CHAPTER-I

INTRODUCTION

1.1 At the beginning of the 20th century a new discipline of analysis 'The functional analysis' developed. The essence of functional analysis lies in the fact that the concepts and methods of elementary analysis as well as of related domains of algebra and geometry are carried over to general objects. The fixed point theory is an important branch of functional analysis.

By a fixed point theorem we shall mean a statement which asserts that under certain conditions a mapping T of a Set X into itself admits one or more fixed points, i.e., points $x \in X$ for which $Tx = x$. Fixed point theorem have found fruitful applications in various areas viz. The theory of non-linear oscillations, fluid flows, initial and boundary value problems, ordinary and partial differential equation etc. These applications have been discussed by Kolmogorov and Fomin [1], Smart [1] and Swaminathan [1].

1.2 PRELIMINARY CONCEPTS

(a) METRIC SPACES

DEFINITION 1. Let $X$ be a nonempty set and $d$ be a function from $X \times X$ into $\mathbb{R}^+$ such that for every $x, y$ and $z \in X$, we have
\(1.2.1\) \(d(x, y) \geq 0,\)

\(1.2.2\) \(d(x, y) = 0 \iff x = y,\)

\(1.2.3\) \(d(x, y) = d(y, x),\)

\(1.2.4\) \(d(x, z) \leq d(x, y) + d(y, z)\)

Then \(d\) is called a metric or the distance function and the pair \((X, d)\) is called the metric space.

**Definition 2.** A sequence \(\{x_n\}\) of points of a metric space is said to converge to a point \(z\) if given \(\varepsilon > 0\), there exists a positive integer \(N(\varepsilon)\) such that

\[d(x_n, z) < \varepsilon, \quad \forall \, n \geq N.\]

This is denoted by \(x_n \rightarrow z\) or \(\lim_{n \to \infty} d(x_n, z) = 0\)

**Definition 3.** A sequence \(\{x_n\}\) is said to be a Cauchy sequence if for each \(\varepsilon > 0,\) there exists a positive integer \(N(\varepsilon)\) such that

\[d(x_n, x_m) < \varepsilon, \quad \forall \, n, m \geq N.\]

A convergent sequence is always a Cauchy sequence, but the converse is not true.

**Definition 4.** A metric space \((X, d)\) is said to be complete if every Cauchy sequence in \(X\) converges to a point in \(X\).

**Definition 5.** Let \((X, d_1)\) and \((Y, d_2)\) be two metric spaces and let \(T\) be a mapping of \(X\) into \(Y,\) \(T\) is called continuous at a point \(x_0\) in \(X\) if for each \(\varepsilon > 0,\) there exists \(\delta > 0\) such that
\[ d_1(x, x_0) < \delta \Rightarrow d_2(Tx, Tx_0) < \epsilon \]

T is said to be continuous, if it is continuous at each point of its domain.

(b) **L-Spaces**

**Definition 1.** (Frechet [1]) Let \( \omega \) denote the set of all non-negative integers and \( C \) be a class consisting of pairs \( \{ \{ x_n : n \in \omega \}, x \} \), where \( \{ x_n : n \in \omega \} \) is a sequence in a non-empty set of \( X \) and \( x \) is a point in \( X \). Then \( C \) is said to be a convergence class (i.e. \( \{ x_n : n \in \omega \} \) converges to \( x \) or \( \lim_{n \to \infty} x_n = x \)) if and only if it satisfies the following conditions

(1.2.5) if \( x_n : x \in X \) for all \( n \in \omega \), then \( \{ x_n : n \in \omega \}, x \) \( \in C \),

(1.2.6) if \( \{ x_n : n \in \omega \} \) converges to \( x \), then so does each subset of \( \{ x_n : n \in \omega \} \).

Then the pair \( (X, C) \) is said to be an L-space.

