Chapter 5

Cohomology of $S_\lambda(L)$

In this Chapter we shall compute the cohomology groups of the complex manifold $S_\lambda(L)$ with values in the structure sheaf. Using this we shall compute the Picard group and the algebraic dimension of $S_\lambda(L)$.

Let $L_i \rightarrow X_i, i = 1, 2$ be holomorphic principal $\mathbb{C}^*$-bundle over complex projective manifolds $X_i$, with $\dim X_i \geq 1$. We assume that the principal $\mathbb{C}^*$-bundle $L_i \rightarrow X_i$ is associated to a negative ample line bundle $\bar{L}_i \rightarrow X_i$. Further we assume that $X_i$ is arithmetically Cohen-Macaulay for the projective embedding determined by the ample line bundle $\bar{L}_i^*$. This means that the cone $\tilde{L}_i$ over $X_i$ is a Cohen-Macaulay affine analytic space.

We apply the Künneth formula established by A. Cassa [7, Teorema 3] to obtain the following lemma. We refer to Theorem 2.2.8 for the Künneth formula. Let $L = L_1 \times L_2$. Then $\mathcal{O}_{L_1 \times L_2} = p_{r_1}^* \mathcal{O}_{L_1} \hat{\otimes} p_{r_2}^* \mathcal{O}_{L_2}$, where $p_{r_i}^*$ denotes the projection $L_1 \times L_2 \rightarrow L_i, i = 1, 2$. Here we denote the structure sheaf of an analytic variety $Y$ by $\mathcal{O}_Y$.

Lemma 5.0.4. Let $\bar{L}_i \rightarrow X_i$ be negative ample holomorphic line bundle over a smooth projective variety $X_i, i = 1, 2$. Assume that $X_i$ is arithmetically Cohen-Macaulay for the projective embedding determined by the ample line bundle $\bar{L}_i^*$. Then, $H^i(L; \mathcal{O}_L) = 0$ for $i \neq 0, \dim X_1, \dim X_2, \dim X_1 + \dim X_2$. Also, $H^0(L; \mathcal{O}_L) \cong H^0(L_1; \mathcal{O}_{L_1}) \hat{\otimes} H^0(L_2; \mathcal{O}_{L_2})$.

Proof. By Corollary 2.1.8 we have $H^i(L_i; \mathcal{O}_{L_i}) = 0$ for $i \neq 0, \dim X_i$. Rest of the proof now follow readily by the Künneth formula 2.2.8.

Remark 5.0.5. We remark that the vanishing of the cohomology groups $H^q(L; \mathcal{O}_L)$ for $0 < q < \min\{\dim X_1, \dim X_2\}$ in the Lemma 5.0.4 (ii) follows from [2, Ch. I, Theorem 3.6]. To see this, set $\tilde{L} := \tilde{L}_1 \times \tilde{L}_2 \setminus A$ where $A$ is the closed analytic space $A = \tilde{L}_1 \times \{a_2\} \cup \{a_1\} \times \tilde{L}_2$. The ideal
Let $\mathcal{I} \subset \mathcal{O}_L$ of $A$ equals $\mathcal{I}_1 \mathcal{I}_2$ where $\mathcal{I}_1, \mathcal{I}_2$ are the ideals of the components $A_1 := \hat{L}_1 \times \{a_2\}, A_2 := \{a_1\} \times \hat{L}_2$ of $A$. Then $\text{depth}_A \mathcal{O}_L = \text{depth}_2 \mathcal{O}_L = \min_j \{\text{depth}_{\mathcal{I}_j} \mathcal{O}_L\} = \min\{\dim X_1 + 1, \dim X_2 + 1\}$. Thus we see that $\text{depth}_A \mathcal{O}_L = \min\{\dim X_1 + 1, \dim X_2 + 1\}$. Therefore $H^q(\hat{L}_1 \times \hat{L}_2; \mathcal{O}_{\hat{L}_1 \times \hat{L}_2}) \cong H^q(L; \mathcal{O}_L)$ if $q < \min\{\dim X_1, \dim X_2\}$ by [2, Ch. I, Theorem 3.6] where the isomorphism is induced by the inclusion. Since $\hat{L}_1 \times \hat{L}_2$ is Stein, the cohomology groups $H^q(L; \mathcal{O}_L)$ vanish for $1 \leq q < \min\{\dim X_1, \dim X_2\}$.

Note that the hypothesis of the above lemma are satisfied in the case when $X_i, i = 1, 2$ are flag manifolds $G_i/P_i$ where $G_i$ is a semi-simple complex Lie group over $\mathbb{C}$, $P_i$ is a parabolic subgroup and $\hat{L}_i$ any negative ample line bundle, over $G_i/P_i$. This follows because the flag manifold $X_i$ are arithmetically Cohen-Macaulay for the projective embedding determined by any ample line bundles. For this fact refer to the Section 2.1.2. If we assume that $L$ itself is very ample, then it is not possible to blow-down $X$. However, in this case, the following lemma allows one to compute the cohomology groups of $L$.

**Lemma 5.0.6.** Let $E$ be any holomorphic principal $\mathbb{C}^*$-bundle over a complex manifold $X$. Let $E^*$ be the dual to $E$. Then $E \cong E^*$ as complex manifolds. In particular, $H^p_\partial(E) \cong H^p_\partial(E^*)$.

**Proof.** Let $\psi : E \longrightarrow E^*$ be the map $v \mapsto v^*$ where $v^*(\lambda v) = \lambda \in \mathbb{C}$. Then $\psi$ is a biholomorphism. \qed

Suppose that $\alpha_\lambda$ is an admissible $\mathbb{C}$-action on $L \longrightarrow X$ of scalar type, or diagonal type, or linear type. It is understood that in the case of diagonal type, there is a standard $T_i$-action on $X_i, i = 1, 2$, and that $X_i = G_i/P_i$ and $L_i$ negative ample in the case of linear type action. Denote by $\nu_\lambda$ (or more briefly $\nu$) the holomorphic vector field on $L$ associated to the $\mathbb{C}$-action. Thus the $\mathbb{C}$-action is just the flow associated to $\nu$. We shall denote by $\mathcal{O}_v^{\nu}$ the sheaf of germs of local holomorphic functions which are constant along the $\mathbb{C}$-orbits. Thus $\mathcal{O}_v^{\nu}$ is isomorphic to $\pi_\lambda^*(\mathcal{O}_{S_\lambda(L)})$. One has an exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_v^{\nu} \rightarrow \mathcal{O}_L \xrightarrow{\nu} \mathcal{O}_L \rightarrow 0. \quad (8)$$

