CHAPTER-II

SOME RELATED FIXED POINT THEOREMS ON THREE L-SPACES

2.1 In the following, the function \( d \) in all the L-spaces \((X,d)\) considered will have
the properties that \( d(x, y) = d(y, x) \) for all \( x, y \in X \) and \( d(x, y) = 0 \) iff \( x = y \).

Generalizing a result of Kanan [1], following theorem was proved by Jain,
Shrivastava and Fisher [1].

**THEOREM A.** Let \((X, d), (Y, \rho)\) and \((Z, \sigma)\) be \(d\)-complete L-spaces. If \( T \) is an
orbitally continuous mapping of \( X \) into \( Y \), \( S \) is an orbitally continuous mapping of \( Y \)
into \( Z \) and \( R \) is a mapping of \( Z \) into \( X \) satisfying the inequalities

\[
\begin{align*}
(2.1.1) \quad d(RSTx, RSTx') & \leq c \max \{ d(x, x'), d(x, RSTx), d(x', RSTx'), \\
& \quad d(x', RSTx), \rho(Tx, Tx'), \sigma(STx, STx') \} \\
\end{align*}
\]

\[
\begin{align*}
(2.1.2) \quad \rho(TRSy, TRSy') & \leq c \max \{ \rho(y, y'), \rho(y, TRSy), \rho(y', TRSy'), \\
& \quad \rho(y', TRSy), \sigma(Sy, Sy'), d(RSy, RSy') \} \\
\end{align*}
\]

\[
\begin{align*}
(2.1.3) \quad \sigma(STRz, STRz') & \leq c \max \{ \sigma(z, z'), \sigma(z, STRz), \sigma(z', STRz'), \\
& \quad \sigma(z', STRz), d(Rz, Rz'), \rho(TRz, TRz') \} \\
\end{align*}
\]

for all \( x, x' \) in \( X \), \( y, y' \) in \( Y \) and \( z, z' \) in \( Z \) where \( 0 \leq c < 1 \). Then \( RST \) has a unique
fixed point \( u \) in \( X \), \( TRS \) has a unique fixed point \( v \) in \( Y \) and \( STR \) has a unique fixed
point \( w \) in \( Z \). Further \( Tu = v \), \( Sv = w \) and \( Rw = u \).
We now obtain the following related fixed point theorems:

**THEOREM 1.** Let \((X, d), (Y, \rho)\) and \((Z, \sigma)\) be \(d\)-complete \(L\)-spaces. If \(T\) is an orbitally continuous mapping of \(X\) into \(Y\), \(S\) is an orbitally continuous mapping of \(Y\) into \(Z\) and \(R\) is a mapping of \(Z\) into \(X\) satisfying the inequalities

\[
\begin{align*}
(2.1.4) \quad & d \left( RSTx, \ RSy \right) \leq c \max \{ d \left( x, RS\right), d \left( x, RSTx\right), \rho \left( y, Tx\right), \sigma \left( Sy, STx\right) \} \\
(2.1.5) \quad & \rho \left( TRSy, \ TRz \right) \leq c \max \{ \rho \left( y, TRz\right), \rho \left( y, TRSy\right), \sigma \left( z, Sy\right), d \left( Rz, RSy\right) \} \\
(2.1.6) \quad & \sigma \left( STRz, \ STx \right) \leq c \max \{ \sigma \left( z, STx\right), \sigma \left( z, STRz\right), d \left( x, Rz\right), \rho \left( Tx, TRz\right) \}
\end{align*}
\]

for all \(x\) in \(X\), \(y\) in \(Y\) and \(z\) in \(Z\) where \(0 \leq c < 1\). Then \(RST\) has a unique fixed point \(u\) in \(X\), \(TRS\) has a unique fixed point \(v\) in \(Y\) and \(STR\) has a unique fixed point \(w\) in \(Z\). Further, \(Tu = v, \ Sv = w\) and \(Rw = u\).

**PROOF.** Let \(x_0\) be an arbitrary point in \(X\). Define sequences \(\{x_n\}, \{y_n\}\) and \(\{z_n\}\) in \(X\), \(Y\) and \(Z\) respectively by

\[
x_n = (RST)^n x_0, \quad y_n = Tx_{n-1}, \quad z_n = Sy_n
\]

for \(n = 1, 2, 3, \ldots\).