**Definition 2.** (Kasahara [1]). Let \( X \) be an L-space and let \( d \) be a non-negative extended real valued function on \( X \times X \). Then \( X \) is said to be \( d \)-complete, if each sequence \( \{ x_n : n \in \omega \} \) in \( X \) with \( \sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty \) converges to at least one point in \( X \).

**Definition 3.** Let \( X \) be an L-space and \( f \) be a mapping of \( X \) into itself. If for each \( x \in X \), \( \lim_{i \to \infty} f^{n_i}(x) = a \in X \) implies \( \lim_{i \to \infty} f(f^{n_i}(x)) = f(a) \), then \( f \) is said to be orbitally continuous.
(c) **BANACH SPACES**

**DEFINITION 1.** A normed linear space is a linear space $X$ in which to each vector $x$ there corresponds a real number, denoted by $\| x \|$ and called the norm of $x$, is such a manner that

(1.2.7) \[ \| x \| \geq 0 \quad \text{and} \quad \| x \| = 0 \quad \text{iff} \quad x = 0, \]

(1.2.8) \[ \| x + y \| < \| x \| + \| y \|, \]

(1.2.9) \[ \| ax \| = | a | \| x \|. \]

Every normed linear space $X$ is a metric space with respect to the metric $d$ defined by $d(x, y) = \| x - y \|$.

**DEFINITION 2.** A Banach space $X$ is a complete normed linear space. Completeness means that if \[ \lim_{m,n \to \infty} \| x_m - x_n \| = 0 \] whenever $\{ x_n \} \in X$, then there exists $x \in X$ such that \[ \lim_{n \to \infty} \| x_n - x \| = 0. \]

(d) **NON-ARCHIMEDEAN PROBABILISTIC METRIC SPACE**

**DEFINITION 1.** Let $X$ be any nonempty set and $D$ be the set of all left continuous distribution functions. An order pair $(X, f)$ is called a non-Archiimedean probabilistic metric space (briefly, a N.A. PM-space) if $f$ is a mapping from $X \times X$ into $D$ satisfying the following conditions: (We shall denote the distribution function $f(x, y)$ by $Fx, y$ for all $x, y \in X$);
(1.2.10) \( F_x, y(t) = 1 \) for all \( t > 0 \) if and only if \( x = y \),

(1.2.11) \( F_x, y = F_y, x \),

(1.2.12) \( F_x, y(0) = 0 \),

(1.2.13) If \( F_x, y(t_1) = 1 \) and \( F_y, z(t_2) = 1 \), then \( F_x, z(\max\{t_1, t_2\}) = 1 \).

**DEFINITION 2.** A t-norm is a function \( \Delta : [0, 1] \times [0, 1] \rightarrow [0, 1] \) which is associative, commutative, non-decreasing in each coordinate and \( \Delta(a, 1) = a \) for every \( a \in [0, 1] \).

**DEFINITION 3.** A N.A. Menger PM-spae is an order triple \((X, f, \Delta)\), where \( \Delta \) is a t-norm and \((X, f)\) is a non Archimedean PM-space satisfying the following condition :

(1.2.14) \( F_x, z(\max\{t_1, t_2\}) \geq \Delta(F_x, y(t_1), F_y, z(t_2)) \) for all \( x, y, z \in X \) and \( t_1, t_2 \geq 0 \).

**DEFINITION 4.** A PM-space \((X, f)\) is said to be of type \((C)_g\) if there exists a \( g \in \Omega \) such that \( g(F_x, y(t)) \leq g(F_x, z(t)) + g(F_z, y(t)) \) for all \( x, y, z \in X \) and \( t \geq 0 \), where \( \Omega = \{ g : g : [0, 1] \rightarrow [0, \infty) \text{ is continuous, strictly decreasing, } g(1) = 0 \text{ and } g(0) < \infty \} \).

