Chapter 3

The function space $\mathcal{F}_d$

Let $\Omega$ be a domain in $\mathbb{R}^N$. In this chapter we identify an important subspace of the weak- $L^d(\Omega)$ spaces for $d > 1$ and characterize these subspaces with the behaviour of $t^{\frac{1}{d}} f^*(t)$ at 0 and $\infty$. Later we show that these subspaces have a close connection with the existence of eigenvalues for certain weighted eigenvalue problems for the Laplacian and the $p$-Laplacian.

The main results presented in this chapter are published in [10] as a joint work with Marcello Lucia and Mythily Ramaswamy.

For $1 < d < \infty$, it is well known that $C_c^\infty(\Omega)$ is dense in the Lebesgue space $L^d(\Omega)$, whereas in $L^1(\Omega)$ the closure of $C_c^\infty(\Omega)$ defines a proper closed subspace of $L^1(\Omega)$, namely, the space of functions vanishing on $\partial \Omega$ and at $\infty$ (if the domain is unbounded). A similar situation occurs in the Lorentz spaces. More precisely, for $(d, q) \in [1, \infty) \times [1, \infty)$, $C_c^\infty(\Omega)$ is dense in the Lorentz space $L(d, q)$. However, for $d > 1$, the closure of $C_c^\infty(\Omega)$ in $L(d, \infty)$ defines a proper subspace that will henceforth be denoted by $\mathcal{F}_d$:

$$\mathcal{F}_d := C_c^\infty(\Omega)^{\|\cdot\|_{L(d, \infty)}} \subset L(d, \infty).$$

Equivalently the space $\mathcal{F}_d$ can also be defined as the closure of the set of all simple functions on $\Omega$. More precisely, let

$$S_1 := \left\{ f = \sum_{j=1}^{m} c_j \chi_{E_j}, m \in \mathbb{N}, c_i \in \mathbb{R} \setminus \{0\} \right\},$$

where $E_1, \ldots, E_m$ are bounded measurable sets and $\chi_E$ stands for the characteristic function of the set $E$. Then $S_1$ is dense in $L(d, q)$ for $1 \leq d, q < \infty$ (see [36]). Now using the definition of $\mathcal{F}_d$ and the continuous embedding of
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$L(d, q)$ into $L(d, \infty)$, we can see that

$$S_1^{-\cdot \|_{(d, \infty)}} = F_d.$$ 

In certain situations we may have to consider the following type of step functions:

$$S_2 := \left\{ f = \sum_{j=1}^{m} c_j \chi_{E_j}, m \in \mathbb{N}, c_i \in \mathbb{R} \setminus \{0\} \right\},$$

where $E_j$ is a measurable set of finite measure, not necessarily bounded. However, as the distribution function of $\chi_E$ depends only on the measure of $E$ and not on the boundedness of $E$, we see that, $S_1 \subset S_2 \subset L(d, q)$. Hence

$$S_2^{-\cdot \|_{(d, \infty)}} = S_1^{-\cdot \|_{(d, \infty)}} = F_d.$$

Throughout this chapter we assume that $d > 1$. Thus by Remark 2.2.5, $F_d$ is contained in $L^1_{\text{loc}}(\Omega)$. The following proposition shows that $F_d$ is bigger than the Lorentz space $L(d, q)$ for any finite $q$, however strictly smaller than $L(d, \infty)$.

**Proposition 3.0.13.** \( i \) For each \( 1 \leq q < \infty \), \( L(d, q) \subset F_d \).

\( ii \) For each \( a \in \Omega \subset \mathbb{R}^N \) and \( d < N \), the Hardy potential \( h(x) = |x-a|^{-d} \) does not belong to \( F_{\frac{N}{d}} \).

**Proof.** \( i \) Let \( f \in L(d, q) \). Since $C^\infty_c(\Omega)$ is dense in $L(d, q)$, there exists a sequence $f_n \in C^\infty_c(\Omega)$ such that $\lim_{n \to \infty} \|f - f_n\|_{(d, q)} = 0$. By (a) of Proposition 2.2.3 we have

$$\|f - f_n\|_{(d, \infty)} \leq C \|f - f_n\|_{(d, q)},$$

hence by the definition $f \in F_d$.

