CHAPTER-VII

FIXED POINT THEOREMS IN NON-ARCHIMEDEAN
MENGER PM - SPACES

7.1 The aim of this section is to obtain a common fixed point theorem in non-
Archimedean Menger PM-space. Our result includes several results for commuting,
weakly commuting and compatible mappings on metric spaces and PM-spaces given
by Ciric [3], Jungck ([2], [3]), Jungck, Murthy and Cho [1], Sehgal and Bharucha
- Reid [1], Sessa [1], Singh and Pant [1], Stojakovic [1] and Cho, Ha and Chang [1].

Throughout this chapter, let \((X, f, \Delta)\) be a complete N.A. Menger PM - space
of type \((D)_g\) with a continuous strictly increasing \(t\) - norm \(\Delta\).

Let \(\phi: [0, +\infty) \to [0, +\infty)\) be a function satisfying the following condition

\[
(7.1.1) \quad \phi \text{ is upper - semi continuous from the right and } \phi(t) < t \text{ for all } t > 0.
\]

Before stating the theorem, we need the following :

**LEMMA 1.** Let \(A, S : X \to X\) be mappings. If \(A\) and \(S\) are compatible of type \((A)\)
and \(Sz = Az\) for some \(z \in X\), then \(SAz = AAz = ASz = SSz\).

**LEMMA 2.** Let \(A, S : X \to X\) be compatible mappings of type \((A)\) and let \(\{x_n\}\) be
a sequence in \(X\) such that

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z \text{ for some } z \in X.
\]

Then we have the following
(7.1.2) \[
\lim_{n \to \infty} ASx_n = Sz \text{ if } S \text{ is continuous at } z.
\]

(7.1.3) \[
SAz = ASz \text{ and } Sz = Az \text{ if } A \text{ and } S \text{ are continuous at } z.
\]

**Lemma 3.** If a function \( \phi : [0, +\infty) \to [0, +\infty) \) satisfies the condition (7.1.1), then we have

(7.1.4) \[
\lim_{n \to \infty} \phi^n(t) = 0, \text{ where } \phi^n(t) \text{ is the } n^{th} \text{ iteration of } \phi(t).
\]

(7.1.5) If \( \{t_n\} \) is a non-decreasing sequence of real numbers and \( t_{n+1} \leq \phi(t_n) \),

\[
\text{then } \lim_{n \to \infty} t_n = 0.
\]

In particular, if \( t \leq \phi(t) \) for all \( t \geq 0 \), then \( t = 0 \).

**Lemma 4.** Let \( \{y_n\} \) be a sequence in \( X \). Such that \( \lim_{n \to \infty} Fy_n, y_{n+1}(t) = 1 \) for all \( t > 0 \). If the sequence \( \{y_n\} \) is not a Cauchy sequence in \( X \), then there exist \( \epsilon > 0 \), \( t_0 > 0 \), two sequences \( \{m_i\}, \{n_i\} \) of positive integers such that

(7.1.6) \[
m_i > n_i + 1 \text{ and } n_i \to \infty \text{ as } i \to \infty,
\]

(7.1.7) \[
Fy_{m_i}, y_{m_i}(t_0) < 1 - \epsilon_0 \text{ and } Fy_{m_i-1}, y_{m_i-1}(t_0) \geq 1 - \epsilon_0, \text{ i = 1, 2,} \ldots
\]

Let \( A, B, S, T : X \to X \) be mappings such that

(7.1.8) \[
A(X) \subset T(X) \text{ and } B(X) \subset S(X)
\]

(7.1.9) \[
g(F_{Ax, By}(t)) < \phi \left\{ \max \{ g(F_{Sx, Ty}(t)), g(F_{Sx, Ax}(t)), g(F_{Ty, By}(t)) \right\}
\]

\[
\frac{1}{2} \left( g(F_{Sx, By}(t)) + g(F_{Ty, Ax}(t)) \right),
\]

\[
\frac{1}{2} \left( g(F_{Sx, Ax}(t)) + g(F_{Ty, By}(t)) \right)
\]
for all \( t > 0 \), where a function \( \phi : [0, +\infty) \to [0, +\infty) \) satisfies the condition (7.1.1). Then by (7.1.8), since \( A(X) \subseteq T(X) \), for any \( x_0 \in X \), there exists a point \( x_1 \in X \) such that \( Ax_0 = Tx_1 \). Since \( B(X) \subseteq S(X) \), for this point \( x_1 \), we can choose a point \( x_2 \in X \) such that \( Bx_1 = Sx_2 \) and so on. Inductively we can define a sequence \( \{y_n\} \) in \( X \) such that

\[
(7.1.10) \quad y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}

\text{for } n = 0, 1, 2, \ldots \ldots \ldots \ldots
\]