**DEFINITION 5.** A N.A. Menger PM-space \((X, f, \Delta)\) is said to be of type \((D)_g\) if there exists a \( g \in \Omega \) such that

\[
g(\Delta(s, t)) \leq g(s) + g(t) \text{ for all } s, t \in [0, 1].
\]
SOME OTHER CONCEPTS

CONTRACTION, CONTRACTIVE AND NON-EXPANSIVE MAPPINGS

**DEFINITION 1.** A mapping $T$ of a metric space $(X, d)$ into itself is called a contraction if $d(Tx, Ty) \leq kd(x, y)$, $\forall x, y \in X$ and $0 \leq k < 1$.

**DEFINITION 2.** A mapping $T$ of a metric space $(X, d)$ into itself is called contractive if $d(Tx, Ty) < d(x, y)$, $\forall x, y \in X$, $x \neq y$.

**DEFINITION 3.** A mapping $T$ of a metric space $(X, d)$ into itself is said to be non-expansive if $d(Tx, Ty) \leq d(x, y)$, $\forall x, y \in X$.

We note that

Contraction $\Rightarrow$ Contractive $\Rightarrow$ non-expansive, and all such mappings are continuous.

ORBITALLY COMPLETE METRIC SPACE AND ORBITALLY CONTINUOUS MAPPINGS

**DEFINITION 4.** Let $P$, $Q$ and $T$ be self maps on a metric space $(X, d)$. For a point $x_o \in X$ if there exists a sequence $\{x_n\}$ such that

$$Tx_{2n+1} = Qx_{2n+1}, \quad n = 0, 1, 2, \ldots \ldots \ldots \text{then } O(P, Q, T, x_o) = \{ Tx_n : n = 1, 2, \ldots \ldots \} \text{ is called the Orbit of } (P, Q, T) \text{ at } x_o.$$

**DEFINITION 5.** $T$ is said to be orbitally continuous at $x_o$ if and only if it is continuous on closure of $O(P, Q, T, x_o)$. It is clear that every continuous map is
orbitally continuous. Ciric [2] has shown that orbitally continuous map need not to be continuous.

**DEFINITION 6.** X is called orbitally complete at \( x_n \), if and only if every Cauchy sequence in \( O (P, Q, T, x_o) \) converges in X.

**WEAKLY COMMUTING MAPPINGS**

**DEFINITION 7.** Let A and S be two self-mappings of a metric space \( (X, d) \). Then \( \{A, S\} \) is said to be a weakly commuting pair on X, if \( d (SAx, ASx) \leq d (Ax, Sx), \forall x \in X \). We can similarly define weak commutativity in a normed linear space. Clearly a commuting pair is weakly commuting but the converse is not true in general (Rhoades and Sessa [1]).

**COMPATIBLE MAPPINGS**

**DEFINITION 8.** (Jungck, G. [3]) Let S and T be mappings from a normed space \( (X, \|.|\|) \) into itself. The mappings S and T are said to be compatible, if

\[
\lim_{n \to \infty} \| STx_n - TSx_n \| = 0 \quad \text{whenever} \quad \{x_n\} \text{ is a sequence in } X \quad \text{such that} \quad \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \quad \text{for some} \quad t \in X.
\]

**DEFINITION 9.** (Jungck, G. [3]) let S and T be mappings from a normed space \( (X, \|.|\|) \) into itself. The mappings S and T are said to be compatible of Type (A) if
\[ \lim_{n \to \infty} \| TSx_n - SSx_n \| = 0, \quad \lim_{n \to \infty} \| STx_n - TTx_n \| = 0, \]

whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for \( t \in X \).

**DEFINITION 10.** (Jungck, G. [3]) Let \( S \) and \( T \) be mapping from a normed space \( (X, \| \cdot \|) \) into itself. The mapping \( S \) and \( T \) are said to be compatible of Type (B) if

\[
\lim_{n \to \infty} \| STx_n - TTx_n \| \leq \frac{1}{2} \left[ \lim_{n \to \infty} \| STx_n - St \| + \lim_{n \to \infty} \| St - SSx_n \| \right],
\]

\[
\lim_{n \to \infty} \| TSx_n - SSx_n \| \leq \frac{1}{2} \left[ \lim_{n \to \infty} \| TSx_n - Tt \| + \lim_{n \to \infty} \| Tt - TTx_n \| \right]
\]

whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \).

