CHAPTER VIII

FIXED POINTS FOR COMPATIBLE MAPPINGS OF TYPE (C)

8.1 In a recent paper, Pathak, Cho, Kang and Madharia [1] have introduced the concept of compatible mappings of type (C) in normed spaces. They have also established a common fixed point theorem for two pairs of these mappings. The results of the present chapter are motivated by their work.

Let \( R^+ \) be the set of all non-negative reals and \( F \) be the family of mapping \( \phi \) from \( R^+ \) into \( R^+ \) such that \( \phi \) is upper semi continuous non-decreasing in each coordinate variable, and \( \phi(t) < t \) for any \( t > 0 \).

Let \( A, B, S \) and \( T \) be mappings from a normed space \( (X, ||.||) \) into itself such that

\[
(8.1.1) \quad A(X) \subset T(X) \text{ and } B(X) \subset S(X)
\]

\[
(8.1.2) \quad ||Ax-By||^p \leq \phi \left( \max \left( ||Sx-Ty||^p, ||Sx-Ax||^p, ||By-Ty||^p, ||Ty-Ax||^p, ||Sx-By||^p \right) \right)
\]

for all \( x, y \in X \), where \( \phi \in F \) and \( p \geq 1 \). Then by (8.1.1), since \( A(X) \subset T(X) \), for an arbitrary point \( x_0 \in X \) there exists a point \( x_1 \in X \) such that \( Ax_0 = Tx_1 \), since \( B(X) \subset S(X) \), for this point \( x_1 \in X \), we can choose a point \( x_2 \in X \) such that \( Bx_1 = Sx_2 \) and so on. Inductively, we can define a sequence \( \{y_n\} \) in \( X \) such that

\[
(8.1.3) \quad y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}
\]
for every \( n = 0, 1, 2, \ldots \).

For our first theorem, we need the following Lemmas:

**Lemma 1.** Let \( S \) and \( T \) be CC-mappings from a normed space \( (X, \| \cdot \|) \) into itself. Suppose that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \). Then we have the following:

\[
\begin{align*}
\text{(8.1.4) } & \quad \lim_{n \to \infty} TTx_n = St \text{ if } S \text{ is continuous at } t, \\
\text{(8.1.5) } & \quad \lim_{n \to \infty} SSx_n = Tt \text{ if } T \text{ is continuous at } t, \\
\text{(8.1.6) } & \quad STt = TSt \text{ and } St = Tt \text{ if } S \text{ and } T \text{ are continuous at } t.
\end{align*}
\]

**Lemma 2.** For any \( t > 0 \), \( \gamma(t) < t \) if and only if \( \lim_{n \to \infty} \gamma^n(t) = 0 \), where \( \gamma^n \) denotes the \( n \)-times composition of \( \gamma \).

**Lemma 3.** Let \( A, B, S \) and \( T \) be mappings from a normed space \( (X, \| \cdot \|) \) into itself satisfying \( (8.1.1) \) and \( (8.1.2) \). Then \( \lim_{n \to \infty} \| y_n - y_{n+1} \| = 0 \), where \( \{y_n\} \) is a sequence in \( X \) defined by \( (8.1.3) \).

**Proof.** By \( (8.1.2) \) and \( (8.1.3) \), we have

\[
\| y_{2n} - y_{2n+1} \|^p = \| Ax_{2n} - Bx_{2n+1} \|^p \\
\leq \phi \left\{ \max \left( \| y_{2n-1} - y_{2n} \|^p, \| y_{2n-1} - y_{2n} \|^p, \| y_{2n} - y_{2n+1} \|^p, \\
\| y_{2n} - y_{2n} \|^p, \| y_{2n-1} - y_{2n} \|^p \right) \right\}
\]
If \( \| y_{2n} - y_{2n+1} \| > \| y_{2n-1} - y_{2n} \| \) in the above inequality, then we have
\[
\| y_{2n} - y_{2n+1} \|^p \leq \phi \left( \max \left( \| y_{2n} - y_{2n+1} \|^p, \| y_{2n} - y_{2n+1} \|^p, \| y_{2n} - y_{2n+1} \|^p, \| y_{2n} - y_{2n+1} \|^p \right) \right)
\]
\[
= \phi \left( \| y_{2n} - y_{2n+1} \|^p \right)
\]
which is a contradiction. Thus we have
\[
\| y_{2n} - y_{2n+1} \|^p \leq \phi \left( \| y_{2n-1} - y_{2n} \|^p \right)
\]
Similarly, we have
\[
\| y_{2n+1} - y_{2n+2} \|^p \leq \phi \left( \| y_{2n} - y_{2n+1} \|^p \right)
\]
But this follows that
\[
\| y_n - y_{n+1} \|^p \leq \phi \left( \| y_{n-1} - y_n \|^p \right) \leq \ldots \leq \phi^p \left( \| y_0 - y_1 \|^p \right)
\]
and so from Lemma 2, we have
\[
\lim_{n \to \infty} \| y_n - y_{n+1} \| = 0
\]
This completes the proof.