**Lemma 5.** Let \( A, B, S, T : X \to X \) be mappings satisfying the conditions (7.1.8) and (7.1.9), then the sequence \( \{y_n\} \) defined by (7.1.10) such that

\[
\lim_{n \to \infty} g(Fy_n, y_{n+1}(t)) = 0, \text{ for all } t > 0
\]

is a Cauchy sequence in \( X \).

**Proof.** Since \( g \in \Omega \), it follows that \( \lim_{n \to \infty} Fy_n, y_{n+1}(t) = 1 \) for all \( t > 0 \), if and only if \( \lim_{n \to \infty} g(Fy_n, y_{n+1}(t)) = 0 \), for all \( t > 0 \).

By Lemma 4, if \( \{y_n\} \) is not a Cauchy sequence in \( X \), there exist \( \epsilon_0 > 0 \), \( t_0 > 0 \) and two sequences \( \{m_i\}, \{n_i\} \) of positive integers such that (7.1.6) holds and

\[
(7.1.11) \quad g(Fy_m, y_n(t_0)) > g(1 - \epsilon_0) \quad \text{and} \quad g(Fy_{m-1}, y_n(t_0)) \leq g(1 - \epsilon_0),
\]

\( i = 1, 2, 3, \ldots \ldots \ldots \ldots \)

Thus we have
\[ g (1 - \varepsilon_0) < g (Fy_{m_i}, y_{m_i} (t_0)) \]
\[ \leq g (Fy_{m_i}, y_{m_i-1} (t_0)) + g (Fy_{m_i-1}, y_{m_i-1} (t_0)) \]

(7.1.12) \[ \leq g (1 - \varepsilon_0) + g (Fy_{m_i}, y_{m_i-1} (t_0)) \]

Letting \( i \to \infty \) in (7.1.12), we have

(7.1.13) \[ \lim_{i \to \infty} g (Fy_{m_i}, y_{m_i} (t_0)) = g (1 - \varepsilon_0) \]

On the other hand, we have

(7.1.14) \[ g (1 - \varepsilon_0) < g (Fy_{m_i}, y_{m_i} (t_0)) \]
\[ \leq g (Fy_{m_i}, y_{m_i+1} (t_0)) + g (Fy_{m_i+1}, y_{m_i} (t_0)) \]

Now, consider \( g (Fy_{m_i+1}, y_{m_i} (t_0)) \) in (7.1.14).

Without loss of generality, assume that both \( n_i \) and \( m_i \) are even. Then by condition (7.1.9), we have

\[ g (Fy_{m_i+1}, y_{m_i} (t_0)) = g (FAX_{m_i}, B_{m_i+1} (t_0)) \]
\[ \leq \phi [ \max \{ g (FSX_{m_i}, TX_{m_i+1} (t_0)), \]
\[ g (FSX_{m_i}, AX_{m_i} (t_0)), g (FTX_{m_i+1}, BX_{m_i+1} (t_0)) ] \]

(7.1.15) \[ \frac{1}{2} (g (FSX_{m_i}, BX_{m_i+1} (t_0) + g (FTX_{m_i+1}, AX_{m_i} (t_0))), \]
\[ \frac{1}{2} (g (FSX_{m_i}, AX_{m_i} (t_0) + g (FTX_{m_i+1}, BX_{m_i+1} (t_0)))) ] \]
\[ \leq \phi \left[ \max \{ g\left(Fy_{m-1}, y_n(t_0)\right), \right. \]
\[ g\left(Fy_{m-1}, y_n(t_0)\right), g\left(Fy_n, y_{n+1}(t_0)\right), \]
\[ \left. \frac{1}{2} \left( g\left(Fy_{m-1}, y_{n+1}(t_0)\right) + g\left(Fy_n, y_m(t_0)\right)\right) \right\} \]