\( ii \) From (2.2) it is clear that $h \in L\left(\frac{N}{d}, \infty\right)$. Let $f \in S_1$, $f = \sum_{i=1}^{m} c_i \chi_{E_i}$ with $c_i \in \mathbb{R}$. We show that $\|h - f\|_{(d, \infty)} \geq c > 0$ for some constant $c$ independent of $f$, which will conclude the proof, since $S_1$ is dense in $F_{\frac{N}{d}}$. 

Fix a ball $B(a, r) \subset \Omega$. Let $c_0 = \max_{1 \leq i \leq m} |c_i|$. Let $s > 0$.

$$\alpha_{h-f}(s) \geq \left| \left\{ x \in B(a; r) : |(h - f)(x)| > s \right\} \right|$$

$$\geq \sum_{i=1}^{m} \left\{ \left\{ x \in B(a, r) \cap E_i : |x - a|^{-d} - c_i > s \right\} \right|$$

$$+ \left\{ x \in B(a, r) \setminus \left( \bigcup_{i=1}^{m} E_i \right) : |x - a|^{-d} > s \right\}.$$

Notice that if $x \in B(a; r) \setminus \bigcup_{i=1}^{m} E_i$ and

$$|x - a|^{-d} < (c_0 + s)^{-1},$$

then we have

$$s < c_0 + s < |x - a|^{-d}. \tag{3.1}$$

Similarly, if $x \in B(a; r) \cap E_i$ and if (3.1) holds, then

$$|c_i| + s \leq c_0 + s < |x - a|^{-d}.$$

Thus

$$s < |x - a|^{-d} - |c_i| \leq |x - a|^{-d} - c_i. \tag{3.3}$$

Consequently, from (3.2) and (3.3), we see that

$$\alpha_{h-f}(s) \geq \left| \left\{ x \in B(a; r) : |x - a|^{-d} < (c_0 + s)^{-1} \right\} \right|.$$

Let $s_0 > 0$ be such that $(c_0 + s)^{-1} < r^d, \forall s > s_0$. Thus, from the above inequality, we get

$$\alpha_{h-f}(s) \geq \omega_N (c_0 + s)^{\frac{N}{d}}, \forall s > s_0.$$

Now,

$$\|h - f\|_{(\frac{N}{d}, \infty)} = \sup_{s > 0} \left\{ s \left( \alpha_{h-f}(s) \right)^{\frac{d}{N}} \right\} \geq \sup_{s > 0} \left\{ s \omega_N^{\frac{d}{N}} (c_0 + s)^{-1} \right\} \geq \omega_N^{\frac{d}{N}}.$$

**Remark 3.0.14.** From (i) it is clear that the usual Lebesgue space $L^d(\Omega)$ is contained in $F_d$ and from (ii), by taking $h(x) = |x - a|^{-\frac{d}{2}}$, for some $a \in \Omega$, we see that $F_d$ is a proper subspace of weak-$L^d(\Omega)$, for $d > 1$.

In the following lemma we give an equivalent condition for a function to
be in the space $F_d$, in terms of splitting of the function.

**Lemma 3.0.15.** Let $f \in L(d, \infty)$. Then $f \in F_d$ if and only if for each $\varepsilon > 0$, there exists $f_\varepsilon \in L^\infty(\Omega)$ such that $|\text{supp} \, (f_\varepsilon)| < \infty$ and $\|f - f_\varepsilon\|_{(d, \infty)} < \varepsilon$.

**Proof.** Let $f \in F_d$ and let $\varepsilon > 0$ be given. Then by the definition of the space $F_d$, we get $f_\varepsilon \in C_0^\infty(\Omega)$ such that $\|f - f_\varepsilon\|_{(d, \infty)} < \varepsilon$. Clearly $f_\varepsilon \in L^\infty(\Omega)$ and $|\text{supp} \, (f_\varepsilon)| < \infty$. Conversely, for a given $\varepsilon > 0$ let $f_\varepsilon \in L^\infty(\Omega)$ such that $\|f - f_\varepsilon\|_{(d, \infty)} < \frac{\varepsilon}{2}$ and $|\text{supp} (f_\varepsilon)| < \infty$. Thus $f_\varepsilon \in L^2(\Omega)$ and hence we can find $\varphi_\varepsilon \in C_0^\infty(\Omega)$ such that

$$\|f_\varepsilon - \varphi_\varepsilon\|_{(d, \infty)} \leq \|\varphi_\varepsilon - f_\varepsilon\|_d < \frac{\varepsilon}{2}.$$  