Since the fibre of $\pi_\lambda : L \longrightarrow S_\lambda(L)$ is Stein, we see that $H^q(L; \mathcal{O}_v^{\nu}) \cong H^q(S_\lambda(L); \mathcal{O}_{S_\lambda(L)})$ for all $q$. Thus, the exact sequence (8) leads to the following long exact sequence:

$$0 \rightarrow H^0(S_\lambda(L); \mathcal{O}_{S_\lambda(L)}) \rightarrow H^0(L; \mathcal{O}_L) \rightarrow H^0(L; \mathcal{O}_L) \rightarrow H^1(S_\lambda(L); \mathcal{O}_{S_\lambda(L)}) \rightarrow \cdots$$
Proof. The only assertion which remains to be explained is that the constant function 1 is not in the image of \(v\).

\[\cdots \rightarrow H^{q-1}(L; \mathcal{O}_L) \rightarrow H^{q}(S\lambda(L); \mathcal{O}_{S\lambda(L)}) \rightarrow H^{q}(L; \mathcal{O}_L) \rightarrow H^{q}(L; \mathcal{O}_L) \rightarrow H^{q+1}(L; \mathcal{O}_L) \rightarrow H^{q+1}(L; \mathcal{O}_L) \rightarrow \cdots\]  

\( (9) \)

**Theorem 5.0.7.** Suppose that \(L = L_1 \times L_2\) where the \(L_i\) satisfy the hypotheses of Lemma 5.0.4. Suppose that \(1 \leq \dim X_1 \leq \dim X_2\). Then \(H^q(S\lambda(L); \mathcal{O}) = 0\) provided \(q \notin \{0, 1, \dim X_i, \dim X_i + 1, \dim X_1 + \dim X_2, \dim X_1 + \dim X_2 + 1; i = 1, 2\}\). Moreover one has \(\mathbb{C} \subset H^1(S\lambda(L); \mathcal{O})\), given by the constant functions in \(H^0(L; \mathcal{O})\).

**Proof.** The only assertion which remains to be explained is that the constant function 1 is not in the image of \(v_* : H^0(L; \mathcal{O}) \rightarrow H^0(L; \mathcal{O})\). All other assertions follow trivially from the long exact sequence \((9)\) and the Lemma 5.0.4.

Suppose that \(f : L \rightarrow \mathbb{C}\) is such that \(v(f) = 1\). This means that \(\frac{d}{dz}|_{z=0}(f \circ \mu_p)(z) = 1\) for all \(p \in L, z \in \mathbb{C}\), where \(\mu_p : \mathbb{C} \rightarrow L\) is the map \(z \mapsto \alpha_\lambda(z).p = z.p\). Since \(\mu_{w.p}(z) = z.(w.p) = (z + w).p = \mu_p(z + w)\), it follows that \(\frac{d}{dz}|_{z=w}(f \circ \mu_p) = 1\) for all \(w \in \mathbb{C}\). Hence \(f \circ \mu_p(z) = z + f(p)\). This means that the complex hypersurface \(Z(f) := f^{-1}(0) \subset L\) meets each fibre at exactly one point. It follows that the projection \(L \rightarrow S\lambda(L)\) restricts to a bijection \(Z(f) \rightarrow S\lambda(L)\).

In fact, since \(v(f) \neq 0\) we see that \(Z(f)\) is smooth and since \(v_p\) is tangent to the fibres of the projection \(L \rightarrow S\lambda(L)\) for all \(p \in Z(f)\), we see that the bijective morphism of complex analytic manifolds \(Z(f) \rightarrow S\lambda(L)\) is an immersion. It follows that \(Z(f) \rightarrow S\lambda(L)\) is a biholomorphism. Thus \(Z(f)\) is a compact complex analytic sub manifold of \(L \subset \hat{L}\). Since \(\hat{L}\) is Stein, this is a contradiction. \(\square\)

### 5.1 Picard Group

For a complex manifold \(Y\), the group of isomorphism class of line bundle on \(Y\) is isomorphic to the cohomology group \(H^1(Y, \mathcal{O}^*) =: \text{Pic}(Y)\). We denote the kernel of the natural map \(H^1(Y, \mathcal{O}) \rightarrow H^1(Y, \mathcal{O}^*)\) by \(\text{Pic}^0(Y)\). The vector space \(\text{Pic}^0(Y)\) is isomorphic to the class of line bundles with trivial Chern class. Our next result concerns the Picard group \(\text{Pic}(S\lambda(L))\).

**Proposition 5.1.1.** Let \(L_i \rightarrow X_i\) be as in the Theorem 5.0.4. Suppose that \(X_i\) is simply connected. Then \(\text{Pic}^0(S\lambda(L)) \cong \mathbb{C}^l\) for some \(l \geq 1\).

**Proof.** Since \(\hat{L}\) is negative ample, \(c_1(L_i) \in H^2(X_i; \mathbb{Z})\) is a non-torsion element. Clearly \(H^1(S\lambda(L); \mathbb{Z}) = 0\) by a straightforward argument involving the Serre spectral sequence associated to the principal \(\mathbb{S}^1 \times \mathbb{S}^1\)-bundle with projection \(S(L) \rightarrow X_1 \times X_2\). Using the exact sequence \(0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 1\)
we see that $\text{Pic}^0(S_\lambda(L)) \cong H^1(S_\lambda(L); O) \cong \mathbb{C}$. Now $l \geq 1$ by the Theorem 5.0.7.

The above proposition is applicable when $X_i = G_i/P_i$ and $\tilde{L}_i$ are negative ample. However, in this case we have the following stronger result.

**Theorem 5.1.2.** Let $X_i = G_i/P_i$ where $P_i$ is any parabolic subgroup and let $\tilde{L}_i \to X_i$ be a negative line bundle, $i = 1, 2$. We assume that, when $X_i = \mathbb{P}^1$, the bundle $\tilde{L}_i$ is a generator of $\text{Pic}(X_i)$. Then $\text{Pic}^0(S_\lambda(L)) \cong \mathbb{C}$.

If the $P_i$ are maximal parabolics and the $L_i$ are generators of $\text{Pic}(X_i) \cong \mathbb{Z}$, then $\text{Pic}(S_\lambda(L)) \cong \text{Pic}^0(S_\lambda(L)) \cong \mathbb{C}$.

**Proof.** It is easy to see that $H^1(S(L); \mathbb{Z}) = 0$ and that, when $P_i$ are maximal parabolics and $L_i$ generators of $\text{Pic}(X_i) \cong \mathbb{Z}$, $S(L)$ is 2-connected. If $\dim X_i > 1$ for $i = 1, 2$, then $H^1(L; O) = 0$ by Theorem 5.0.4 and so we need only show that $\text{ker}(H^1(L; O) \to H^0(L; O))$ is isomorphic to $\mathbb{C}$. In case $\dim X_i = 1$—equivalently $X_i = \mathbb{P}^1$—$\tilde{L}_i$ is the tautological bundle by our hypothesis. Thus $L_i = \mathbb{C}^2 \setminus \{0\}$. In this case we need to also show that $\text{ker}(H^1(L; O) \to H^0(L; O))$ is zero. Note that the theorem is known due to Loeb and Nicolau [19, Theorem 2] when both the $X_i$ are projective spaces and the $L_i$ are negative ample generators—in particular when both $X_i = \mathbb{P}^1$.