**LEMMA 4.** Let A, B, S and T be mappings from a normed space \((X, \| . \|)\) into itself satisfying the conditions (8.1.1) and (8.1.2). Then the sequence \(\{y_n\}\) defined by (8.1.3) is a Cauchy sequence in X.

**PROOF.** By virtue of Lemma 3, it is sufficient to show that \(\{y_{2n}\}\) is a Cauchy sequence in X. Suppose to the contrary that there is an \(\epsilon > 0\) such that for each even integer 2k, there exists even integers 2m(k) and 2n(k) with 2m(k) > 2n(k) ≥ 2k such that
(8.1.7) \[ \| y_{2n(k)} - y_{2n(k)} \| > \varepsilon. \]

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (8.1.7), that is

(8.1.8) \[ \| y_{2n(k)} - y_{2m(k)-2} \| \leq \varepsilon \quad \text{and} \quad \| y_{2n(k)} - y_{2m(k)} \| > \varepsilon. \]

Then, for each even integer $2k$, we have

\[ \varepsilon < \| y_{2n(k)} - y_{2m(k)} \| \]

\[ \leq \| y_{2n(k)} - y_{2m(k)-2} \| + \| y_{2m(k)-2} - y_{2m(k)-1} \| + \| y_{2m(k)-1} - y_{2m(k)} \| \]

It follows from Lemma 3 and (8.1.8) that, as $k \to \infty$

(8.1.9) \[ \| y_{2n(k)} - y_{2m(k)} \| \to \varepsilon \]

By Triangle inequality, we have

\[ \| y_{2n(k)} - y_{2m(k)-1} \| - \| y_{2n(k)} - y_{2m(k)} \| \leq \| y_{2m(k)-1} - y_{2m(k)} \| \]

and

\[ \| y_{2n(k)+1} - y_{2m(k)-1} \| - \| y_{2n(k)} - y_{2m(k)} \| \leq \| y_{2m(k)-1} - y_{2m(k)} \| + \| y_{2n(k)} - y_{2m(k)+1} \| \]

From Lemma 3 and (8.1.9), it follows that as $k \to \infty$

(8.1.10) \[ \| y_{2n(k)} - y_{2m(k)-1} \| \to \varepsilon \quad \text{and} \quad \| y_{2n(k)+1} - y_{2m(k)-1} \| \to \varepsilon \]

Therefore, by (8.1.2) and (8.1.3), we have

\[ \| y_{2n(k)} - y_{2m(k)} \| \leq \| y_{2n(k)} - y_{2n(k)+1} \| + \| A x_{2n(k)} - B x_{2n(k)+1} \| \]

\[ \leq \| y_{2n(k)} - y_{2n(k)+1} \| + \left[ \phi \left( \max \left( \| y_{2n(k)} - y_{2n(k)} \|^{p}, \| y_{2n(k)} - y_{2n(k)+1} \|^{p}, \| y_{2n(k)} - y_{2n(k)} \|^{p}, \| y_{2n(k)+1} - y_{2n(k)+1} \|^{p} \right) \right]^{\frac{1}{p}} \]
Since $\phi \in F$, by Lemma 3, (8.1.9) and (8.1.10), we have, as $k \to \infty$

\[(8.1.11)\]

$$\varepsilon \leq \left\{ \phi \left( \max \left( \varepsilon^p, 0, 0, \varepsilon^p, \varepsilon^p \right) \right) \right\}^{1/p} < \varepsilon$$

which is a contradiction, therefore, $\{y_{2n}\}$ is a Cauchy sequence in $X$. This completes the proof.

Our theorem runs as follows

**THEOREM 1.** Let $A$, $B$, $S$ and $T$ be mappings from a Banach space $(X, \|\cdot\|)$ into itself satisfying the condition (8.1.1) and (8.1.2). Suppose that one of $A$, $B$, $S$ and $T$ is continuous and the pairs $\{A, S\}$ and $\{B, T\}$ are CC-mappings. Then $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$.

**PROOF.** Let $\{y_n\}$ be the sequence in $X$ defined by (8.1.3). By Lemma 4, $\{y_n\}$ is a Cauchy sequence in $X$ and hence it converges to a point $z$ in $X$. Consequently, subsequence\, $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converges to point $z$.

Now, suppose that $A$ is continuous, since $A$ and $S$ are CC-mappings, it follows from Lemma 1 that $ASx_{2n}$, $SSx_{2n} \to Az$ as $n \to \infty$.