Applying inequality (2.1.5), we have

\[
\begin{align*}
\rho \left( y_n, y_{n+1} \right) & = \rho \left( TRSy_n, TRz_{n,1} \right) \\
& \leq c \max \{ \rho \left( y_n, TRz_{n-1}\right), \rho \left( y_n, TRSy_n\right), \sigma \left( z_{n-1}, Sy_n\right), d \left( Rz_{n-1}, RSy_n\right) \} \\
& = c \max \{ \rho \left( y_n, y_n\right), \rho \left( y_n, y_{n+1}\right), \sigma \left( z_{n-1}, z_n\right), d \left( x_{n-1}, x_n\right) \}
\end{align*}
\]

and so
\[(2.1.7) \quad \rho (y_n, y_{n+1}) \leq c \max \{d(x_{n-1}, x_n), \sigma (z_{n-1}, z_n)\}\]

Applying inequality (2.1.6), we have
\[
\sigma (z_n, z_{n+1}) = \sigma (STz_n, STx_{n-1})
\leq c \max \{\sigma (z_n, STx_{n-1}), \sigma (z_n, STLz_n), d(x_{n-1}, Rz_n), \rho (Tx_{n-1}, TRz_n)\}
= c \max \{\sigma (z_n, z_n), \sigma (z_n, z_{n+1}), d(x_{n-1}, x_n), \rho (y_n, y_{n+1})\}
\]
and so

\[(2.1.8) \quad \sigma (z_n, z_{n+1}) \leq c \max \{d(x_{n-1}, x_n), \sigma (z_{n-1}, z_n)\}\]
on using inequality (2.1.7).

Applying inequality (2.1.4), we have
\[
d(x_n, x_{n+1}) = d(RSTx_n, RSy_n)
\leq c \max \{d(x_n, RSy_n), d(x_n, RSTx_n), \rho (y_n, Tx_n), \sigma (Sy_n, STx_n)\}
= c \max \{d(x_n, x_n), d(x_n, x_{n+1}), \rho (y_n, y_{n+1}), \sigma (z_n, z_{n+1})\}
\]
and so

\[(2.1.9) \quad d(x_n, x_{n+1}) \leq c \max \{d(x_{n-1}, x_n), \sigma (z_{n-1}, z_n)\}\]
on using inequality (2.1.7) and (2.1.8).

It now follows easily by induction on using inequalities (2.1.7), (2.1.8) and (2.1.9) that
\[
d(x_n, x_{n+1}) \leq c^{n-1} \max \{d(x_1, x_2), \sigma (z_1, z_2)\},
\rho (y_n, y_{n+1}) \leq c^{n-1} \max \{d(x_1, x_2), \sigma (z_1, z_2)\},
\sigma (z_n, z_{n+1}) \leq c^{n-1} \max \{d(x_1, x_2), \sigma (z_1, z_2)\}.
\]
Thus,

\[ \sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \frac{1}{(1 - c)} \max \{ d(x_1, x_2), \sigma(z_1, z_2) \}, \]

\[ \sum_{n=1}^{\infty} \rho(y_n, y_{n+1}) \leq \frac{1}{(1 - c)} \max \{ d(x_1, x_2), \sigma(z_1, z_2) \}, \]

\[ \sum_{n=1}^{\infty} \sigma(z_n, z_{n+1}) \leq \frac{1}{(1 - c)} \max \{ d(x_1, x_2), \sigma(z_1, z_2) \}. \]

Since X is a d-complete L-space, it follows that the sequence \{x_n\} has a limit u.

Similarly, the sequences \{y_n\} and \{z_n\} have limits v and w in Y and Z respectively.

Since T and S are orbitally continuous, we have

\[ \lim_{n \to \infty} y_n = \lim_{n \to \infty} Tx_{n-1} = Tu = v, \]

\[ \lim_{n \to \infty} z_n = \lim_{n \to \infty} Sy_n = Sv = w. \]

Applying inequality (2.1.4) again, we have

\[ d(RSTu, x_n) = d(RSTu, RSTx_{n-1}) \]

\[ \leq c \max \{ d(u, x_0), d(u, RSTu), \rho(Tx_{n-1}, Tu), \sigma(STx_{n-1}, STu) \} \]

Letting n tend to infinity, it follows that

\[ d(RSTu, u) \leq cd(u, RSTu) \]

and so

\[ RSTu = u, \quad \text{since } c < 1. \]

We now have

\[ TRSv = TRSTu = Tu = v \]
and so

$$\text{STR}_w = \text{STR}_v = v = w$$

Hence $v$ and $w$ are fixed points of TRS and STR respectively.