The validity of the theorem for the case when $\lambda$ is of diagonal type implies its validity in the linear case as well. This is because one has a family $\{L/C_{\lambda_0}\}$ of complex manifolds parametrized by $\epsilon \in \mathbb{C}$ defined by $\lambda_0 = \lambda + \lambda_{0,\epsilon}$, where $S_{\lambda_0}(L) = L/C_{\lambda_0} \cong L/C_{\lambda}$ if $\epsilon \neq 0$ and $\lambda_0 := \lambda$ is of diagonal type. (See §3.) The semi-continuity property ([15, Theorem 6, §4]) for $\dim H^1(S_{\lambda_0}(L); O)$ implies that $\dim H^1(S_{\lambda_0}(L); O) \leq \dim H^1(S_{\lambda}(L); O)$. But Theorem 5.0.7 says that $\dim H^1(S_{\lambda}(L); O) \geq 1$ and so equality must hold, if $H^1(S_{\lambda}(L); O) \cong \mathbb{C}$. Therefore we may (and do) assume that the complex structure is of diagonal type.

First we show that $\text{coker}(v_x : H^0(L; O) \to H^0(L; O))$ is 1-dimensional, generated by the constant functions. Consider the commuting diagram where $\tilde{v}$ is the holomorphic vector field defined by the action of $\mathbb{C}$ given by $\lambda(\varpi_1, \varpi_2)$ on $V(\varpi_1, \varpi_2)$. Note that $\tilde{v}_x = v_x$ if $x \in L$.

\[
\begin{array}{ccc}
H^0(V(\varpi_1, \varpi_2); O) & \stackrel{\tilde{v}}{\longrightarrow} & H^0(V(\varpi_1, \varpi_2); O) \\
\downarrow & & \downarrow \\
H^0(L; O) & \stackrel{v}{\longrightarrow} & H^0(L; O)
\end{array}
\]

By Hartog’s theorem, $H^0(V(\varpi_1, \varpi_2); O) \cong H^0(V(\varpi_1) \times V(\varpi_2); O)$. Also, since $\tilde{L}_i$ is normal at its vertex [25], again by Hartog’s theorem, $H^0(L; O) \cong H^0(L_1 \times L_2; O)$. Since $L_i \subset V(\varpi_i)$ are closed sub varieties, it follows that the
both vertical arrows, which are induced by the inclusion of $L$ in $V(\varpi_1, \varpi_2)$, are surjective. From what has been shown in the proof of Theorem 5.0.7, we know that the constant functions are not in the cokernel of $\tilde{v}_*$. So it suffices to show that $\text{coker}(\tilde{v}_*)$ is 1-dimensional. This was established in the course of proof of Theorem 2 of [19]. For the sake of completeness we sketch the proof. We identify $V(\varpi_i)$ with $C^r$, where $r_i := \dim V(\varpi_i)$, by choosing a basis for $V(\varpi_i)$ consisting of weight vectors. Let $r = r_1 + r_2$ so that $C^r \cong V(\varpi_1) \times V(\varpi_2)$. The problem is reduced to the following: Given a holomorphic function $f : C^r \to \mathbb{C}$ with $f(0) = 0$, solve for a holomorphic function $\phi$ satisfying the equation

$$\sum_i b_i z_j \frac{\partial \phi}{\partial z_i} = f,$$

where we may (and do) assume that $\phi(0) = 0$. In view of the Observation made preceding the statement of Theorem 4.2.1, we need only to consider the case where $(b_i) = (\lambda_{\mu}, \lambda_{\nu})_{\mu \in I_{\varpi(1)}}, \nu \in I_{\varpi(2)} \in C^r$ satisfies the weak hyperbolicity condition of type $(r_1, r_2)$. Denote by $z^m$ the monomial $z_1^{m_1} \cdots z_n^{m_r}$ where $m = (m_1, \ldots, m_r)$ and by $|m|$ its degree $\sum_{1 \leq j \leq r} m_j$. Let $f(z) = \sum_{|m| > 0} a_m z^m \in H^0(C^r; \mathcal{O})$. Then $\phi(z) = \sum a_m / (b, m) z^m$ where $b, m = \sum b_j m_j$ is the unique solution of Equation (10). Note that weak hyperbolicity and the fact that $|m| > 0$ imply that $b, m \neq 0$, and, $b, m \to \infty$ as $m \to \infty$. Therefore $\phi$ is a convergent power series and so $\phi \in H^0(C^r; \mathcal{O})$.

It remains to show that, when $X_1 = \mathbb{P}^1$, $L_1 = C^2 \setminus \{0\}$, and $\dim X_2 > 1$, the homomorphism $v_* : H^1(L; \mathcal{O}) \to H^1(L; \mathcal{O})$ is injective. Let $z_j, 1 \leq j \leq r$, denote the coordinates of $C^2 \times V(\varpi_2)$ with respect to a basis consisting of $\tilde{T}$-weight vectors. Since $\dim X_2 > 1$, we have $H^1(L_2, \mathcal{O}) = 0$. Also $H^1(L_1; \mathcal{O}) = H^1(C^2 \setminus \{0\}; \mathcal{O})$ is the space $\mathcal{A}$ of convergent power series $\sum a_{m_1, m_2} m_1 z_1^{m_1} z_2^{m_2}$ without constant terms.

By Lemma 5.0.4 and Küneth formula 2.2.8, $H^1(L; \mathcal{O}) = \mathcal{A} \otimes H^0(L_2; \mathcal{O}) \cong \mathcal{A} \otimes H^0(L_2; \mathcal{O})$. Let $\mathcal{I} \subset H^0(V(\varpi_2); \mathcal{O})$ denote the ideal of functions vanishing on $L_2$ so that $H^0(L_2; \mathcal{O}) = H^0(\varpi_2; \mathcal{O})/\mathcal{I}$. One has the commuting diagram

\[
\begin{array}{ccc}
0 & \to & \mathcal{A} \otimes \mathcal{I} \\
\downarrow \tilde{v}_* & & \downarrow \tilde{v}_* \\
0 & \to & \mathcal{A} \otimes H^0(V(\varpi_2); \mathcal{O})
\end{array}
\]

\[
\begin{array}{ccc}
0 & \to & \mathcal{A} \otimes \mathcal{I} \\
\downarrow \tilde{v}_* & & \downarrow \tilde{v}_* \\
0 & \to & \mathcal{A} \otimes H^0(V(\varpi_2); \mathcal{O}) \\
\downarrow \tilde{v}_* & & \downarrow \tilde{v}_* \\
0 & \to & \mathcal{A} \otimes H^0(L_2; \mathcal{O})
\end{array}
\]

where the rows are exact. Theorem 2 of [19] implies that

$$\tilde{v}_* : \mathcal{A} \otimes H^0(V(\varpi_2); \mathcal{O}) \to \mathcal{A} \otimes H^0(V(\varpi_2); \mathcal{O})$$