By (8.1.2), we have

$$\|ASx_{2n} - Bx_{2n+1}\|^p \leq \phi \left\{ \max \left( \|SSx_{2n} - Tx_{2n+1}\|^p, \|SSx_{2n} - ASx_{2n}\|^p, \|Tx_{2n+1} - ASx_{2n}\|^p, \|SSx_{2n} - Bx_{2n+1}\|^p \right) \right\}$$

By letting $n \to \infty$, in the above inequality, we have
\[ \| A z - z \|_p^p \leq \phi \{ \max ( \| A z - z \|_p^p, 0, 0, \| z - A z \|_p^p, \| A z - z \|_p^p ) \} \]

i.e.,
\[ \| A z - z \|_p^p < \| A z - z \|_p^p, \]
which is a contradiction. Thus we have
\[ A z = z. \]
Since \( A(X) \subset T(X) \), there exist a point \( u \in X \) such that \( z = A z = T u \). Again by (8.1.2), we have
\[
\| A S x_{2n} - B u \|_p^p \leq \phi \{ \max ( \| S S x_{2n} - T u \|_p^p, \| S S x_{2n} - A S x_{2n} \|_p^p, \| T u - B u \|_p^p, \\
\| T u - A S x_{2n} \|_p^p, \| S S x_{2n} - B u \|_p^p ) \}
\]
By letting \( n \rightarrow \infty \), since \( \phi \in F \), we obtain
\[
\| z - B u \|_p^p \leq \phi \{ \max (0, 0, \| z - B u \|_p^p, 0, \| z - B u \|_p^p ) \}
\]
\[ \leq \| z - B u \|_p^p \]
which implies that \( z = B u \). Since \( B \) and \( T \) are CC-mappings and \( T u = B u = z \) by Lemma 1, \( T B u = B T u \) and hence \( T z = T B u = B T u = B z \). Moreover, by (8.1.2), we have
\[
\| A x_{2n} - B z \|_p^p \leq \phi \{ \max ( \| S x_{2n} - T z \|_p^p, \| S x_{2n} - A x_{2n} \|_p^p, \| T z - B z \|_p^p, \\
\| T z - A x_{2n} \|_p^p, \| S x_{2n} - B z \|_p^p ) \}
\]
By letting \( n \rightarrow \infty \), we obtain
\[
\| z - B z \|_p^p \leq \phi \{ \max ( \| z - B z \|_p^p, 0, 0, \| B z - z \|_p^p, \| z - B z \|_p^p ) \}
\]
\[ \leq \| z - B z \|_p^p \]
which means that \( z = B z \). Since \( B(X) \subset S(X) \), there exists a point \( v \in X \) such that
\[ z = B z = S v. \]
By using (8.1.2), we have
$$\| \text{Av} - z \|_p^p = \| \text{Av} - Bz \|_p^p$$

$$\leq \phi \{ \max ( \| \text{Sv} - Tz \|_p^p, \| \text{Sv} - \text{Av} \|_p^p, \| Tz - Bz \|_p^p, \| Tz - \text{Av} \|_p^p, \| \text{Sv} - Bz \|_p^p ) \}$$

$$\leq \phi \{ \max (0, \| z - \text{Av} \|_p^p, 0, \| z - \text{Av} \|_p^p, 0 ) \}$$

$$\leq \| z - \text{Av} \|_p^p$$

so that $\text{Av} = z$. Since $A$ and $S$ are CC-mappings and $\text{Av} = \text{Sv} = z$, we have $S\text{Av} = A\text{Sv}$ and hence $Sz = S\text{Av} = A\text{Sv} = Az$. Therefore, $z$ is a common fixed point of $A, B, S$ and $T$.

Now, suppose that $S$ is continuous. Since $A$ and $S$ are CC-mappings it follows from Lemma 1 that $S\text{Ax}_{2n}, A\text{Ax}_{2n} \to Sz$ as $n \to \infty$.

By (8.1.2), we have

$$\| A\text{Ax}_{2n} - Bx_{2n+1} \|_p^p \leq \phi \{ \max ( \| S\text{Ax}_{2n} - T_{2n+1} \|_p^p, \| S\text{Ax}_{2n} - A\text{Ax}_{2n} \|_p^p, \| T_{2n+1} - Bx_{2n+1} \|_p^p, \| T_{2n+1} - A\text{Ax}_{2n} \|_p^p, \| S\text{Ax}_{2n} - Bx_{2n+1} \|_p^p ) \}$$