Now

$$\|f - \varphi_\varepsilon\|_{(d, \infty)} \leq \|f - f_\varepsilon\|_{(d, \infty)} + \|f_\varepsilon - \varphi_\varepsilon\|_{(d, \infty)} < \varepsilon.$$  

Hence $f \in F_d$ by the definition of $F_d$. \qed

The next lemma gives a necessary condition for a function to be in $F_d$ in terms of splitting of the domain $\Omega$.

**Lemma 3.0.16.** If $f \in F_d$, then for every $\varepsilon > 0$, there exists bounded set $\Omega_\varepsilon \subset \Omega$ such that $\|f_{\chi_{\Omega_\varepsilon}}\|_{(d, \infty)} < \varepsilon$.

**Proof.** For a given $\varepsilon > 0$, we choose $f_\varepsilon \in C_0^\infty(\Omega)$ such that $\|f - f_\varepsilon\|_{(d, \infty)} < \varepsilon$. Now the set $\Omega_\varepsilon = \{x : f_\varepsilon(x) \neq 0\}$ is bounded. Further, using (iii) of Proposition 2.1.4 we obtain

$$\|f_{\chi_{\Omega_\varepsilon}}\|_{(d, \infty)} = \|(f - f_\varepsilon)_{\chi_{\Omega_\varepsilon}}\|_{(d, \infty)} \leq \|f - f_\varepsilon\|_{(d, \infty)} < \varepsilon.$$  

\qed

The above characterization of the space $F_d$ is very useful for determining whether a function belongs $F_d$ or not. In most of the cases, we take $\Omega_\varepsilon$ to be certain level sets of the function, as it is easy to check whether the level sets is of finite measure or not.

Next we list a few useful characterization of the space $F_d$ in terms of the decreasing rearrangement and the distribution function.

**Proposition 3.0.17.** Let $f \in M_0(\Omega)$. Then the following statements are equivalent:

(i) $f \in F_d$. 


(ii) \( f^*(t) = o(t^{1 \over 2}) \) at 0 and \( \infty \). i.e,
\[
\lim_{t \to 0^+} t^{1 \over 2} f^*(t) = 0 = \lim_{t \to \infty} t^{1 \over 2} f^*(t).
\] (3.4)

(iii) \( \alpha_f(s) = o(s^{-d}) \) at 0 and \( \infty \). i.e,
\[
\lim_{s \to 0^+} s^d \alpha_f(s) = 0 = \lim_{s \to \infty} s^d \alpha_f(s).
\] (3.5)

Proof. (i)\( \Rightarrow \) (ii)
Let \( f \in \mathcal{F}_d \). For a given \( \varepsilon > 0 \) write \( f = f_\varepsilon + (f - f_\varepsilon) \) where \( f_\varepsilon \in C_0^\infty(\Omega) \) is such that \( \|f - f_\varepsilon\|_{(d,\infty)} < \varepsilon \). From (d) of Proposition 2.1.5, we have
\[
(f_1 + f_2)^*(t_1 + t_2) \leq f_1^*(t_1) + f_2^*(t_2),
\]
for any \( t_1, t_2 > 0 \). Thus we deduce that
\[
f^*(2t) \leq (f - f_\varepsilon)^*(t) + f_\varepsilon^*(t) .
\]
Hence
\[
(2t)^{1 \over 2} f^*(2t) \leq 2^{1 \over 2} \left\{ t^{1 \over 2} (f - f_\varepsilon)^*(t) + t^{1 \over 2} f_\varepsilon^*(t) \right\} \leq 2^{1 \over 2} \left\{ \varepsilon + t^{1 \over 2} f_\varepsilon^*(t) \right\} .
\]
Since \( f_\varepsilon \in C_0^\infty(\Omega) \), the function \( f_\varepsilon^* \) has also compact support and therefore satisfies (3.4). Thus \((2t)^{1 \over 2} f^*(2t)\) can be made arbitrarily small for large and small values of \( t \), showing that (3.4) holds for any \( f \in \mathcal{F}_d \).