**DEFINITION 11.** Let \( S \) and \( T \) be mappings from a normed space \( (X, \| \cdot \|) \) into itself. The mappings \( S \) and \( T \) are said to be compatible of type (C) if

\[
\lim_{n \to \infty} \| STx_n - TTx_n \| \leq \frac{1}{3} \left[ \lim_{n \to \infty} \| STx_n - St \| + \lim_{n \to \infty} \| St - SSx_n \| + \lim_{n \to \infty} \| St - TTx_n \| \right],
\]

\[
\lim_{n \to \infty} \| TSx_n - SSx_n \| \leq \frac{1}{3} \left[ \lim_{n \to \infty} \| TSx_n - Tt \| + \lim_{n \to \infty} \| Tt - TTx_n \| + \lim_{n \to \infty} \| SSx_n - Tt \| \right]
\]

whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \), for some \( t \in X \).
INTIMATE MAPPINGS

DEFINITION 12. Let $S$ and $T$ be self-maps of metric space $(X, d)$. The pair

$\{S, T\}$ is said to be $T$-intimate, iff $\alpha \ d(TS_{x_n}, TX_n) \leq \alpha \ d(SS_{x_n}, SX_n)$

where $\alpha = \limsup$ or $\liminf$, $\{x_n\}$ is a sequence in $X$ such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t,$$

for some $t \in X$.

Intimate mapping condition is more improved than the well known mappings conditions such as weakly commuting ([Sessa [1]], compatible ([Jungck [3]]), $D$-compatibility ([Sharma and Sahu [1]]), Semi compatibility ([Cho, Sharma and Sahu [1] ] and R-commutativity ([Pant [1]]). Infact, newly defined mapping is a generalization of the compatibility of type (A) mapping condition considered by Murthy, Chang, Cho and Sharma [1]. The most important feature of Intimate mapping condition is that for all the above said mapping pairs, it is necessary to commute at coincidence point but for intimate mapping condition such necessity is not required i.e. the mapping pairs does not necessarily commute at coincidence point.

1.3 In 1912, Brouwer [1], perhaps for the first time introduced the concept of fixed point. Schauder [1] extended Brouwer’s result to compact convex sets in Banach space.

Banach [1], a great polish mathematician, proved an important result on fixed points in 1922 which is known as Banach’s contraction principle. It states that every contraction mapping of complete metric space into itself has a unique fixed point.
Chu and Diaz [1], Sehgal [1], Holmes [1], Guseman [1], Browder [1], and Hardy and Rogers [1] have generalized Banach contraction principal. Later on Rakotch [1], Ciric [1], Boyd and Wong [1], Iseki [3], Ray and Rhoades [1], Achari ([1], [3]), Maiti and Pal ([1], [2]), Khazanchi [1], Das and Gupta [1], Juggi [2], Pachpatte [3], Jain and Dixit [1] and others have obtained more general and interesting results in this field.

Rhoades, through his papers ([1] - [4]) has done a very difficult task and thus made a creditable contribution in the field of fixed point theory where he has compared and summarized most of the contraction type mappings.

1.4 COMMON FIXED POINT THEOREMS IN METRIC SPACES

A point \( x \in X \) is said to be common fixed point of self mappings \( T_1, T_2, \ldots, T_k \), on the metric space \( X \) if \( T_1x = T_2x = \ldots = T_kx = x \).

For the first time Kannan [1] gave a sufficient condition for the existence of a unique common fixed point for a pair of mappings defined on a complete metric space. His result was extended by Singh, S.P. [1], Rus [1], Iseki [1], Yen [1] and Sehgal [2].

Jungck [1] gave a result for a pair of continuous and commutating self-mappings of a complete metric space. His result was extended by Singh, S.L. [1], Yeh [2] for three continuous self mappings. Pachpatte [1], Jain and Rajoriya [1] also obtained some results for three mappings in complete metric space. Recently,
Khan and Swaleh [1], Khan and Imadad [1] and Rhoades, Sessa and Khan [1] have proved common fixed points of three mappings satisfying condition weaker than commuting.