by (7.1.13), (7.1.14) and (7.1.15), letting \( i \to \infty \) in (7.1.15), we have
\[ g\left(1-\varepsilon_0\right) \leq \phi \left[ \max \{ g\left(1-\varepsilon_0\right), 0, 0, g\left(1-\varepsilon_0\right), 0 \} \right] \]
\[ = \phi \left( g\left(1-\varepsilon_0\right) \right) \]
\[ < g\left(1-\varepsilon_0\right), \]

which is a contradiction. Therefore, \( \{y_n\} \) is a Cauchy sequence in \( X \).

Now, we are ready to give our theorem.

**THEOREM 1.** Let \( A, B, S, T : X \to X \) be mappings satisfying the conditions (7.1.8), (7.1.9), (7.1.16) and (7.1.17)

(7.1.16) \( S \) and \( T \) is continuous,

(7.1.17) the pair \( A, S \) and \( B, T \) are compatible of type (A). Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**PROOF.** If we prove \( \lim_{n \to \infty} g\left(Fy_n, y_{n+1}(t)\right) = 0 \) for all \( t > 0 \), then by Lemma 5, the sequence \( \{y_n\} \) defined by (7.1.10) is a Cauchy sequence in \( X \).
Now, we prove $\lim_{n \to \infty} g(Fy_n, y_{n+1}(t)) = 0$ for all $t > 0$.

In fact by (7.1.9) and (7.1.10), we have

$$g(Fy_{2n}, y_{2n+1}(t)) = g(FAx_{2n}, Bx_{2n+1}(t)),$$

$$\leq \phi \left[ \max \left\{ g(FSx_{2n}, Tx_{2n+1}(t)),
\begin{array}{ll}
g(FSx_{2n}, Ax_{2n}(t)), & g(FTx_{2n+1}, Bx_{2n+1}(t)), \\
\frac{1}{2} \left( g(FSx_{2n}, Bx_{2n+1}(t)) + g(FTx_{2n+1}, Bx_{2n}(t)) \right), \\
\frac{1}{2} \left( g(FSx_{2n}, Ax_{2n}(t)) + g(FTx_{2n+1}, Bx_{2n+1}(t)) \right) \right\} \right]$$

$$\leq \phi \left[ \max \left\{ g(Fy_{2n-1}, y_{2n}(t)), g(Fy_{2n-1}, y_{2n}(t)),
\begin{array}{ll}
g(Fy_{2n}, y_{2n+1}(t)), & \frac{1}{2} \left( g(Fy_{2n-1}, y_{2n+1}(t)) + g(Fy_{2n}, y_{2n+1}(t)) \right), \\
\frac{1}{2} \left( g(Fy_{2n-1}, y_{2n}(t)) + g(Fy_{2n}, y_{2n+1}(t)) \right) \right\} \right]$$

$$\leq \phi \left[ \max \left\{ g(Fy_{2n-1}, y_{2n}(t)), g(Fy_{2n}, y_{2n+1}(t)),
\begin{array}{ll}
\frac{1}{2} \left( g(Fy_{2n-1}, y_{2n}(t)) + g(Fy_{2n}, y_{2n+1}(t)) \right), \\
\frac{1}{2} \left( g(Fy_{2n-1}, y_{2n}(t)) + g(Fy_{2n}, y_{2n+1}(t)) \right) \right\} \right]$$

If $g(Fy_{2n-1}, y_{2n}(t)) \leq g(Fy_{2n}, y_{2n+1}(t))$ for all $t > 0$, 

Then, by (7.1.9),
\[ g(Fy_{2n}, y_{2n+1}(t)) \leq \phi \left( g(Fy_{2n}, y_{2n+1}(t)) \right), \]
which means that, by Lemma 3, \( g(Fy_{2n}, y_{2n+1}(t)) = 0 \) for all \( t > 0 \). Similarly, we have \( g(Fy_{2n+1}, y_{2n+2}(t)) = 0 \) for all \( t > 0 \).

