
\]

This shows that \( \lim_{n \to \infty} \lambda_n^{-1} = 0 \) and hence the sequence \( \{\lambda_n\} \) of eigenvalues of (4.1) is unbounded. \( \Box \)

**Remark 4.4.4.** If \( g \in \mathcal{F}_{\frac{\Omega}{2}} \), then using a similar technique as in Lemma 4.3.3 one can verify that the map \( G' : D_0^{1,2}(\Omega) \to [D_0^{1,2}(\Omega)]' \) defined by

\[
\langle G'(u), v \rangle = \int_{\Omega} guv
\]

is a compact operator. Indeed, later we show that the map \( G' \) is the derivative of \( G \). Now as the map \( (\Delta)^{-1} : [D_0^{1,2}(\Omega)]' \to D_0^{1,2}(\Omega) \) is bounded and linear, the map \( L = (\Delta)^{-1} G' \) is a compact operator on \( D_0^{1,2}(\Omega) \). Also \( L \) is a self
adjoint operator. Note that problem (4.2) is equivalent to solving:

\[ u - \lambda L(u) = 0. \]  

(4.37)

Thus the reciprocal of a nonzero eigenvalue of \( L \) is an eigenvalue of (4.1). By the spectral theorem for the self adjoint compact operators we have an orthonormal basis for the \( \text{Ker}(L)^\perp \). Now if \( g \neq 0 \), using Lemma 4.4.1, we obtain an infinite linearly independent set in \( \text{Ker}(L)^\perp \). Thus the set of all eigenvalues of (4.1) is infinite, since eigenspace corresponding to each eigenvalue of \( L \) is finite dimensional. Since 0 is the only possible limit point of eigenvalues of \( L \), the set of all eigenvalues of (4.1) is also unbounded.

4.5 Miscellaneous remarks

Remark 4.5.1. In [10], we have considered the existence of positive solutions for the following nonlinear problem:

\[ -\Delta u + V(x)u = \lambda g(x)|u|^{p-2}u, \quad u|_{\partial \Omega} = 0, \]  

(4.38)

for some appropriate value of the parameter \( \lambda \), where \( \Omega \) is a non-empty open connected subset of \( \mathbb{R}^N \) with \( N \geq 2 \) and \( V, g \in L^1_{\text{loc}}(\Omega) \). Indeed we prove the existence of a positive solution of (4.38), using a similar variational technique as in Theorem 4.3.5, when \( p \in (2, 2^*) \) and

\[ V \geq 0, \quad g^+ \in \mathcal{F}_{\frac{p}{2}} \setminus \{0\} \text{ with } \frac{1}{p} + \frac{p}{2^*} = 1. \]  

(4.39)

Remark 4.5.2. A natural question is whether one can get the existence of a principal eigenfunction when \( g^+ \notin \mathcal{F}_{\frac{2}{2^*}} \).

(a) Tertikas in [74] introduced the notion of subcritical potential and using the concentration compactness lemma, he showed that for every subcritical potential \( g \), the problem (4.1) admits a principal eigenvalue (see [74, Corollary 3.6]). He also showed that such existence still holds if \( g = \frac{1}{|x|^2} + g_1(x) \), where \( g_1(x) > 0 \) is any subcritical potential (see [74, Theorem 1.7]). One can verify that all positive functions in \( \mathcal{F}_{\frac{N}{2^*}} \) are subcritical potentials. Clearly for any positive weight \( g_1 \in \mathcal{F}_{\frac{N}{2^*}}, \frac{1}{|x|^2} + g_1(x) \) cannot be in \( \mathcal{F}_{\frac{N}{2^*}} \), but (4.1) still admits a principal eigenvalue when \( g(x) = \frac{1}{|x|^2} + g_1(x) \).

(b) In [25] using the concentration compactness lemma, Chabrowski proved
the existence of a positive principal eigenvalue for (4.1) in bounded domains for certain weights of the form $g(x) = \frac{m(x)}{|x|^2}$, where $m \in C(\Omega)$ and $m(0)\lambda_1(g) < \left(\frac{N-2}{2}\right)^2$. There are weights satisfying Chabrowski’s conditions but not lie in $\mathcal{F}_{\frac{N}{2}}$. 