1.5. FIXED POINTS FOR L-SPACES


1.6 FIXED POINT IN NON-ARCHIMEDEAN MENGER PM-SPACES

In 1978, Istratescu [1] established the existence of fixed point of mappings on non-Archimedean Menger space and thus he extended the corresponding results of Sehgal and Bharucha-Reid [1] and Sherwood [1] on a Menger space. Chang [1] and Singh and Pant [1] also proved some fixed point theorems on a non-Archimedean Menger space.
Recently Cho, Ha and Chang [1] have defined the concept of compatible mappings and compatible mapping of type (A) in non-Archimedean Menger Probabilistic Metric spaces and studied the existence problems of common fixed points for compatible mappings of type (A).

1.7 **FIXED POINTS FOR CC-MAPPINGS**

Murthy, Cho and Fisher [1] proved some fixed point theorems of Gregus type for compatible mappings of type (A) in Banach spaces. Pathak and Khan [1] obtained some common fixed point theorems of Gregus type (Diviccaro [1], Fisher and Sessa [1] and Gregus [1]) for compatible mapping of type (B) in Banach spaces. Recently, Pathak, Cho, Kang and Madharia [1] have derived some relations between these mappings. They have also proved a common fixed point theorem of Gregus type for compatible mappings of type (C) in Banach spaces.

1.8 **FIXED POINTS FOR INTIMATE MAPPINGS**

Jungck ([3] - [5]) introduced compatible mappings and obtained several fixed point theorems for them. Later on Cho [1], Murthy, Chang, Cho and Sharma [1], Jungck, Murthy and Cho [1] proved some common fixed point theorems for compatible mappings of type (A). Intimate mapping is a generalization of compatibility of type (A).

1.9 Chapter II is devoted to the study of some fixed point theorem in three $L\cdot$-spaces. Recently, Jain, Shrivastava and Fisher [1] have generalized a result of
Kanan [1] from two L-spaces to three L-spaces. In the present chapter, two related fixed point theorems in three L-spaces are obtained.

The concept of related fixed point theorem on two metric spaces was introduced by Fisher ([4], [5]) Popa [1] also obtained some interesting results in this direction. Nung [1], Jain, Sahu and Fisher [1] generalized the result of Fisher ([4], [5]) from two metric spaces to three metric spaces. Inspired by the work of Popa [1], we have proved some results on three metric spaces in Chapter III.

Chapter IV consists of two sections. In section 1, we obtain a fixed point theorem on four metric spaces which includes the results of Fisher [5] and Jain, Sahu and Fisher [1]. The results of section 2 is inspired by the work of Nung [1].

Chapter V deals with intimate mappings in complete metric space. The result of section 1 extends the results of Prasad [1], Pathak [1], and Jungck and Pathak [1]. In section 2, we mention two theorems on the convergence of self-mappings on a metric space and the existence of their fixed points.

In Chapter VI, we obtain some fixed point theorems for three mappings under weak commutativity condition. In section 1, we generalize the results of Pachpatte [1] and Jain and Rajoriya [1] by replacing commutativity condition by weak commutativity and using only orbital continuity in place of continuity. In section 2, we prove some more results for different type of mappings.

Chapter VII is devoted to the study of some common fixed point theorems in non-Archimedean Menger PM-spaces. Our first theorem includes several results for
commuting, weakly commuting and compatible mappings on metric spaces and PM-spaces by Ciric [2], Jungeck [3], Sehgal and Bharucha-Reid [1], Sessa [1], Singh and Pant [1], Stojakovic [1] and Cho, Ha and Chang [1].

Finally, in chapter VIII, we confine our attention to prove some fixed point theorems for compatible mapping of type (C). Which are motivated by the work of Pathak, Cho, Kang and Madharia [1].