By letting $n \to \infty$, we have

$$\| Sz - z \|_p^p \leq \phi \{ \max ( \| Sz - z \|_p^p, 0, 0, \| z - Sz \|_p^p, \| Sz - z \|_p^p ) \}$$

$$\leq \| z - Sz \|_p^p,$$

which is a contradiction. Thus we have $Sz = z$. Since $B(X) \subset S(X)$ there exists a point $u \in X$ such that $z = Sz = Bu$. Again by (8.1.2), we have

$$\| Az - Bx_{2n+1} \|_p^p \leq \phi \{ \max ( \| Sz - T_{2n+1} \|_p^p, \| Sz - Az \|_p^p, \| T_{2n+1} - Bx_{2n+1} \|_p^p, \| T_{2n+1} - Az \|_p^p, \| Sz - Bx_{2n+1} \|_p^p ) \}$$
By letting \( n \to \infty \), since \( \phi \in F \) we obtain
\[
\| A z - z \|_p^p \leq \phi \left\{ \max \left( 0, \| z - A z \|_p^p, 0, \| z \|_p^p \right) \right\} \\
\leq \| z - A z \|_p^p
\]
which means that \( z = A z \). Since \( A(X) \subset T(X) \), there exists a point \( u \in X \) such that \( z = A z = T u \). By using (8.1.2), we have
\[
\| A x_{2n} - B u \|_p^p \leq \phi \left\{ \max \left( \| S x_{2n} - T u \|_p^p, \| S x_{2n} - A x_{2n} \|_p^p, \| T u - B u \|_p^p, \right. \right. \\
\left. \left. \| T u - A x_{2n} \|_p^p, \| S x_{2n} - B u \|_p^p \right) \right\}
\]
By letting \( n \to \infty \), since \( \phi \in F \), we obtain
\[
\| z - B u \|_p^p \leq \phi \left\{ \max \left( 0, 0, \| z - B u \|_p^p, 0, \| z - B u \|_p^p \right) \right\} \\
\leq \| z - B u \|_p^p
\]
which implies that \( z = B u \). Since \( B \) and \( T \) are \( C C \) mappings and \( T u = B u = z \), by Lemma 1, \( T B u = B T u \) and hence \( T z = T B u = B T u = B z \). Moreover by (8.1.2), we have
\[
\| A x_{2n} - B z \|_p^p \leq \phi \left\{ \max \left( \| S x_{2n} - T z \|_p^p, \| S x_{2n} - A x_{2n} \|_p^p, \| T z - B z \|_p^p, \right. \right. \\
\left. \left. \| T z - A x_{2n} \|_p^p, \| S x_{2n} - B z \|_p^p \right) \right\}
\]
By letting \( n \to \infty \), since \( \phi \in F \), we obtain
\[
\| z - B z \|_p^p \leq \phi \left\{ \max \left( \| z - B z \|_p^p, 0, \| z - B z \|_p^p, \| B z - z \|_p^p, \| z - B z \|_p^p \right) \right\} \\
\leq \| z - B z \|_p^p
\]
which means that \( z = B z \). Since \( B(X) \subset S(X) \), there exist a point \( v \in X \) such that \( z = B z = S v \). By using (8.1.2), we have
\[ \| Sv - z \|^p = \| AAx_{2n} - Bz \|^p \]
\[ \leq \phi \{ \max ( \| SAx_{2n} - Tz \|^p, \| SAx_{2n} - AAx_{2n} \|^p, \| Tz - Bz \|^p, \| Tz - AAx_{2n} \|^p, \| SAx_{2n} - Bz \|^p ) \} \]

By letting \( n \to \infty \), since \( \phi \in F \), we obtain
\[ \| Sv - z \|^p \leq \phi \{ \max ( \| Sv - z \|^p, 0, 0, \| z - Sv \|^p, \| Sv - z \|^p ) \} \]
\[ \leq \| Sv - z \|^p \]
so that \( Sv = z \). Since \( A \) and \( S \) are CC-mappings and \( Sv = Av = z \). \( SAv = ASv \) and so \( Az = ASz = SAz = Sz \). Therefore, \( z \) is a common fixed point of \( A, B, S \) and \( T \).

Similarly, we can also the proof of the same conclusion, when \( B \) or \( T \) is continuous.

It follows from (8.1.2) that \( z \) is a unique common fixed point of \( A, B, S \) and \( T \). This complete the proof.

8.2 Before stating our next theorem, we need the following:

Let \( A, B, S \) and \( T \) be mappings from a normed space \( (X, \| \cdot \|) \) into itself such that

\[ A(X) \subset T(X) \text{ and } B(X) \subset S(X) \]

\[ \| Ax - By \|^p \leq \phi \{ \max ( \| Sx - Ty \|^p, \| Sx - Ax \|^p, \| Ty - By \|^p, \| Sx - By \|^p, \| Ty - Ax \|^p, \| Sx - Ax \|^p, \| Ty - By \|^p, \| Sx - By \|^p, \| Ty - Ax \|^p ) \} \]
for all \( x, y \) in \( X \), where \( P \geq 1 \) and \( \phi \in F \). Then by (8.2.1), since \( A(X) \subset T(X) \), for an arbitrary point \( x_0 \in X \) there exists a point \( x_1 \in X \), such that \( Ax_0 = Tx_1 \). Since \( B(X) \subset S(X) \), for this point \( x_1 \in X \), we can choose a point \( x_2 \in X \), such that \( Bx_1 = Sx_2 \) and so on. Inductively, we can define a sequence \( \{y_n\} \) in \( X \) such that

\[
y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \quad \text{for every} \quad n = 0, 1, 2, \ldots.
\]

**Lemma 5.** Let \( A, B, S \) and \( T \) be mappings from a normed space \( (X, ||||) \) into itself satisfying the condition (8.2.1) and (8.2.2). Then \( \lim_{n \to \infty} ||y_n - y_{n+1}|| = 0 \), where \( \{y_n\} \) is a sequence in \( X \) defined by (8.2.3).