Thus we have
\[ \lim_{n \to \infty} g(Fy_n, y_{n+1}(t)) = 0 \quad \text{for all} \quad t > 0. \]

On the other hand, if \( g(Fy_{2n+1}, y_{2n}(t)) \geq g(Fy_{2n}, y_{2n+1}(t)) \), then, by (7.1.9), we have \( g(Fy_{2n}, y_{2n+1}(t)) \leq \phi \left( g(Fy_{2n-1}, y_{2n}(t)) \right) \), for all \( t > 0 \).

Similarly, \( g(Fy_{2n+1}, y_{2n+2}(t)) \leq g(Fy_{2n}, y_{2n+1}(t)) \), for all \( t > 0 \).

Thus we have \( g(Fy_n, y_{n+1}(t)) \leq \phi \left( g(Fy_{n-1}, y_{n}(t)) \right) \) for all \( t > 0 \) and \( n = 1, 2, 3, \ldots \).

Therefore by Lemma 3, \( \lim_{n \to \infty} g(Fy_n, y_{n+1}(t)) = 0 \), for all \( t > 0 \),

which implies that \( \{y_n\} \) is a Cauchy sequence in \( X \) by Lemma 5. Since \( (X, f, \Delta) \) is complete, the sequence \( \{y_n\} \) converges to a point \( z \in X \) and so the subsequence \( \{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\}, \{Tx_{2n+1}\} \) of \( \{y_n\} \) also converges to the limit \( z \).

Now, suppose that \( T \) is continuous. Since \( B \) and \( T \) are compatible of type (A), by Lemma 2, \( BTx_{2n+1}, TTx_{2n+1}, \to Tz \) as \( n \to \infty \).

Putting \( x = x_{2n} \) and \( y = Tx_{2n+1} \) in (7.1.9), we have
\[ g(FAx_{2n}, BTx_{2n+1}(t)) \leq \phi \left[ \max \left( g(FSx_{2n}, TTx_{2n+1}(t)) \right) \right], \]
\[ g(FSx_{2n}, Ax_{2n}(t)), g(FTTx_{2n+1}, BTx_{2n+1}(t)), \]
\[ 1 \over 2 \left( g(FS_{2n}, BT_{2n+1}(t)) + g(FTT_{2n+1}, Ax_{2n}(t)) \right), \]

\[ 1 \over 2 \left( g(FS_{2n}, Ax_{2n}(t)) + g(FTT_{2n+1}, Bx_{2n+1}(t)) \right) \]

for all \( t > 0 \). Letting \( n \to \infty \) in (7.1.18), we have

\[ g(Fz, Tz, (t)) \leq \phi \left[ \max \{ g(Fz, Tz(t)), g(Fz, z(t)), g(Fz, Tz(t)) \right] \]

\[ = \phi(g(Fz, Tz(t))) \]

for all \( t > 0 \), which means that \( g(Fz, Tz(t)) = 0 \) for all \( t > 0 \), by Lemma 3 and so we have \( Tz = z \). Again, replacing \( x \) by \( x_{2n} \) and \( y \) by \( z \) in (7.1.9), we have

\[ g(FAx_{2n}, Bz(t)) \leq \phi \left[ \max \{ g(FS_{2n}, Tz(t)), g(FS_{2n}, Ax_{2n}(t)), g(Fz, Bz(t)), \right] \]

\[ 1 \over 2 \left( g(FS_{2n}, Bz(t)) + g(FTz, Ax_{2n}(t)) \right), \]

\[ 1 \over 2 \left( g(FS_{2n}, Ax_{2n}(t)) + g(FTZ, Bz(t)) \right) \]
for all $t > 0$, which implies that $g(Fz, Bz(t)) \leq \phi(g(Fz, Bz(t)))$ for all $t > 0$ and so we have $Bz = z$. Since $B(X) \subseteq S(X)$, there exists a point $w \in X$ such that $Bz = Sw = z$.

By using (7.1.9) again, we have

$$g(FAw, z(t)) = g(FAw, Bz(t))$$

$$< \phi[\max\{g(FSw, Tz(t)), g(FSw, Aw(t)), g(FTz, Bz(t)), 1/2(g(FSw, Bz(t)) + g(FTz, Aw(t))), 1/2(g(FSw, Aw(t)) + g(FTz, Bz(t)))\}]$$

$$\leq \phi(g(FAw, z(t)))$$

for all $t > 0$. Which means that $Aw = z$.