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is an isomorphism. As before, this is equivalent to showing that Equation (10) has a (unique) solution $\phi$ without constant term when $f = \sum_{m} c_{m}z^{m} \in A \hat{\otimes} H^{0}(V(\varphi_{2}); O)$, is any convergent power series in $z_{1}^{-1}, z_{2}^{-1}, z_{j}, 3 \leq z_{j} \leq r$, where the sum ranges over $m = (m_{1}, m_{2}, \ldots, m_{r}) \in \mathbb{Z}^{r}, m_{1}, m_{2} < 0, m_{j} \geq 0, \forall j \geq 3$. It is clear that $\phi(z) = \sum_{m} c_{m}/(b.m)z^{m}$ is the unique formal solution. Note that weak hyperbolicity condition implies that $b.m \neq 0$ and $b.m \to \infty$ as $\sum_{j \geq 1}|m_{j}| \to \infty$. So $\phi(z)$ is a well-defined convergent power series in the variables $z_{1}^{-1}, z_{2}^{-1}, z_{j}, j \geq 3$ and is divisible by $z_{1}^{-1}z_{2}^{-1}$. Hence $\phi \in A \hat{\otimes} H^{0}(V(\varphi_{2}); O)$ and so $\tilde{v}_{z} : A \hat{\otimes} H^{0}(V(\varphi_{2}); O) \to A \hat{\otimes} H^{0}(V(\varphi_{2}); O)$ is an isomorphism. The ideal $I$ is stable under the action of $T_{2}$, and so is generated as an ideal by (finitely many) polynomials in $z_{3}, \ldots, z_{n}$ which are $T_{2}$-weight vectors. In particular, the generators are certain homogeneous polynomials $h(z_{3}, \ldots, z_{n})$ such that $\tilde{v}_{z}(z_{1}^{m_{1}}z_{2}^{m_{2}}h) = b.mz_{1}^{m_{1}}z_{2}^{m_{2}}h \forall m_{1}, m_{2} \in \mathbb{Z}$ where $z^{m}$ is any monomial that occurs in $z_{1}^{m_{1}}z_{2}^{m_{2}}h$. It follows easily that $\tilde{v}_{z}$ maps $A \hat{\otimes} I$ isomorphically onto itself. A straightforward argument involving diagram chase now shows that $v_{z} : A \hat{\otimes} H^{0}(L_{2}; O) \to A \hat{\otimes} H^{0}(L_{2}; O)$ is an isomorphism. This completes the proof. 

\begin{rem}
In the case when $X_{1}$ is any projective space $\mathbb{P}^{r_{1}}$ and $L_{1}$ is the tautological bundle over $X_{1}$, then the map $v^{*} : H^{r_{1}}(L, O) \to H^{r_{1}}(L, O)$ is an isomorphism. This can be shown as similar to the case $r_{1} = 1$ which is done in the last part of the previous theorem. Using this, as in the Theorem 5.0.7, we can deduce that $H^{r_{1}}(S_{\lambda}(L), O) = 0, r_{1} > 1$. Moreover if $|r_{1} - r_{2}| > 2$ then we can further deduce that $H^{r_{1}+1}(S_{\lambda}(L), O) = 0$. This is slight improvement of the Theorem 5.0.7.
\end{rem}

Assume that $P_{i} \subset G_{i}$ are maximal parabolics and the $L_{i}$ are the negative ample generators of the $Pic(G_{i}/P_{i}) \cong \mathbb{Z}$. We have the following description of the principal $\mathbb{C}$-bundles over $S_{\lambda}(L)$. Let $z \neq 0$. Let $\{g_{i,j}\}$ be a 1-cocyle defining the principal $\mathbb{C}$-bundle $L \to S_{\lambda}(L)$. Then the $\mathbb{C}$-bundle $L_{z}$ representing the element $z [L] \in H^{1}(S_{\lambda}(L); O)$ is defined by the cocycle $\{g_{i,j}\}$ for any $z \in \mathbb{C}$. We denote the corresponding $\mathbb{C}$-bundle by $L_{z}$. Note that the total space and the projection are the same as that of $L$. The $\mathbb{C}$-action on $L_{z}$ is related to that on $L$ where $w.v \in L_{z}$ equals $(w/v).v = \alpha_{\lambda}(w/z)(v) \in L$ for $w \in \mathbb{C}, v \in L$. The vector field corresponding to the $\mathbb{C}$-action on $L_{z}$ is given by $(1/z)v_{\lambda}$. Of course, when $z = 0, L_{z}$ is just the product bundle.

We shall denote the line bundle (i.e. rank 1 vector bundle) corresponding to $L_{z}$ by $E_{z}$. Observe that if $z \neq 0$

$$E_{z} = L_{z} \times_{\mathbb{C}} \mathbb{C}, \text{ where } (w,v,t) \sim (v, \exp(2\pi \sqrt{-1}w)t), \ w, t \in \mathbb{C}, v \in L_{z}.$$ 

If $z \neq 0$, any cross-section $\sigma : S_{\lambda}(L) \to E_{z} = L_{z} \times_{\mathbb{C}} \mathbb{C}$ corresponds to a
holomorphic function $h_\sigma : L \rightarrow \mathbb{C}$ which satisfies the following:

$$h_\sigma(w,v) = \exp(-2\pi\sqrt{-1}w)h_\sigma(v)$$

(11)

for all $v \in L_z, w \in \mathbb{C}$. Equivalently, this means that

$$h_\sigma(\alpha_\lambda(w)v) = \exp(-2\pi\sqrt{-1}wz)h_\sigma(v)$$

for $w \in \mathbb{C}$ and $v \in L$.

This implies that

$$v_\lambda(h_\sigma) = -2\pi\sqrt{-1}zh_\sigma.$$  

(12)

Conversely, if $h$ satisfies (11), then it determines a unique cross-section of $E_z$ over $S_\lambda(L)$.

### 5.2 Algebraic Dimension

For a complex manifolds $Y$, we shall denote the field of meromorphic functions on $Y$ by $\mathcal{M}(Y)$. The algebraic dimension of a complex manifold $Y$ is the transcendence degree $\text{tr.deg}_C(\mathcal{M}(Y))$. We have the following result concerning the field of meromorphic functions on $S_\lambda(L)$ with diagonal type complex structure. The proof will be given after some preliminary observations. In the end of the section we shall give explicitly the algebraic dimension of $S_\lambda(L)$.

**Theorem 5.2.1.** Let $L_i$ be the negative ample generator of $\text{Pic}(G_i/P_i) \cong \mathbb{Z}$ where $P_i$ is a maximal parabolic subgroup of $G_i$, $i=1,2$. Assume that $S_\lambda(L)$ is of diagonal type. Then the field $\kappa(S_\lambda(L))$ of meromorphic functions of $S_\lambda(L)$ is purely transcendental over $\mathbb{C}$. The transcendence degree of $\kappa(S_\lambda(L))$ is less than $\dim S_\lambda(L)$.