**Proof.** By (8.2.2), we have

\[
||Ax_{2n} - Bx_{2n+1}||^2 \leq \phi \{ \max (||Sx_{2n} - Tx_{2n+1}||^2, ||Sx_{2n} - Ax_{2n}||^2/2, ||Tx_{2n+1} - Bx_{2n+1}||^2),
\]

\[
||Sx_{2n} - Bx_{2n+1}||^2/2, ||Tx_{2n+1} - Ax_{2n}||^2/2,
\]

\[
||Sx_{2n} - Ax_{2n}||^2/2, ||Tx_{2n+1} - Bx_{2n+1}||^2/2,
\]

\[
||Sx_{2n} - Bx_{2n+1}||^2/2, ||Tx_{2n+1} - Ax_{2n}||^2/2 \}
\]

or

\[
||y_{2n} - y_{2n+1}||^2 \leq \phi \{ \max (||y_{2n-1} - y_{2n}||^2, ||y_{2n-1} - y_{2n}||^2/2, ||y_{2n} - y_{2n+1}||^2/2),
\]

\[
||y_{2n-1} - y_{2n+1}||^2/2, ||y_{2n} - y_{2n}||^2/2, ||y_{2n-1} - y_{2n}||^2/2,
\]

\[
||y_{2n} - y_{2n+1}||^2/2, ||y_{2n-1} - y_{2n+1}||^2/2, ||y_{2n} - y_{2n}||^2/2 \}
\]

If \( ||y_{2n} - y_{2n+1}|| > ||y_{2n-1} - y_{2n}|| \) in the above inequality, then we have
\[ \| y_{2n} - y_{2n+1} \|^p \leq \phi \left( \max \left( \| y_{2n} - y_{2n+1} \|^p, \| y_{2n} - y_{2n+1} \|^p, 0, \| y_{2n} - y_{2n+1} \|^p, 0 \right) \right) \]

= \phi \left( \| y_{2n} - y_{2n+1} \|^p \right),

which is a contradiction. Thus, we have

\[ \| y_{2n} - y_{2n+1} \|^p \leq \phi \left( \| y_{2n} - y_{2n+1} \|^p \right) \]

Similarly we have \( \| y_{2n+1} - y_{2n+2} \|^p \leq \phi \left( \| y_{2n} - y_{2n+1} \|^p \right) \)

But this follows that

\[ \| y_n - y_{n+1} \|^p \leq \phi \left( \| y_n - y_n \|^p \right) \leq \cdots \cdots \leq \phi^n \left( \| y_0 - y_1 \|^p \right) \]

and so from Lemma 2, we have

\[ \lim_{n \to \infty} \| y_n - y_{n+1} \| = 0 \]

This completes the proof.

**Lemma 6.** Let \( A, B, S \) and \( T \) be mappings from a normed space \( (X, \| \|) \) into itself satisfying the condition \((8.2.1)\) and \((8.2.2)\). Then the sequence \( \{y_n\} \) defined by \((8.2.3)\) is a Cauchy sequence in \( X \).

**Proof.** Following the proof of Lemma 4, and we have

\[ \| Y_{2n(k)} - Y_{2m(k)} \| \leq \| Y_{2n(k)} - Y_{2n(k)+1} \| + \| Y_{2n(k)+1} - Y_{2m(k)} \| \]

\[ \quad = \| Y_{2n(k)} - Y_{2n(k)+1} \| + \| Ax_{2n(k)} - Bx_{2n(k)+1} \| \]

\[ \quad \leq \| Y_{2n(k)} - Y_{2n(k)+1} \| + \phi \left( \max \left( \| Sx_{2n(k)} - Tx_{2n(k)+1} \|^p, \| Sx_{2n(k)} - Ax_{2n(k)} \|^\frac{p}{2}, \| Tx_{2n(k)+1} - Ax_{2n(k)+1} \|^\frac{p}{2}, \| Sx_{2n(k)} - Bx_{2n(k)+1} \|^\frac{p}{2}, \| Tx_{2n(k)+1} - Ax_{2n(k)+1} \|^\frac{p}{2} \right) \]
\[
\begin{align*}
\| S_{2m(k)} - A_{2m(k)} \|^{p/2} &\leq \| T_{2m(k)+1} - B_{2m(k)+1} \|^{p/2}, \\
\| S_{2m(k)} - B_{2m(k)+1} \|^{p/2} &\leq \| T_{2m(k)+1} - A_{2m(k)} \|^{p/2}\}
\end{align*}
\]

\[
\begin{align*}
\phi \{ \max (\| y_{2m(k)+1} - y_{2m(k)} \|^{p}, \\
\| y_{2m(k)-1} - y_{2m(k)} \|^{p/2}, \\
\| y_{2m(k)-1} - y_{2m(k)+1} \|^{p/2}, \\
\| y_{2m(k)-1} - y_{2m(k)} \|^{p/2} \} \}^{Y_p}
\end{align*}
\]