Since $A$ and $S$ are compatible mappings of type (A) and $Aw = Sw = z$, by Lemma 1, $Az = ASw = SSw = Sz$. Again by using (7.1.9) we have $Az = z$. Therefore $Az = Bz = Sz = Tz = z$, that is, $z$ is a common fixed point of the mappings $A, B, S, T$. The uniqueness of the common fixed point $z$ follows easily from (7.1.9).

7.2 Now, we prove another analogous result for different type of mappings.

Before stating it, we require the following:

**Lemma 6.** Let $A, B, S, T : X \to X$ be mappings satisfying the condition (7.1.8) and
(7.2.1) \[ g(FAx, By(t)) < \phi \left[ \max \{ g(FSx, Ty(t)), g(FSx, Ax(t)) \} \right. \]
\[ \left. g(FTy, By(t)), \frac{1}{2} g(FSx, By(t)), \right. \]
\[ g(FTy, Ax(t)) \]

for all \( t > 0 \), where a function \( \phi : [0, +\infty) \to [0, +\infty) \) satisfies the condition (7.1.1).

Then the sequence \( \{y_n\} \) defined by (7.1.10) such that \( \lim_{n \to \infty} g(Fy_n, y_{n+1}(t)) = 0 \)
for all \( t > 0 \), is a Cauchy sequence in \( X \).

**PROOF.** Since \( g \in \Omega \), it follows that \( \lim_{n \to \infty} Fy_n, y_{n+1}(t) = 1 \), for all \( t > 0 \), if and only if \( \lim_{n \to \infty} g(Fy_n, y_{n+1}(t)) = 0 \) for all \( t > 0 \). By Lemma 4, if \( \{y_n\} \) is not a Cauchy sequence in \( X \), there exists \( \varepsilon_o > 0, t_0 > 0 \) and two sequences \( \{m_i\}, \{n_i\} \) of positive integers such that (7.1.6) holds and

(7.2.2) \[ g(Fy_{m_i}, y_{n_i}(t_0)) > g(1-\varepsilon_o) \quad \text{and} \quad g(Fy_{m_{i-1}}, y_{n_{i}}(t_0)) \leq g(1-\varepsilon_o), \]
\[ i = 1, 2, 3, \ldots \]

Thus we have

\[ g(1-\varepsilon_o) < g(Fy_{m_i}, y_{n_i}(t_0)) \]
\[ \leq g(Fy_{m_i}, y_{m_{i-1}}(t_0)) + g(Fy_{m_{i-1}}, y_{n_i}(t_0)) \]
\[ \leq g(1-\varepsilon_o) + g(Fy_{m_i}, y_{m_{i-1}}(t_0)) \]

(7.2.3) \[ \lim_{i \to \infty} g(Fy_{m_i}, y_{n_i}(t_0)) = g(1-\varepsilon_o) \]

Letting \( i \to \infty \) in (7.2.3), we have

(7.2.4) \[ \lim_{i \to \infty} g(Fy_{m_i}, y_{n_i}(t_0)) = g(1-\varepsilon_o) \]
On the other hand, we have

\[(7.2.5) \quad g(1-\varepsilon_0) < g(Fy_m, y_n(t_o)) \leq g(Fy_n, y_{n+1}(t_o)) + g(Fy_{n+1}, y_m(t_o)) \]

Now, consider \(g(Fy_{n+1}, y_m(t_o))\) in (7.2.5). Without loss of generality, assume that both \(m_i\) and \(n_i\) are even. Then by condition (7.2.1), we have

\[g(Fy_{n+1}, y_m(t_o)) = g(FAx_m, Bx_{n+1}(t_o)) \leq \Phi \left[ \max \{ g(FSx_m, Tx_{n+1}(t_o)), \ g((FSx_m, Ax_m(t_o)), g(FT_{n+1}, Bx_{n+1}(t_o)), \right] \]

\[(7.2.6) \quad \frac{1}{2} \left( g(FSx_m, Bx_{n+1}(t_o)), g(FT_{n+1}, Ax_m(t_o)) \right) \]

\[= \Phi \left[ \max \{ g(Fy_{m-1}, y_n(t_o)), g(Fy_{m-1}, y_{n+1}(t_o)), \}ight] \]