Let $U_i$ denote the opposite big cell, namely the $B_i^-$-orbit of $X_i = G_i/P_i$ the identity coset where $B_i^-$ is the Borel subgroup of $G_i$ opposed to $B_i$. One knows that $U_i$ is a Zariski dense open subset of $X_i$ and is isomorphic to $\mathbb{C}^{r_i}$ where $r_i$ is the number of positive roots in the unipotent part $P_{i,u}$ of $P_i$. The bundle $\pi_i : L_i \rightarrow X_i$ is trivial over $U_i$ and so $\tilde{U}_i := \pi_i^{-1}(U_i)$ is isomorphic to $\mathbb{C}^{r_i} \times \mathbb{C}^\ast$. We shall now describe a specific isomorphism which will be used in the proof of the above theorem.

Consider the projective imbedding $X_i \subset \mathbb{P}(V(\varpi_i))$. Let $v_0 \in V(\varpi_i)$ be a highest weight vector so that $P_i$ stabilizes $\mathbb{C}v_0$; equivalently, $\pi_i(v_0)$ is the identity coset in $X_i$. Let $Q_i \subset P_i$ be the isotropy at $v_0 \in V(\varpi_i)$ for the $G_i$ so that $G_i/Q_i = L_i$. The Levi part of $P_i$ is equal to centralizer of a one-dimensional torus $Z$ contained in $T$ and projects onto $P_i/Q_i \cong \mathbb{C}^\ast$, the structure group of $L_i \rightarrow X_i$. 

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Let $F_i \in H^0(X_i; \mathcal{L}_i^*) = V(\varpi_i)^*$ be the lowest weight vector such that $F_i(v_0) = 1$. Then $U_i \subset X_i$ is precisely the locus $F_i \neq 0$ and $F_i|_{U_i} : \mathbb{C}v \rightarrow \mathbb{C}$ is an isomorphism of vector spaces for $v \in U_i$. We denote also by $F_i$ the restriction of $F_i$ to $U_i$.

Let $Y_{\beta}$ be the Chevalley basis element of $\text{Lie}(G_i)$ of weight $-\beta, \beta \in R^+(G_i)$. We shall denote by $X_{\beta} \in \text{Lie}(G_i)$ the Chevalley basis element of weight $\beta \in R^+(G_i)$. Recall that $H_{\beta} := [X_{\beta}, Y_{\beta}] \in \text{Lie}(T)$ is non-zero whereas $[X_{\beta}, Y_{\beta}] = 0$ if $\beta \neq \beta'$.

Let $R_{P_i} \subset R^+(G_i)$ denote the set of positive roots of $G_i$ complementary to positive roots of Levi part of $P_i$ and fix an ordering on it. (Thus $\beta \in R_{P_i}$ if and only if $-\beta$ is not a root of $P_i$.) Let $r_i = |R_{P_i}| = \dim X_i$. Then $\text{Lie}(P_{i,u}^-) \cong \mathbb{C}^{r_i}$ where $P_{i,u}^-$ denotes the unipotent radical of the parabolic opposed to $P_i$. Observe that $P_i \cap P_{i,u}^- = \{1\}$. The exponential map defines an algebraic isomorphism $\theta : \mathbb{C}^{r_i} \cong \text{Lie}(P_{i,u}^-) \longrightarrow U_i$ where $\theta((y_{\beta})_{\beta \in R_{P_i}}) = (\prod_{\beta \in R_{P_i}} \exp(y_{\beta}Y_{\beta})).P_i \in G_i/P_i$. It is understood that, here and in the sequel, the product is carried out according to the ordering on $R_{P_i}$.

If $v \in \mathbb{C}v_0$, then $\theta$ factors through the map $\theta_v : \mathbb{C}^{r_i} \cong \text{Lie}(P_{i,u}^-) \longrightarrow \tilde{U}_i$ defined by $(y_{\beta})_{\beta \in R_{P_i}} \mapsto \prod \exp(y_{\beta}Y_{\beta}).v$. Moreover, $F_i$ is constant—equal to $F_i(v)$—on the image of $\theta_v$.

We define $\tilde{\theta} : \mathbb{C}^{r_i} \times \mathbb{C}^* \cong \text{Lie}(P_{i,u}^-) \times \mathbb{C}^* \cong P_{i,u}^- \times \mathbb{C}^* \longrightarrow \tilde{U}_i$ to be $\tilde{\theta}((y_{\beta}), z) = (\prod \exp(y_{\beta}Y_{\beta})).zv_0 = \theta(zv_0((y_{\beta})))$. This is an isomorphism. We obtain coordinate functions $z, y_{\beta}, \beta \in R_{P_i}$ by composing $\tilde{\theta}$ with projections $\mathbb{C}^{r_i} \times \mathbb{C}^* \longrightarrow \mathbb{C}$. Note that $F_i(\tilde{\theta}((y_{\beta}), z))) = z$. Thus the coordinate function $z$ is identified with $F_i$.

Since $F_i$ is the lowest weight vector (of weight $-\varpi_i$), $Y_{\beta}F_i = 0$ for all $\beta \in R^+(G_i)$. Define $F_{i,\beta} := X_{\beta}(F_i), \beta \in R_{P_i}$. Then $Y_{\beta}(F_{i,\beta}) = -[X_{\beta}, Y_{\beta}]F_i = -H_{\beta}(F_i) = \varpi_i(H_{\beta})F_i$ for all $\beta \in R_{P_i}$. Note that $\varpi_i(H_{\beta}) \neq 0$ as $H_{\beta} \in R_{P_i}$. If $\beta', \beta \in R_{P_i}$ are unequal, then $Y_{\beta'}F_{i,\beta} = 0$. It follows that $Y_{\beta'}^m(F_{i,\beta}) = 0$ unless $\beta' = \beta$ and $m = 1$.

The following result is well-known to experts in standard monomial theory. (See [18].)

**Lemma 5.2.2.** With the above notations, the map $\tilde{U}_i \longrightarrow \mathbb{C}^{r_i} \times \mathbb{C}^*$ defined as $v \mapsto ((F_{i,\beta}(v))_{\beta \in R_{P_i}}, F_i(v)), v \in \tilde{U}_i$, is an algebraic isomorphism for $i = 1, 2$.

**Proof.** It is easily verified that $\frac{\partial f}{\partial y_{\beta}}|_{v_0} = Y_{\beta}(f)(v_0)$ for any local holomorphic function defined in a neighborhood of $v_0$. (Cf. [18].)