Since \( \phi \) is upper semi continuous, as \( k \to \infty \), we have

\[
\begin{align*}
\varepsilon &\leq \phi \{ \max (\varepsilon^p, 0, \varepsilon^p, 0, \varepsilon^p) \}^{Y_p} \\
&\leq \phi(\varepsilon^p)^{Y_p} \\
&< \varepsilon
\end{align*}
\]

which is a contradiction. Therefore, \( \{y_{2n}\} \) is a Cauchy sequence in \( X \) and so is \( \{y_n\} \).

This completes the proof.

Our final theorem is as follows:

**THEOREM 2.** Let \( A, B, S \) and \( T \) be mappings from a Banach space \( (X, \| \cdot \|) \) into itself satisfying the conditions (8.2.1) and (8.2.2). Suppose that one of \( A, B, S \) and \( T \) is continuous and the pairs \( \{A, S\} \) and \( \{B, T\} \) are CC-mappings. Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**PROOF.** Let \( \{y_n\} \) be the sequence in \( X \) defined by (8.2.3). By Lemma 6, \( \{y_n\} \) is a Cauchy sequence in \( X \) and hence it is converges to a point \( z \) in \( X \). Consequently
subsequences \{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\}, and \{Tx_{2n+1}\}, of \{y_n\} also converges to the point z.

Now, suppose that A is continuous, since A and S are CC-mappings, it follows from Lemma 1, that \(ASx_{2n}, SSx_{2n} \rightarrow Az\) as \(n \rightarrow \infty\)

By (8.2.2), we have

\[
\| ASx_{2n} - Bx_{2n+1} \|_p \leq \phi \{ \max (\| SSx_{2n} - Tx_{2n+1} \|_p, \\
\| SSx_{2n} - ASx_{2n} \|_{\frac{p}{2}} \| Tx_{2n+1} - Bx_{2n+1} \|_{\frac{p}{2}}, \\
\| SSx_{2n} - ASx_{2n} \|_{\frac{p}{2}} \| Tx_{2n+1} - ASx_{2n} \|_{\frac{p}{2}}, \\
\| SSx_{2n} - ASx_{2n} \|_{\frac{p}{2}} \| Tx_{2n+1} - ASx_{2n} \|_{\frac{p}{2}} ) \}
\]

By letting \(n \rightarrow \infty\) in the above inequality, we have

\[
\| Az - z \|_p \leq \phi \{ \max (\| Az - z \|_p, 0, \| Az - z \|_p, 0, 0 ) ) \}
\]
i.e.

\[
\| Az - z \| \leq [ \phi \{ \max (\| Az - z \|_p, 0, \| Az - z \|_p, 0, 0 ) \} ]^{\frac{1}{p}}
\]

\[
\leq [ \phi \{ \| Az - z \|_p \} ]^{\frac{1}{p}}
\]

\[
< \| Az - z \|
\]

which is a contradiction. Thus we have \(Az = z\), Since \(A(X) \subset T(X)\), there exists \(u \in X\) such that \(z = Az = Tu\). Again by (8.2.2), we have

\[
\| ASx_{2n} - Bu \|_p \leq \phi \{ \max (\| SSx_{2n} - Tu \|_p, \| SSx_{2n} - ASx_{2n} \|_{\frac{p}{2}} \\
\| Tu - Bu \|_{\frac{p}{2}}, \| SSx_{2n} - Bu \|_{\frac{p}{2}} \| Tu - ASx_{2n} \|_{\frac{p}{2}}, \\
\| SSx_{2n} - ASx_{2n} \|_{\frac{p}{2}} \| Tu - Bu \|_{\frac{p}{2}}, \\
\| SSx_{2n} - Bu \|_{\frac{p}{2}} \| Tu - ASx_{2n} \|_{\frac{p}{2}} ) \}
\]
By letting $n \to \infty$, since $\phi \in F$, we obtain

$$\| z - Bu \|_p^p \leq \phi \{ \max 0, 0, 0, 0, 0 \} = \phi(0) = 0$$

which implies that $z = Bu$. Since $B$ and $T$ are $CC$ - mappings and $Tu = Bu = z$, by Lemma 1. $TBu = BTu$ and hence $Tz = TBu = BTu = Bz$. Moreover, by (8.2.2), we have

$$\| Ax_{2n} - Bz \|_p^p \leq \phi \{ \max (\| Sx_{2n} - Tz \|_p^p, \| Sx_{2n} - Ax_{2n} \|^{p/2}_2, \| Tz - Bz \|^{p/2}_2, \| Sx_{2n} - Bz \|^{p/2}_2, \| Tz - Ax_{2n} \|^{p/2}_2, \| Sx_{2n} - Ax_{2n} \|^{p/2}_2, \| Tz - Bz \|^{p/2}_2) \}$$