We now prove the uniqueness of the fixed point $u$. Suppose that RST has a second fixed point $u'$. Then using inequality (2.1.4), we have

$$d(u, u') = d(RSTu, RSTu')$$

$$\leq c \max \{ d(u, u'), d(u, u), \rho(Tu, Tu'), \sigma(STu, STu') \}$$

(2.1.10) $$= c \max \{ \rho(Tu, Tu'), \sigma(STu, STu') \}$$

Further, using inequality (2.1.5), we have

$$\rho(Tu, Tu') = \rho(TRSTu, TRSTu')$$

$$\leq c \max \{ \rho(Tu, Tu'), \rho(Tu, Tu), \sigma(STu, STu'), d(u, u') \}$$

(2.1.11) $$= c \max \{ d(u, u'), \sigma(STu, STu') \}$$

Finally, using inequality (2.1.6), we have

$$\sigma(STu, STu') = \sigma(STRSTu, STRSTu')$$

$$\leq c \max \{ \sigma(STu, STu'), \sigma(STu, STu), d(u, u'), \rho(Tu, Tu') \}$$

(2.1.12) $$= c \max \{ d(u, u'), \rho(Tu, Tu') \}$$

Now from inequalities (2.1.10), (2.1.11) and (2.1.12), we have

$$d(u, u') \leq c^2 d(u, u').$$
since \( c < 1 \), it follows that \( u = u' \) and the uniqueness of \( u \) follows. Similarly, it can be proved that \( v \) is the unique fixed point of \( TRS \) and \( w \) is the unique fixed point of \( STR \).

This completes the proof of the theorem.

**THEOREM 2.** Let \((X, d), (Y, ρ)\) and \((Z, σ)\) be \(d\)-complete \(L\)-spaces and suppose \( T \) is a mapping of \( X \) into \( Y \), \( S \) is a mapping of \( Y \) into \( Z \) and \( R \) is a mapping of \( Z \) into \( X \) satisfying the inequalities

\[
\text{(2.1.13)} \quad d^2(\text{RSy}, \text{RSTx}) \leq c \max \{d(x, \text{RSy}) \ ρ(y, T x), ρ(y, T x) \ d(x, \text{RSTx}),
\quad d(x, \text{RSTx}) \ σ(Sy, STx), σ(Sy, STx) \ d(x, \text{Rsy}) \}
\]

\[
\text{(2.1.14)} \quad ρ^2(\text{TRz}, \text{TRSy}) \leq c \max \{ρ(y, \text{TRz}) \ σ(z, Sy), σ(z, Sy) \ ρ(y, \text{TRSy}),
\quad ρ(y, \text{TRSy}) \ d(Rz, RSy), d(Rz, RSy) \ ρ(y, \text{TRz}) \}
\]

\[
\text{(2.1.15)} \quad σ^2(\text{STx}, \text{STRz}) \leq c \max \{σ(z, \text{STx}) \ d(x, Rz), d(x, Rz) \ σ(z, \text{STRz}),
\quad σ(z, \text{STRz}) \ ρ(Tx, TRz), ρ(Tx, TRz) \ σ(z, \text{STx}) \}
\]

for all \( x \) in \( X \), \( y \) in \( Y \) and \( z \) in \( Z \), where \( 0 ≤ c < 1 \). If one of the mapping \( R \), \( S \), \( T \) is orbitally continuous, then \( RST \) has a unique fixed point \( u \) in \( X \), \( TRS \) has a unique fixed point \( v \) in \( Y \) and \( STR \) has a unique fixed point \( w \) in \( Z \). Further, \( Tu = v \), \( Sv = w \) and \( Rw = u \).

**PROOF.** Let \( x_o \) be an arbitrary point in \( X \) and define sequences \( \{x_n\} \), \( \{y_n\} \) and \( \{z_n\} \) in \( X \), \( Y \) and \( Z \) respectively as in the proof of Theorem 1.
Applying inequality (2.1.14), we have

\[ \rho^2(y_n, y_{n+1}) = \rho^2(\text{TR} z_{n-1}, \text{TR} y_n) \]
\[ \leq c \max \{ \rho(y_n, y_n) \sigma(z_{n-1}, z_n), \sigma(z_{n-1}, z_n) \rho(y_n, y_{n+1}), \]
\[ \rho(y_n, y_{n+1}) \ d(x_{n-1}, x_n), d(x_{n-1}, x_n) \rho(y_n, y_n) \} \]
\[ = c \max \{ \sigma(z_{n-1}, z_n) \rho(y_n, y_{n+1}), \rho(y_n, y_{n+1}) d(x_{n-1}, x_n) \} \]

and so

(2.1.16) \[ \rho(y_n, y_{n+1}) \leq c \max \{ d(x_{n-1}, x_n), \sigma(z_{n-1}, z_n) \} \]