(ii)\( \Rightarrow \) (iii)
Let (ii) hold. Thus for given \( \varepsilon > 0 \), there exist \( t_1, t_2 > 0 \) such that
\[
t^{1 \over 2} f^*(t) < \varepsilon, \quad \forall t \in (0, t_1) \cup (t_2, \infty).
\] (3.6)
Let \( s_1 = \varepsilon (t_1)^{1 \over 2} \) and \( s_2 = \varepsilon (t_2)^{1 \over 2} \). Note that,
\[
\text{If } s \in (0, s_2) \cup (s_1, \infty), \text{ then } t = \left( \frac{s}{\varepsilon} \right)^{d} \in (0, t_1) \cup (t_2, \infty).
\]
Now using (3.6) and (2.3) with \( c = \varepsilon \), we get
\[
s(\alpha_f(s))^{1 \over 2} < \varepsilon, \quad \forall s \in (0, s_2) \cup (s_1, \infty).
\]
This shows that \( \alpha_f(s) = o(s^{-d}) \) at 0 and \( \infty \).

(iii)\( \Rightarrow \) (i)
Assume that (iii) holds. Thus for given \( \varepsilon > 0 \), there exist \( s_1, s_2 \) such that
\[
s(\alpha_f(s))^{1 \over 2} < \varepsilon, \quad \forall s \in (0, s_1] \cup [s_2, \infty).
\] (3.7)
Let
\[ A_\varepsilon := \{ x : s_1 < |f(x)| < s_2 \}, \quad f_\varepsilon := f \chi_{A_\varepsilon}, \quad g = f \chi_{A_\varepsilon} \]
Note that \(|A_\varepsilon| \leq \alpha_f(s_1) < \infty\) and \(f_\varepsilon \in L^\infty(\Omega)\). Thus by Proposition 3.0.15 it is enough to prove that
\[ \|f - f_\varepsilon\|_{(d,\infty)} = \|g\|_{(d,\infty)} < \varepsilon. \]
Observe that, for \(s \in (s_1, s_2)\), \(\alpha_g(s) = \alpha_f(s_2)\) and hence
\[ s(\alpha_g(s))^{\frac{1}{d}} < s_2(\alpha_f(s_2))^{\frac{1}{d}} < \varepsilon, \quad \forall \, s \in (s_1, s_2). \] (3.8)
Since \(|g| \leq |f|\), we have \(\alpha_g(s) \leq \alpha_f(s), \quad \forall \, s > 0\). Now by combining (3.7) and (3.8) we get
\[ s(\alpha_g(s))^{\frac{1}{d}} < \varepsilon, \quad \forall \, s > 0. \]
Hence by Lemma 2.2.2 we get \(\|g\|_{(d,\infty)} < \varepsilon. \)

Recall the following definition:

**Definition 3.0.18.** We say that, a norm \(\|\cdot\|_X\) on a function space \(X\) is absolutely continuous with respect to the measure \(\mu\), if for each \(g \in X\), \(\|g\chi_E\| \to 0\) as \(\mu(E) \to 0\).

Now using Proposition 3.0.17, we have the following corollary:

**Corollary 3.0.19.** The norm in \(F_d\) is absolutely continuous with respect to the Lebesgue measure in \(\mathbb{R}^N\).

**Proof.** Let \(g \in F_d\). Now for \(E \subset \Omega\), note that \((g\chi_E)^*(t) = 0\) for \(t > |E|\) and \(g^*(t) \geq (g\chi_E)^*(t)\). Therefore
\[ \|g\chi_E\|_{(d,\infty)} = \sup_{t > 0} \left\{ t^{\frac{1}{d}} (g\chi_E)^*(t) \right\} \leq \sup_{0<t\leq|E|} \left\{ t^{\frac{1}{d}} g^*(t) \right\}, \]
Since \(g \in F_d\), by the above characterization \(t^{\frac{1}{d}} g^*(t) \to 0\) as \(t \to 0\). Thus it is immediate that \(\|g\chi_E\|_{(d,\infty)} \to 0\) as \(|E| \to 0\).