Let $y = \tilde{\theta}((y_{\gamma}), z) = \prod_{\gamma \in R_{P_1}} \exp(y_{\gamma}Y_{\gamma}) \in P_{i,\gamma}^-$. Denote by $t_y : \tilde{U}_i \longrightarrow \tilde{U}_i$ the left multiplication by $y$. If $v = y.v_0 \in \tilde{U}_i$, then $(\partial/\partial y_{\beta}|_v)(f)$ equals
Hence \((\partial/\partial y_\beta)|_{v_0}(f\circ l_y)\). Taking \(f = F_{i,\beta}, \beta \in R_{P_i}\) a straightforward computation using the observation made preceding the lemma, we see that \((\partial/\partial y_\beta|_v)(F_{i,\gamma}) = Y_\beta|_{v_0}(F_{i,\gamma} \circ l_y) = F_i(v)\varpi_i(H_\beta)\delta_{\beta,\gamma}\) (Kronecker \(\delta\)). We also have
\[
(\partial/\partial y_\beta|_v)(F_i) = 0 \text{ for all } v \in \tilde{U}_i.
\]
Hence \((\partial/\partial y_\beta)|_{v}(F_{i,\gamma}/F_i) = \varpi_i(H_\beta)\delta_{\beta,\gamma}\). Thus the Jacobian matrix relating the \(F_{i,\beta}/F_i\) and the \(y_\beta, \beta \in R_{P_i}\), is a diagonal matrix of constant functions. The diagonal entries are non-zero as \(\varpi_i(H_\beta) \neq 0\) for \(\beta \in R_{P_i}\) and since \(F_i\) is nowhere vanishing, the lemma follows. \(\square\)

We shall use the coordinate functions \(F_i, F_{i,\beta}, \beta \in R_{P_i}\), to write Taylor expansion for analytic functions on \(\tilde{U}_i\). In particular, the coordinate ring of the affine variety \(\tilde{U}_i\) is just the algebra \(\mathbb{C}[F_{i,\beta}, \beta \in R_{P_i}][F_i, F_i^{-1}]\). The projective normality \([25]\) of \(G_i/P_i\) implies that \(\mathbb{C}[\tilde{L}_i] = \bigoplus_{r \geq 0} H^0(X_i; L_i^{-r}) = \bigoplus_{r \geq 0} V(r\varpi_i)^*\). Since \(\tilde{U}_i\) is defined by the non-vanishing of \(F_i\), we see that \(\mathbb{C}[U_i] = \mathbb{C}[\tilde{L}_i][1/F_i]\).

**Example 5.2.3.** In the particular case of Grassmannian, we shall describe the lemma explicitly as follow.

Let \(X\) be the Grassmannian \(G_{4,2}\), the space of all vector subspace of dim 2 in \(\mathbb{C}^4\). Let \(M_{4,2}\) be the space of \(4 \times 2\) matrices of rank 2. Let \(\sigma : M_{4,2} \to G_{4,2}\) be the projection map, where \(\pi(A)\) is the subspace of \(\mathbb{C}^4\) generated by the column vectors of \(A\). The flag manifold \(X\) can be identified with the quotient \(M_{4,2}/\sim\), where \(\sim\) is the relation in which \(A \sim B\) if and only if there exit \(C \in GL(2,\mathbb{C})\) such that \(A = BC\). We have the Plücker embedding \(\theta : G_{4,2} \hookrightarrow \mathbb{P}(\wedge^2 \mathbb{C}^4)\); \(\theta(A) = v_1 \wedge v_2\) where \(v_1\) and \(v_2\) are column vectors of \(A\). Let \(I(4, 2) = \{I = (i_1, i_2) \mid 1 \leq i_1 < i_2 \leq 4\}\). We denote the homogeneous coordinates of points in \(\mathbb{P}(\wedge^2 \mathbb{C}^4)\) by \((p_I, I \in I(4, 2))\). Then for \(A \in G_{4,2}, p_I\) is 2-minor of \(A\) with row indices \(I = (i_1, i_2)\). Fix \(I_0 = (1, 2)\). We denote by the open subset \(U_{I_0} \subset G_{4,2}\) the locus \(p_{I_0} \neq 0\). Every element \(A \in U_{I_0}\) has unique representation of the form

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
x_1 & x_2 \\
x_3 & x_4
\end{bmatrix}; \quad x_i \in \mathbb{C} \text{ for } 1 \leq i \leq 4.
\]

This gives a structure of affine space to \(U_{I_0}\) with coordinate function \(x_1, x_2, x_3, x_4\). The functions \(p_I/p_{I_0}, I \in I(4, 2)\), gives a well defined functions on \(U_{I_0} \cong \mathbb{C}^4\). These functions can be expressed in terms of coordinate \(x_1, x_2, x_3, x_4\) as follows: \(p_{13}/p_{12} = x_2, p_{14}/p_{12} = x_4, p_{23}/p_{12} = -x_1, p_{24}/p_{12} = \ldots\)
−x_3, p_{34}/p_{12} = x_1x_4−x_2x_3. This give a new coordinate system on U_{i_0} with coordinate functions \( p_{13}/p_{12}, p_{14}/p_{12}, p_{23}/p_{12} \) and \( p_{24}/p_{12} \). Let \( \pi : E(\wedge^2\gamma_{4,2}) \rightarrow G_{4,2} \) be the holomorphic principal \( \mathbb{C}^* \)-bundle over \( G_{4,2} \), corresponding to the tautological bundle \( \wedge^2\gamma_{4,2} \). The Plücker embedding \( \tilde{\theta} : G_{4,2} \rightarrow \mathbb{P}(\wedge^2\mathbb{C}^4) \) can be “lifted” to an embedding \( E(\wedge^2\gamma_{4,2}) \hookrightarrow \wedge^2\mathbb{C}^4 \setminus \{0\} \). Let \( \tilde{U}_{i_0} := \pi^{-1}(U_{i_0}) \) be the open subset of \( E(\wedge^2\gamma_{4,2}) \). Then \( \tilde{U}_{i_0} \cong \mathbb{C}^* \times U_{i_0} \). The functions \( p_{12}, p_{13}/p_{12}, p_{14}/p_{12}, p_{23}/p_{12} \) and \( p_{24}/p_{12} \) gives a coordinate functions on \( \tilde{U}_{i_0} \). Since \( p_{12} \neq 0 \) on \( \tilde{U}_{i_0} \), the functions \( p_{12}, p_{13}, p_{14}, p_{23} \) and \( p_{24} \) gives a new coordinate system on \( \tilde{U}_{i_0} \).

Now let \( X = X_1 \times X_2 \) and \( \tilde{T} = \tilde{T}_1 \times \tilde{T}_2 \cong (\mathbb{C}^*)^N, N = n_1 + n_2 \), where the isomorphism is as chosen in §3. Let \( d_i > 0, i = 1, 2 \), be chosen as in Proposition 4.1.1 so that the \( \tilde{T}_i \)-action on \( L_i \rightarrow G_i/P_i = d_i \)-standard. Let \( \lambda = \lambda_s \in \text{Lie}(\tilde{T}) \). Suppose that \( \lambda \) satisfies the weak hyperbolicity condition of type \((n_1, n_2)\).