Applying inequality (2.1.15), we have

\[ \sigma^2(z_n, z_{n+1}) = \sigma^2(\text{ST} x_{n-1}, \text{ST} z_n) \]
\[ \leq c \max \{ \sigma(z_n, z_n) d(x_{n-1}, x_n), d(x_{n-1}, x_n) \sigma(z_n, z_{n+1}), \]
\[ \sigma(z_n, z_{n+1}) \rho(y_n, y_{n+1}), \rho(y_n, y_{n+1}) \sigma(z_n, z_n) \} \]
\[ = c \max \{ d(x_{n-1}, x_n) \sigma(z_n, z_{n+1}), \sigma(z_n, z_{n+1}) \rho(y_n, y_{n+1}) \} \]

and so

\[ \sigma(z_n, z_{n+1}) \leq c \max \{ d(x_{n-1}, x_n), \rho(y_n, y_{n+1}) \} \]

(2.1.17) \[ \leq c \max \{ d(x_{n-1}, x_n), \sigma(z_{n-1}, z_n) \} \]

on using inequality (2.1.16).

Applying inequality (2.1.13), we have

\[ d^2(x_n, x_{n+1}) = d^2(\text{RS} y_n, \text{RST} x_n) \]
\[ \leq c \max \{ d(x_n, x_n) \rho(y_n, y_{n+1}), \rho(y_n, y_{n+1}) d(x_n, x_{n+1}), \]
\[ d(x_n, x_{n+1}) \sigma(z_n, z_{n+1}), \sigma(z_n, z_{n+1}) d(x_n, x_n) \} \]
\[ = c \max \{ \rho(y_n, y_{n+1}) \ d(x_n, x_{n+1}), d(x_n, x_{n+1}) \sigma(z_n, z_{n+1}) \} \]
and so
\[
    d(x_n, x_{n+1}) \leq c \max \{ \rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1}) \}
\]

(2.1.18)

\[
    \leq c \max \{ d(x_{n-1}, x_n), \sigma(z_{n-1}, z_n) \}
\]
on using inequality (2.1.16) and (2.1.17).

It now follows easily by induction on using inequalities (2.1.16), (2.1.17) and (2.1.18) that
\[
    d(x_n, x_{n+1}) \leq c^{n-1} \max \{ d(x_1, x_2), \sigma(z_1, z_2) \},
\]
\[
    \rho(y_n, y_{n+1}) \leq c^{n-1} \max \{ d(x_1, x_2), \sigma(z_1, z_2) \},
\]
\[
    \sigma(z_n, z_{n+1}) \leq c^{n-1} \max \{ d(x_1, x_2), \sigma(z_1, z_2) \}.
\]

Thus,
\[
    \sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \frac{1}{1-c} \max \{ d(x_1, x_2), \sigma(z_1, z_2) \},
\]
\[
    \sum_{n=1}^{\infty} \rho(y_n, y_{n+1}) \leq \frac{1}{1-c} \max \{ d(x_1, x_2), \sigma(z_1, z_2) \},
\]
\[
    \sum_{n=1}^{\infty} \sigma(z_n, z_{n+1}) \leq \frac{1}{1-c} \max \{ d(x_1, x_2), \sigma(z_1, z_2) \}.
\]

Since \(X\) is a \(d\)-complete \(L\)-space it follows that the sequence \(\{x_n\}\) has a limit \(u\). Similarly, the sequences \(\{y_n\}\) and \(\{z_n\}\) have limits \(v\) and \(w\) in \(Y\) and \(Z\) respectively.