By (7.2.3), (7.2.4), (7.2.5) and (7.2.6), letting \(i \to \infty\) in (7.2.6), we have

\[g(1-\varepsilon_0) \leq \Phi \left[ \max \{ g(1-\varepsilon_0), 0, 0, \frac{1}{2} g(1-\varepsilon_0), g(1-\varepsilon_0) \} \right] \]

\[= \Phi \left( g(1-\varepsilon_0) \right) < g(1-\varepsilon_0), \]

which is a contradiction. Therefore, \(\{y_n\}\) is a Cauchy sequence in X.

Our theorem is as follows:
**THEOREM 2.** Let $A, B, S, T : X \to X$ be mappings satisfying the conditions (7.1.8), (7.2.1), (7.1.16) and (7.1.17). Then $A, B, S$ and $T$ have a unique common fixed point in $X$.

**PROOF.** If we prove $\lim_{n \to \infty} g(Fy_n, y_{n+1}(t)) = 0$ for all $t > 0$, then by Lemma 6, the sequence $\{y_n\}$ defined by (7.1.10) is a Cauchy sequence in $X$. Now, we prove

$$\lim_{n \to \infty} g(Fy_n, y_{n+1}(t)) = 0 \text{ for all } t > 0.$$

Infact, by (7.2.1), we have

$$g(Fy_{2n}, y_{2n+1}(t)) = g(FAx_{2n}, Bx_{2n+1}(t))$$

$$\leq \phi \left[ \max \{ g(FSx_{2n}, Tx_{2n+1}(t)),
\quad g(FSx_{2n}, Ax_{2n}(t)), g(FTx_{2n+1}, Bx_{2n+1}(t)) \} \right]$$

$$\leq \phi \left[ \max \{ g(Fy_{2n-1}, y_{2n}(t)), g(Fy_{2n-1}, y_{2n}(t)),
\quad g(Fy_{2n}, y_{2n+1}(t)), \frac{1}{2} g(Fy_{2n-1}, y_{2n+1}(t)), g(1) \} \right]$$

$$\leq \phi \left[ \max \{ g(Fy_{2n-1}, y_{2n}(t)), g(Fy_{2n}, y_{2n+1}(t)),
\quad \frac{1}{2} g(Fy_{2n-1}, y_{2n}(t)) + g(Fy_{2n}, y_{2n+1}(t)) \} \right]$$

If $g(Fy_{2n-1}, y_{2n}(t)) \leq g(Fy_{2n}, y_{2n+1}(t))$ for all $t > 0$, then by (7.2.1),

$$g(Fy_{2n}, y_{2n+1}(t)) \leq \phi \left( g(Fy_{2n}, y_{2n+1}(t)) \right),$$
which means that, by Lemma 3, \( g(F_{2n}, y_{2n+1}(t)) = 0 \) for all \( t > 0 \).

Similarly, we have \( g(F_{2n+1}, y_{2n+2}(t)) = 0 \) for all \( t > 0 \). Thus, we have

\[
\lim_{n \to \infty} g(F_n, y_{n+1}(t)) = 0 \text{ for all } t > 0.
\]

On the other hand, if \( g(F_{2n-1}, y_{2n}(t)) \geq g(F_{2n}, y_{2n+1}(t)) \), then, by (7.2.1), We have

\[
g(F_{2n}, y_{2n+1}(t)) \leq \phi(g(F_{2n-1}, y_{2n}(t)) \text{ for all } t > 0.
\]

Similarly,

\[
g(F_{2n+1}, y_{2n+2}(t)) \leq g(F_{2n}, y_{2n+1}(t)) \text{ for all } t > 0.
\]

Thus, we have

\[
g(F_n, y_{n+1}(t)) \leq \phi(g(F_{n-1}, y_n(t)) \text{ for all } t > 0.
\]

and \( n = 1, 2, \ldots \). Therefore by Lemma 3, \( \lim_{n \to \infty} g(F_n, y_{n+1}(t)) = 0 \) for all \( t > 0 \).