By letting $n \to \infty$, in the above inequality, we obtain

$$\| z - Bz \|_p^p \leq \phi \{ \max \| z - Bz \|_p^p, 0, \| z - Bz \|_p^p, 0, 0 \}$$

i.e. $$\| z - Bz \| \leq \left[ \phi \{ \max \| z - Bz \|_p^p, 0, \| z - Bz \|_p^p, 0, 0 \} \right]^{1/p}$$

$$\leq \left[ \phi \{ \| z - Bz \|_p^p \} \right]^{1/p}$$

$$< \| z - Bz \|$$

which means that $z=Bz$. Since $B(X) \subset S(X)$, there exists a point $v \in X$ such that $z = Bz = Sv$. By (8.2.2), we have

$$\| Av - z \|_p^p = \| Av - Bz \|_p^p \leq \phi \{ \max (\| Sv - Tz \|_p^p, \| Su - Av \|^{p/2}_2, \| Tz - Bz \|^{p/2}_2, \| Sv - Bz \|^{p/2}_2, \| Tz - Av \|^{p/2}_2, \| Sv - Av \|^{p/2}_2, \| Tz - Bz \|^{p/2}_2, \| Sv - Bz \|^{p/2}_2, \| Tz - Av \|^{p/2}_2) \}$$

$$\leq \phi \{ \max (0, 0, 0, 0, 0) \} = \phi(0) = 0$$
so that $Av = z$, since $A$ and $S$ are CC-mappings and $Av = Sv = z$, we have $SAv = ASv$ and hence $Sz = SAv = ASz = Az$. Therefore, $z$ is a common fixed point of $A$, $B$, $S$ and $T$.

Now, suppose that $S$ is continuous, since $A$ and $S$ are CC-mappings, it follows from Lemma 1 that $AA_{2n}, SA_{2n} \to Sz$ as $n \to \infty$.

By (8.2.2), we have

$$
|| AA_{2n} - Bx_{2n+1} ||^p \leq \phi \{ \max ( || SA_{2n} - Tx_{2n+1} ||^p , \\
|| SA_{2n} - AAx_{2n} ||^{p/2} || Tx_{2n+1} - Bx_{2n+1} ||^{p/2} , \\
|| SA_{2n} - Bx_{2n+1} ||^{p/2} || Tx_{2n+1} - AAx_{2n} ||^{p/2} , \\
|| SA_{2n} - AAx_{2n} ||^{p/2} || Tx_{2n+1} - Bx_{2n+1} ||^{p/2} , \\
|| SA_{2n} - Bx_{2n+1} ||^{p/2} || Tx_{2n+1} - AAx_{2n} ||^{p/2} ) \} 
$$

By letting $n \to \infty$, we have

$$
|| Sz - z ||^p \leq \phi \{ \max ( || Sz - z ||^p , 0 , || Sz - z ||^p , 0 , || Sz - z ||^p ) \}
$$

i.e.

$$
|| Sz - z || \leq [ \phi \{ \max ( || Sz - z ||^p , 0 , || Sz - z ||^p , 0 , || Sz - z ||^p ) \} ]^{1/p}
$$

$$
\leq [ \phi \{ || Sz - z ||^p \} ]^{1/p}
$$

$$
< || Sz - z ||
$$

which is a contradiction. Thus we have $Sz = z$. Since $B(X) \subset S(X)$, there exists a point $u \in X$ such that $z = Sz = Bu$. Again by (8.2.2) we have

$$
|| Az - Bx_{2n+1} ||^p \leq \phi \{ \max ( || Sz - Tx_{2n+1} ||^p , || Sz - Az ||^{p/2} || Tx_{2n+1} - Bx_{2n+1} ||^{p/2} , \\
|| Sz - Bx_{2n+1} ||^{p/2} || Tx_{2n+1} - Az ||^{p/2} .
$$
\[ \| S_{z} - A_{z} \|_{2}^{\phi} \leq \| T_{x_{2n+1}} - B_{x_{2n+1}} \|_{2}^{\phi}, \]
\[ \| S_{z} - B_{x_{2n+1}} \|_{2}^{\phi} \leq \| T_{x_{2n+1}} - A_{z} \|_{2}^{\phi} \]}