Recall from (6) and (7) that for any weight \( \mu_i \in \Lambda(\varpi_i) \), there exist elements \( \lambda_{\mu_1}, \lambda_{\mu_2} \in \mathbb{C} \) such that for any \( v = (v_1, v_2) \in V_{\mu_1}(\varpi_1) \times V_{\mu_2}(\varpi_2) \), the \( \alpha_\lambda \)-action of \( \mathbb{C} \) is given by \( \alpha_\lambda(z)v = (\exp(z\lambda_{\mu_1})v_1, \exp(z\lambda_{\mu_2})v_2) \). In fact \( \lambda_{\mu_i} = \sum_{m_{i-1}<j\leq m_{i-1}+n_i} d_{\mu,i} j \lambda_j \) where \( d_{\mu,i} \) are certain non-negative integers. It follows that, as observed in the discussion preceding the statement of Theorem 4.2.1, the complex numbers \( \lambda_{\mu_i} \in \mathbb{C}, \mu_i \in \Lambda(\varpi_i), i = 1, 2 \) satisfy weak hyperbolicity condition:

\[
0 \leq \arg(\lambda_{\mu_1}) < \arg(\lambda_{\mu_2}) < \pi, \quad \forall \mu_i \in \Lambda(\varpi_i), i = 1, 2.
\] (13)

We observe that if \( \mu = \mu_1 + \cdots + \mu_r = \nu_1 + \cdots + \nu_r \), where \( \mu_j, \nu_j \in \Lambda(\varpi_j) \), then \( \lambda_{\mu,r} := \sum \lambda_{\mu_i} = \sum \lambda_{\nu_j} \). (This is a straightforward verification using (5) and (6).) Therefore, if \( v \in V(\varpi_i)^{\otimes r} \) is any weight vector of weight \( \mu \), we get, for the diagonal action of \( \mathbb{C} \), \( z.v = \exp(\lambda_{\mu,r} z) v \).

Any finite dimensional \( \tilde{G}_i \)-representation space \( V \) is naturally \( \tilde{G}_1 \times \tilde{G}_2 \)-representation space and is a direct sum of its \( \tilde{T} \)-weight spaces \( V_{\mu} \). If \( V \) arises from a representation of \( G_i \) via \( \tilde{G}_i \rightarrow G_i \), then the \( \tilde{T} \)-weights of \( V \) are the same as \( T \)-weights.

**Definition 5.2.4.** Let \( Z_i(\lambda) \subset \mathbb{C}, i = 1, 2 \), be the abelian subgroup generated by \( \lambda_{\mu} \in \Lambda(\varpi_i) \) and let \( Z(\lambda) := Z_1(\lambda) + Z_2(\lambda) \subset \mathbb{C} \).

The \( \lambda \)-weight of an element \( 0 \neq f \in \text{Hom}(V_{\mu}(\varpi_i); \mathbb{C}) \) is defined to be \( \text{wt}_\lambda(f) := \lambda_{\mu} \). If \( h \in \text{Hom}(V(\varpi_i)^{\otimes r}; \mathbb{C}) \) is weight vector of weight \( -\mu \), (so that \( h \in \text{Hom}(V_{\mu}(\varpi_i)^{\otimes r}; \mathbb{C}) \)) we define the \( \lambda \)-weight of \( h \) to be \( \lambda_{\mu,r} \).

If \( f \in \text{Hom}(V_{\mu}(r \varpi_i); \mathbb{C}) \) is a weight vector (of weight \( -\mu \)), then it is the image of a unique weight vector \( \tilde{f} \in \text{Hom}(V(\varpi_i)^{\otimes r}; \mathbb{C}) \) under the surjection.
induced by the \( \tilde{G}_i \)-inclusion \( V(r_\varpi_i) \hookrightarrow V(\varpi_i)^{\otimes r} = V(r_\varpi_i) \oplus V' \) where \( \tilde{f}|V' = 0 \). We define the \( \lambda \)-weight of \( f \) to be \( \text{wt}_\lambda(f) := \text{wt}_\lambda(\tilde{f}) \).

If \( h_i \in V(r_\varpi_i)^* \subset \mathbb{C}[\hat{L}_i], i = 1, 2, \) are weight vectors, then \( h_1 h_2 \) is a weight vector of \( V(r_1 \varpi_1)^* \otimes V(r_2 \varpi_2)^* \subset \mathbb{C}[\hat{L}_1 \times \hat{L}_2] \) and we have \( \text{wt}_\lambda(h_1 h_2) = \text{wt}_\lambda(h_1) + \text{wt}_\lambda(h_2) \in \mathbb{Z}(\lambda) \). Note that \( \text{wt}_\lambda(f_1 \ldots f_k) = \sum_{1 \leq j \leq k} \text{wt}_\lambda(f_j) \in \mathbb{Z}(\lambda) \) where \( f_j \in \mathbb{C}[\hat{L}_1 \times \hat{L}_2] = \oplus_{r_1, r_2 \geq 0} V(r_1 \varpi_1)^* \otimes V(r_2 \varpi_2)^* \) are weight vectors. Also \( \text{wt}_\lambda(f) \in \mathbb{Z}(\lambda) \) is a non-negative linear combination of \( \lambda_j, 1 \leq j \leq N \) for any \( \tilde{T} \)-weight vector \( f \in \mathbb{C}[\hat{L}_1 \times \hat{L}_2] \).

If \( f \in V(\varpi_i)^* \), it defines a holomorphic function on \( V(\varpi_1) \times V(\varpi_2) \) and hence on \( L \), and denoted by the same symbol \( f \); explicitly \( f(u_1, u_2) = f(u_i), \forall (u_1, u_2) \in L \).

**Lemma 5.2.5.** We keep the above notations. Assume that \( \lambda = \lambda_s \in \text{Lie}(\tilde{T}) = \mathbb{C}^N \). Fix \( \mathbb{C} \)-bases \( B_i \) for \( V(\varpi_i)^* \), consisting of \( \tilde{T} \)-weight vectors. Let \( z_0 \in \mathbb{Z}(\lambda) \). There are only finitely many monomials \( f := f_1 \ldots f_k, f_j \in B_1 \cup B_2 \) having \( \lambda \)-weight \( z_0 \). Furthermore, \( v_\lambda(f) = \text{wt}_\lambda(f) f \).

**Proof.** The first statement is a consequence of weak hyperbolicity (see (13)). Indeed, since \( 0 \leq \arg(\lambda_\mu) < \pi \) for all \( \mu \in \Lambda(\varpi_i), i = 1, 2 \), given any complex number \( z_0 \), there are only finitely many non-negative integers \( c_j \) such that \( \sum c_j \lambda_\mu = z_0 \).