**Remark 3.0.20.** In contrast to the above result, observe that the norm in weak-\(L^d(\Omega)\), for \(d > 1\), is not absolutely continuous with respect to the
Lebesgue measure in $\mathbb{R}^N$. Indeed, for the Hardy potential $h(x) = |x - a|^{-\frac{N}{d}}$ (take $f \equiv 0$, in the proof of (ii) of Proposition 3.0.13), we have

$$\|h \chi_{B(a,r)}\|_{(\frac{N}{d}, \infty)} \geq \omega_N \frac{d}{r^d}, \quad \forall r > 0.$$ 

The following sufficient condition for a function to be in $\mathcal{F}_d$ is similar to a condition of Rozenblum [68] (see (2.19) of [68]).

**Lemma 3.0.21.** Let $h \in L(d, \infty)$ and $h > 0$. If $f$ is such that

$$\int_{\Omega} h^{d-q} |f|^q < \infty,$$

then $f \in L(d, q)$ and hence in $\mathcal{F}_d$.

**Proof.** The result is obvious when $q = d$. For $q > d$, let $g = h^{d-1} f$. Then the above integrability condition yields $g \in L^q(\Omega)$. Thus by property (2.10), we get $h^{1 - \frac{d}{q}} \in L(\frac{dq}{d-q}, \infty)$. Now by Hölder inequality (2.9) we get $f \in L(d, q)$ and hence in $\mathcal{F}_d$ as $L(d, q) \subset \mathcal{F}_d$. $\Box$

**Remark 3.0.22.** Let $g \in L^q(\mathbb{R}^N)$ with $q \geq d$ and let

$$f(x) = |x|^{\left(\frac{1}{q} - \frac{1}{d}\right)N} g(x).$$

Then using the above lemma one can easily verify that $f \in L(d, q)$. In general, if $h \in L(d, \infty)$ with $h > 0$, then $f = g h^{1 - \frac{d}{q}} \in L(d, q)$. Thus one can obtain Lorentz spaces by interpolating Lebesgue and weak-Lebesgue spaces suitably.

Next we show that the functions having faster decay than the Hardy potential $\frac{1}{|x-a|^d}$ at infinity and at all points in $a \in \Omega$ are in $\mathcal{F}_N$. More precisely, we consider a measurable function, say $g$, on $\Omega \subset \mathbb{R}^N$ with $d < N$, satisfying the following conditions:

$$(i) \lim_{|x| \to \infty, x \in \Omega} |x|^d g(x) = 0, \quad (ii) \lim_{x \to a, x \in \Omega} |x-a|^d g(x) = 0, \quad \forall a \in \overline{\Omega}.$$  \hspace{1cm} (3.9)

First we prove the following preparatory lemma.

**Lemma 3.0.23.** Let $g$ be a measurable function satisfying condition (3.9). Then there exist $a_1, \ldots, a_m \in \overline{\Omega}$ with the property: for every $\varepsilon > 0$ there
exists $R := R(\varepsilon) > 0$ such that
\[
|g(x)| < \frac{\varepsilon}{|x|^d} \quad \text{a.e. } x \in \Omega \setminus B(0, R),
\]
\[
|g(x)| < \frac{\varepsilon}{|x - a_i|^d} \quad \text{a.e. } x \in \Omega \cap B(a_i, R^{-1}), \quad i = 1, \ldots, m,
\]
\[
g \in L^\infty(\Omega \setminus A_\varepsilon),
\]
where $A_\varepsilon := \bigcup_{i=1}^m B(a_i, R^{-1}) \cap \Omega$.

Proof. Using condition (i) of (3.9) we can find $r > 0$ such that
\[
|g(x)| < \frac{1}{|x|^d} \quad \text{a.e. } x \in \Omega \setminus B(0, r).
\]
Now for each $a \in \overline{\Omega \cap B(0, r)}$, by condition (ii), there exists $r_a > 0$ such that
\[
|g(x)| < \frac{1}{|x - a|^d} \quad \text{a.e. in } \Omega \cap B(a, r_a).
\]
Since $\Omega \cap B(0, r)$ is compact, there exist $a_1, \ldots, a_m \in \overline{\Omega \cap B(0, r)}$ such that $\Omega \cap B(0, r) \subset \bigcup_{i=1}^m B(a_i, r_a)$.