By letting \( n \to \infty \), since \( \phi \in F \) we obtain
\[ \| A_{z} - z \|^{p} \leq \phi \{ \max 0, 0, 0, 0, 0 \} = \phi(0) = 0 \]
which means that \( z = A_{z} \). Since \( A(X) \subset T(X) \), there exists a point \( u \in X \) such that \( z = A_{z} = T_{u} \). By using (8.2.2), we have
\[ \| A_{x_{2n}} - B_{u} \|^{p} \leq \phi \{ \max ( \| S_{x_{2n}} - T_{u} \|^{p}, \| S_{x_{2n}} - A_{x_{2n}} \|_{2}^{\phi} \| T_{u} - B_{u} \|_{2}^{\phi}, \]
\[ \| S_{x_{2n}} - B_{u} \|_{2}^{\phi} \| T_{u} - A_{x_{2n}} \|_{2}^{\phi}, \]
\[ \| S_{x_{2n}} - A_{x_{2n}} \|_{2}^{\phi} \| T_{u} - B_{u} \|_{2}^{\phi}, \]
\[ \| S_{x_{2n}} - B_{u} \|_{2}^{\phi} \| T_{u} - A_{x_{2n}} \|_{2}^{\phi} \} \]

By letting \( n \to \infty \), since \( \phi \in F \), we obtain
\[ \| z - B_{u} \|^{p} \leq \phi \{ 0, 0, 0, 0, 0 \} = \phi(0) = 0, \]
which implies that \( z = B_{u} \). Since \( B \) and \( T \) are CC-mappings and \( T_{u} = B_{u} = z \), by Lemma 1, \( TB_{u} = B_{T_{u}} \) and hence \( T_{z} = T_{B_{u}} = B_{T_{u}} = B_{z} \). Moreover, by (8.2.2), we have
\[ \| A_{x_{2n}} - B_{z} \|^{p} \leq \phi \{ \max ( \| S_{x_{2n}} - T_{z} \|^{p}, \| S_{x_{2n}} - A_{x_{2n}} \|_{2}^{\phi} \| T_{z} - B_{z} \|_{2}^{\phi}, \]
\[ \| S_{x_{2n}} - B_{z} \|_{2}^{\phi} \| T_{z} - A_{x_{2n}} \|_{2}^{\phi}, \]
\[ \| S_{x_{2n}} - A_{x_{2n}} \|_{2}^{\phi} \| T_{z} - B_{z} \|_{2}^{\phi}, \]
\[ \| S_{x_{2n}} - B_{z} \|_{2}^{\phi} \| T_{z} - A_{x_{2n}} \|_{2}^{\phi} \} \]
By letting \( n \to \infty \), since \( \phi \in \mathcal{F} \), we obtain

\[
\| z - Bz \|^p \leq \phi \left\{ \max \left( \| z - Bz \|^p, 0, \| z - Bz \|^p, 0, 0 \right) \right\}
\]

i.e.

\[
\| z - Bz \| \leq \left[ \phi \left\{ \max \left( \| z - Bz \|^p, 0, \| z - Bz \|^p, 0, 0 \right) \right\} \right]^{\frac{1}{p}}
\]

\[
\leq \left[ \phi \left( \| z - Bz \|^p \right) \right]^{\frac{1}{p}}
\]

\[
< \| z - Bz \|
\]

which means that \( z = Bz \). Since \( B(X) \subseteq S(X) \) there exists a point \( v \in X \) such that \( z = Bz = Sv \). By using (8.2.2), we have

\[
\| Sv - z \|^p = \| AAx_{2n} - Bz \|^p
\]

\[
\leq \phi \left\{ \max \left( \| SAx_{2n} - Tz \|^p, \| SAx_{2n} - AAx_{2n} \|^\frac{p}{2}, \| Tz - Bz \|^\frac{p}{2}, \| SAx_{2n} - Bz \|^\frac{p}{2}, \| Tz - AAx_{2n} \|^\frac{p}{2} \right) \right\}
\]

By letting \( n \to \infty \), since \( \phi \in \mathcal{F} \), we obtain

\[
\| Sv - z \|^p \leq \phi \left\{ \max \left( \| Sv - z \|^p, 0, \| Sv - z \|^p, 0, 0 \right) \right\}
\]

i.e.

\[
\| Sv - z \| \leq \left[ \phi \left\{ \max \left( \| Sv - z \|^p, 0, \| Sv - z \|^p, 0, 0 \right) \right\} \right]^{\frac{1}{p}}
\]

\[
\leq \left[ \phi \left( \| Sv - z \|^p \right) \right]^{\frac{1}{p}}
\]

\[
< \| Sv - z \|
\]
so that $Sv = z$. Since $A$ and $S$ are $CC$-mappings and $Sv = Av = z$, $SAv = ASv$ and so $Az = ASz = SAz = Sz$. Therefore $z$ is a common fixed point of $A$, $B$, $S$ and $T$.

Similarly, we can also the proof of the same conclusion when $B$ or $T$ is continuous. It follows from (8.2.2) that $z$ is a unique common fixed point of $A$, $B$, $S$ and $T$. This completes the proof.