Now suppose that \(S\) is orbitally continuous, then
\[
    \lim_{n \to \infty} Sy_n = \lim_{n \to \infty} z_n
\]
and so

(2.1.19) \[ S_v = w. \]

Applying inequality (2.1.13), we now have

\[
d^2 (RS_v, x_{n+1}) = d^2 (RS_v, RSTx_n) \\
\leq c \max \{ d (x_n, RS_v) \rho (v, Tx_n), \rho (v, Tx_n) \ d (x_n, x_{n+1}), \\
\ d (x_n, x_{n+1}) \sigma (S_v, STx_n), \sigma (S_v, STx_n) \ d (x_n, RS_v) \}\n\]

Letting \( n \) tend to infinity, it follows on using equation (2.1.19) that

\[
d^2 (RS_v, u) \leq 0
\]

and so

(2.1.20) \[ RS_v = u. \]

Using equation (2.1.19), we get

(2.1.21) \[ Rw = u. \]

Using equation (2.1.20) and inequality (2.1.14), we have

\[
\rho^2 (Tu, y_{n+1}) = \rho^2 (TRS_v, TRS_y) \\
\leq c \max \{ \rho (y_n, TRS_v) \sigma (S_v, SY_n), \sigma (S_v, SY_n) \rho (y_n, TRS_y), \\
\ \rho (y_n, TRS_y) \ d (RS_v, RS_y), \\
\ d (RS_v, RS_y) \rho (y_n, TRS_v) \} \]

Letting \( n \) tend to infinity, it follows on using equation (2.1.20) again that

\[
\rho^2 (Tu, v) \leq 0
\]

and so
(2.1.22) \[ Tu = v \]

It now follows from equations (2.1.19), (2.1.21) and (2.1.22) that

\[ \text{TRS}_v = \text{TR}_w = Tu = v, \]
\[ \text{STR}_w = \text{ST}_u = Sv = w, \]
\[ \text{RST}_u = \text{RS}_v = Rw = u. \]

The same results of course will hold if R or T is orbitally continuous instead of S.

We now prove the uniqueness of the fixed point u. Suppose that RST has a second fixed point \( u' \). Then using inequality (2.1.13), we have

\[ d^2(u, u') = d^2(\text{RST}_u, \text{RST}_u') \]
\[ \leq c \max \{ d(u', u) \rho(Tu, Tu'), \rho(Tu, Tu') \ d(u', u'), \]
\[ d(u', u') \sigma(ST_u, ST_u'), \sigma(ST_u, ST_u') \ d(u', u) \} \]
\[ = c \max \{ d(u, u') \rho(Tu, Tu'), \sigma(ST_u, ST_u') \ d(u, u') \} \]

which implies that

(2.1.23) \[ d(u, u') \leq c \max \{ \rho(Tu, Tu'), \sigma(ST_u, ST_u') \} \]

Further, using inequality (2.1.14), we have

\[ \rho^2(Tu, Tu') = \rho^2(\text{TRST}_u, \text{TRST}_u') \]
\[ \leq c \max \{ \rho(Tu', Tu) \sigma(ST_u, ST_u'), \sigma(ST_u, ST_u') \rho(Tu', Tu), \]
\[ \rho(Tu', Tu') \ d(u, u'), d(u, u') \rho(Tu', Tu) \} \]
\[ = c \max \{ \rho(Tu, Tu') \sigma(ST_u, ST_u'), d(u, u') \rho(Tu, Tu') \} \]

which implies that
\[(2.1.24) \quad \rho(T_u, T_u') \leq c \max \{ \sigma(ST_u, ST'_u), d(u, u') \}\]

Inequalities (2.1.23) and (2.1.24) imply that

\[(2.1.25) \quad d(u, u') \leq c \sigma(ST_u, ST'_u)\]

Finally, using inequality (2.1.15), we have

\[\sigma^2(ST_u, ST_u') = \sigma^2(STRST_u, STRST_u')\]

\[\leq c \max \{ \sigma(ST_u, ST_u') d(u, u'), d(u, u') \sigma(ST_u', ST'_u),\]

\[\sigma(ST_u', ST_u') \rho(T_u, T_u'), \rho(T_u, T_u') \sigma(ST_u, ST_u')\}\]

\[= c \max \{ \sigma(ST_u, ST_u') d(u, u'), \rho(T_u, T_u') \sigma(ST_u, ST_u') \}\]

which implies that

\[(2.1.26) \quad \sigma(ST_u, ST_u') \leq c \max \{ d(u, u'), \rho(T_u, T_u') \}\]

It now follows from inequalities (2.1.24), (2.1.25) and (2.1.26) that

\[d(u, u') \leq c \sigma(ST_u, ST_u') \leq c^2 \sigma(ST_u, ST_u')\]

and so \(u = u'\), since \(c < 1\). The fixed point \(u\) of RST is therefore unique.

Similarly it can be proved that \(v\) is the unique fixed point of TRS and \(w\) is the unique fixed point of STR. This completes the proof of the theorem.