Using condition (i) again, for a given $\varepsilon > 0$, we choose $R = R(\varepsilon) \geq r, r_1, \ldots, r_m$ so that
\[
|g(x)| < \frac{\varepsilon}{|x|^d} \quad \text{a.e. in } \Omega \setminus B(0, R).
\]
Applying condition (ii) in each ball $B(a_i, r_a)$ ($i = 1, \ldots, m$), by choosing $R$ larger if necessary, we can satisfy
\[
|g(x)| < \frac{\varepsilon}{|x - a_i|^d}, \quad \text{a.e. in } \Omega \cap B(a_i, R^{-1}) \quad \text{and } g \in L^\infty(\Omega \setminus A_\varepsilon)
\]

\[\square\]

Theorem 3.0.24. Let $\Omega \subset \mathbb{R}^N$ and $d < N$. Let $g : \Omega \to \mathbb{R}$ be as in the previous lemma. Then $g \in \mathcal{F}_{\mathcal{F}_d}$.

Proof. We use condition (iii) of Proposition 3.0.17 to show that $g \in \mathcal{F}_{\mathcal{F}_d}$.
First we compute the distribution function of $g$. For $\varepsilon > 0$, let $R$ be given by the previous lemma. Let $s_1 := \varepsilon R^{-d}$. We will show that
\[
s(\alpha_g(s))^{\frac{1}{d}} < \varepsilon, \quad \forall s < s_1.
\]
Using (3.10), for each $s \in (0, s_1)$ we have,

$$B(0, R) \subset B(0, \left(\frac{\varepsilon}{s}\right)^{1/d}) \quad \text{and} \quad |g(x)| < s, \quad \forall x \in \Omega \setminus B(0, \left(\frac{\varepsilon}{s}\right)^{\frac{3}{d}}). \quad (3.13)$$

Therefore, for each $s \in (0, s_1)$, the distribution function $\alpha_g(s)$ can be estimated as follows:

$$\alpha_g(s) = \left| \{ x \in \Omega \cap B(0, \left(\frac{\varepsilon}{s}\right)^{\frac{1}{d}}) : |f(x)| > s \} \right| \leq \omega_N\left(\frac{\varepsilon}{s}\right)^{\frac{N}{d}},$$

where $\omega_N$ is the volume of unit ball in $\mathbb{R}^N$. Thus

$$s(\alpha_g(s))^{\frac{d}{N}} < C_1 \varepsilon, \quad \forall s < s_1. \quad (3.14)$$

where the constant $C_1$ is independent of $\varepsilon$. Next consider the set $A_\varepsilon = \bigcup_{i=1}^{m} B(a_i, R^{-1}) \cap \Omega$ and let $s_2 := \|g\|_{L^\infty(\Omega; A_\varepsilon)}$. For $s > s_2$, using (3.11) the distribution function can be estimated as follows:

$$\alpha_g(s) = \left| \{ x \in \Omega : |g(x)| > s \} \right| = \left| \{ x \in A_\varepsilon : |g(x)| > s \} \right|$$

$$\leq \sum_{i=1}^{m} \left| \{ x \in B(a_i, R^{-1}) \cap \Omega : |g(x)| > s \} \right|$$

$$\leq \sum_{i=1}^{m} \left| \{ x \in B(a_i, R^{-1}) : \varepsilon|x-a_i|^{-d} > s \} \right|$$

$$= \sum_{i=1}^{m} \omega_N\left(\frac{\varepsilon}{s}\right)^{\frac{N}{d}}.$$

Therefore

$$s(\alpha_g(s))^{\frac{d}{N}} \leq C_2 \varepsilon \quad \forall s > s_2, \quad (3.15)$$

where $C_2$ is independent of $\varepsilon$. Now proof follows using condition (iii) of proposition 3.0.17 together with (3.14) and (3.15). □