As for the second statement, we need only verify this for \( f \in B_i, i = 1, 2 \). Suppose that \( f \in B_1 \) and that \( f \) is of weight \( -\mu, \mu \in \Lambda(\varpi_1) \), say. Then, for any \( (u_1, u_2) \in L \), writing \( u_i = \sum_{\nu \in \Lambda(\varpi_i)} u_\nu \), using linearity and the fact that \( f(u_1, u_2') = f(u_1) \) we have

\[
v_\lambda(f)(u_1, u_2) = \lim_{w \to 0}(f(\alpha_\lambda(w)(u_1, u_2)) - f(u_1, u_2))/w \\
= \lim_{w \to 0}(f(\exp(\lambda_\mu w)u_\mu) - f(u_1))/w \\
= \lim_{w \to 0}(\exp(\lambda_\mu w) - 1)f(u_1)/w \\
= \lambda_\mu f(u_1) \\
= \lambda_\mu f(u_1, u_2).
\]

This completes the proof. \( \Box \)

We assume that \( F_i, F_{i, \beta}, \beta \in R_{P_i}, \) are in \( B_i, i = 1, 2 \).

Let \( \mathcal{M} \subset \mathbb{C}(\hat{U}_1 \times \hat{U}_2) \) denote the multiplicative group of all Laurent monomials in \( F_i, F_{i, \beta}, \beta \in R_{P_i}, i = 1, 2 \). One has a homomorphism \( \text{wt}_\lambda : \mathcal{M} \to \mathbb{Z}(\lambda) \). Denote by \( \mathcal{K} \) the kernel of \( \text{wt}_\lambda \). Evidently, \( \mathcal{M} \) is a free abelian group of rank \( \text{dim } L \).

**Lemma 5.2.6.** With the above notations, \( \text{wt}_\lambda : \mathcal{M} \to \mathbb{Z}(\lambda) \) is surjective. Any \( \mathbb{Z} \)-basis \( h_1, \ldots, h_k \) of \( \mathcal{K} \) is algebraically independent over \( \mathbb{C} \).
Proof. Suppose that \( \nu \in Z_i(\lambda) \). Write \( \nu = \sum a_{\mu} \lambda_\mu \) and choose \( b_{\mu} \in B_i \) to be of weight \( \mu \). Then \( wt_\lambda(\prod b_{\mu}^{\nu_\mu}) = \nu \). On the other hand, \( wt_\lambda(b_{\mu}) \) equals the \( \lambda \)-weight of any monomial in the \( F_i^{-1}, F_i, F_{i,\beta}, \beta \in R_P \) that occurs in \( b_{\mu}|\tilde{U}_i \). The first assertion follows from this.

Let, if possible, \( P(z_1, \ldots, z_k) = 0 \) be a polynomial equation satisfied by \( h_1, \ldots, h_k \). Note that the \( h_j \) are certain Laurent monomials in a transcendence basis of the field \( \mathbb{C}(\tilde{U}_1 \times \tilde{U}_2) \) of rational functions on the affine variety \( \tilde{U}_1 \times \tilde{U}_2 \). Therefore there must exist monomials \( z^m \) and \( z^{m'} \), \( m \neq m' \), occurring in \( P(z_1, \ldots, z_k) \) with non-zero coefficients such that \( h^m = h^{m'} \in \mathbb{C}(\tilde{U}_1 \times \tilde{U}_2) \). Hence \( h^{m-m'} = 1 \). This contradicts the hypothesis that the \( h_j \) are linearly independent in the multiplicative group \( \mathbb{K} \).

We now turn to the proof of Theorem 5.2.1.

Proof of Theorem 5.2.1: By definition, any meromorphic function on \( S_\lambda(L) \) is a quotient \( f/g \) where \( f \) and \( g \) are holomorphic sections of a holomorphic line bundle \( E_2 \). Any holomorphic section \( f : S(L) \to E_2 \) defines a holomorphic function on \( L \), denoted by \( f \), which satisfies Equation (11). By the normality of \( L_1 \times L_2 \), the function \( f \) then extends uniquely to a function on \( \tilde{L}_1 \times \tilde{L}_2 \) which is again denoted \( f \). Thus we may write \( f = \sum_{r,s \geq 0} f_{r,s} \) where \( f_{r,s} \in V(r\varpi_1)^* \otimes V(s\varpi_2)^* \). Now \( v_\lambda f = af \) and \( v_\lambda f_{r,s} \in V(r\varpi_1)^* \otimes V(s\varpi_2)^* \) implies that \( v_\lambda(f_{r,s}) = af_{r,s} \) for all \( r, s \geq 0 \) where \( a = -2\pi\sqrt{-1} \). This implies that \( wt_\lambda(f_{r,s}) = a \) for all \( r, s \geq 0 \). This implies, by Lemma 5.2.5, that \( f_{r,s} = 0 \) for sufficiently large \( r, s \) and so \( f \) is algebraic.

Now writing \( f \) and \( g \) restricted to \( \tilde{U}_1 \times \tilde{U}_2 \) as a polynomial in the the coordinate functions \( F^\pm_i, F_{i,\beta}, i = 1, 2 \), introduced above, it follows easily that \( f/g \) belongs to the field \( \mathbb{C}(\mathcal{K}) \) generated by \( \mathcal{K} \). Evidently \( \mathcal{K} \) and hence the field \( \mathbb{C}(\mathcal{K}) \) is contained in \( \kappa(S_\lambda(L)) \). Therefore \( \kappa(S_\lambda(L)) \) equals \( \mathbb{C}(\mathcal{K}) \).

By Lemma 5.2.6 the field \( \mathbb{C}(\mathcal{K}) \) is purely transcendental over \( \mathbb{C} \).

Finally, since \( Z(\lambda) \) is of rank at least 2 and since \( wt_\lambda : \mathcal{M} \to Z(\lambda) \) is surjective, \( tr.deg(\kappa(S_\lambda(L))) = rank(\mathcal{K}) \leq rank(\mathcal{K}) - 2 = \dim(L) - 2 = \dim(S_\lambda(L)) - 1 \).

Remark 5.2.7. (i) We have actually shown that the transcendence degree of \( \kappa(S_\lambda(L)) \) equals the rank of \( \mathcal{K} \). In the case when \( X_i \) are projective spaces, this was observed by [19]. When \( \lambda \) is of scalar type, \( tr.deg(\kappa(S_\lambda(L))) = \dim(S_\lambda(L)) - 1 \).

(ii) Theorem 5.2.1 implies that any algebraic reduction of \( S_\lambda(L) \) is a rational variety. In the case of scalar type, one has an elliptic curve bundle \( S_\lambda(L) \to X_1 \times X_2 \). (Cf. [29].) Therefore this bundle projection yields an algebraic reduction. In the general case however, it is an interesting problem.
to construct explicit algebraic reductions of these compact complex manifolds. (We refer the reader to [23] and references therein to basic facts about algebraic reductions.)

(iii) We conjecture that \( \kappa(S_{\lambda}(L)) \) is purely transcendental for \( X_i = G_i/P_i \) where \( P_i \) is any parabolic and \( \bar{L}_i \) is any negative ample line bundle over \( X_i \), where \( S_{\lambda}(L) \) has any linear type complex structure.