Which implies that \( \{y_n\} \) is a Cauchy sequence in \( X \) by Lemma 6. Since \( (X, f, \Delta) \) is complete, the sequence converges to a point \( z \in X \) and so the subsequence \( \{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\}, \{Tx_{2n+1}\} \) of \( \{y_n\} \) also converges to the limit \( z \).

Now, suppose that \( T \) is continuous. Since \( B \) and \( T \) are compatible of type \( (A) \), by Lemma 2, \( BT x_{2n+1}, TT x_{2n+1} \to Tz \) as \( n \to \infty \). putting \( x=x_{2n} \) and \( y=Tx_{2n+1} \) in (7.2.1), we have
\[ g (FAx_{2n}, \ BTx_{2n+1} (t)) \leq \phi \left\{ \max \left\{ g (FSx_{2n}, \ TTx_{2n+1} (t)), \right. \right. \]
\[ \left. \left. g (FSx_{2n}, \ Ax_{2n} (t)), g (FTTx_{2n+1}, \ BTx_{2n+1} (t)), \right\} \right\} \]

(7.2.7)
\[ \frac{1}{2} \ g (FSx_{2n}, \ BTx_{2n+1} (t)), g (FTTx_{2n+1}, \ Ax_{2n} (t)) \} \]

for all \( t > 0 \). Letting \( n \to \infty \) in (7.2.7) we have
\[ g (Fz, Tz, (t)) \leq \phi \left\{ \max \left\{ g (Fz, Tz, (t)), \ g (Fz, z (t)), \right. \right. \]
\[ \left. \left. g (FTz, Tz (t)), \frac{1}{2} \ g (Fz, Tz (t)), g (FTz, z (t)) \} \right\} \]

= \phi \ (g (Fz, Tz (t))) for all \( t > 0 \). Which means that \( g (Fz, Tz (t)) = 0 \) for all \( t > 0 \), by Lemma 3, and so we have \( Tz = z \).

Again replacing \( x \) by \( x_{2n} \) and \( y \) by \( z \) in (7.2.1), we have
\[ g (FAx_{2n}, \ Bz (t)) \leq \phi \left\{ \max \left\{ g (FSx_{2n}, \ Tz (t)), \right. \right. \]
\[ \left. \left. g (FSx_{2n}, \ Ax_{2n} (t)), g (FTz, Bz (t)), \right\} \right\} \]

(7.2.8)
\[ \frac{1}{2} \ g (FSx_{2n}, \ Bz (t)), g (FTz, Ax_{2n} (t)) \} \]

for all \( t > 0 \). Letting \( n \to \infty \) in (7.2.8) we have
\[ g (Fz, Bz, (t)) \leq \phi \left\{ \max \left\{ g (Fz, z (t)), g (Fz, z (t)), \right. \right. \]
\[ \left. \left. g (Fz, Bz (t)), \frac{1}{2} \ (g (Fz, Bz (t)), g (Fz, z (t)) \} \right\} \]

for all \( t > 0 \), which implies that \( g (Fz, Bz, t)) \leq \phi \ (g (Fz, Bz (t))) \) for all \( t > 0 \) and

so we have \( Bz = z \).

Since \( B(X) \subset S(X) \), there exists a point \( w \in X \) such that \( Bz = Sw = z \).
By using (7.2.1) again, we have

\[ g(FAw, z(t)) = g(FAw, Bz(t)) \]

\[ \leq \phi \left[ \max \{ g(FSw, Tz(t)), g(FSw, Aw(t)), g(FTz, Bz(t)), \right. \]

\[ \frac{1}{2} g(FSw, Bz(t)), g(FTz, Aw(t)) \right\] \]

\[ \leq \phi \left( g(FAw, z(t)) \right) \text{ for all } t > 0, \text{ which means that } Aw = z. \]

Since A and S are compatible mappings of type (A) and Aw = Sw = z, by Lemma 1

\[ Az = ASw = SSw = Sz. \]

Again by using (7.2.1), we have Az = z. Therefore, Az = Bz = Sz = Tz = z,

that is, z is a common fixed point of the given mappings A, B, S, T. The uniqueness of the common fixed point z follows easily from (7.2.1).