 CHAPTER IV

FIXED POINT THEOREMS IN ORBITALLY COMPLETE METRIC SPACES

4.1 Let \((M,d)\) be a metric space. A mapping \(T\) on \(M\) is orbitally continuous if \(\lim_{i} T^{n_i}x = z\) implies \(\lim_{i} T^Tx = Tz\) for each \(x\) in \(M\). A space \(M\) is said to be orbitally complete if every Cauchy sequence of the form \(\{T^{n_i}x\}_{i=1}^{\infty}\), \(x\) in \(M\), converges in \(M\).

Ciric[37] has obtained the following result regarding the existence of non-unique fixed points for orbitally continuous self-mapping on a orbitally complete metric space:

**THEOREM (A):** Let \((M,d)\) be orbitally complete metric space and \(T\) be a orbitally continuous self-mapping of \(X\), satisfying

\[
(4.1.1) \quad \min \{d(Tx,Ty), d(x,Tx), d(y,Ty)\} \\
- \min \{d(x,Ty), d(y,Tx)\} \leq qd(x,y)
\]

for all \(x, y\) in \(M\) and some \(q \in (0,1)\). Then for each \(x\) in \(M\) the sequence \(\{T^n x\}\) converges to a fixed point of \(T\).

Recently Dhage[38] obtained the following generalization of theorem (A):


THEOREM (B): Let $T: M \to M$ be an orbitally continuous self-map of a metric space $M$ and let $M$ be $T$-orbitally complete. If $T$ satisfies the condition

$$(4.1.2) \quad \min \{d(Tx, Ty), d(x, Tx), d(y, Ty)\}$$

$$+ \min \{d(x, Ty), d(y, Tx)\} \leq pd(x, y) + qd(x, Tx),$$

for all $x, y$ in $M$ and $a, p, q$ are real numbers such that $0 < p + q < 1$, then for each $x$ in $M$, the sequence $\{T^n x\}$ converges to a fixed point of $T$.

4.2 Our purpose is to improve upon the above results and establish some results of the maps with non-unique fixed points for different kind of mappings. In fact we[145] prove:

THEOREM 1: Let $T: M \to M$ be an orbitally continuous self-map of a metric space $M$ and let $M$ be $T$-orbitally complete. If $T$ satisfies the condition

$$(4.2.1) \quad \min \{d(Tx, Ty), d(x, Tx), d(y, Ty)\}$$

$$+ \min \{d(x, Ty), d(y, Tx)\}$$

$$\leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, Ty) + a_4 d(y, Tx)$$

$$+ a_5 d(x, T^2 x) + a_6 d(y, T^2 y) + a_7 d(Tx, T^2 x)$$

$$+ a_8 d(Ty, T^2 y) + a_9 d(x, y).$$
for all \( x, y \) in \( M \) and \( a, a_i \) are real numbers such that

(i) \( 0 < a_i < 1 \) and \( \sum_{i=1}^{g} a_i < 1 \)

(ii) \( a_1 + a_2 + 2a_3 + 2a_5 + a_6 + a_7 + a_9 < 1 \),

then for each \( x \) in \( M \), the sequence \( \{T^n x\} \) converges to a fixed point of \( T \).

Proof: Let \( x \) in \( M \) be arbitrary, then we define a sequence \( \{x_n\} \) by

\[
(4.2.2) \quad x_0 = x, \quad x_1 = Tx_0, \quad x_2 = Tx_1, \ldots, \quad x_n = Tx_{n-1}.
\]

If for some \( n \) in \( N \), \( x_n = x_{n+1} \), then \( \{x_n\} \) is a Cauchy sequence and the limit of \( \{x_n\} \) is a fixed point of \( T \). Suppose that \( x_n \neq x_{n+1} \) for each \( n=0,1,2,\ldots, \)
then for \( x = x_{n-1} \) and \( y = x_{n} \), by (4.2.1) we have

\[
\begin{align*}
\min \{d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\
+ a \min \{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} \\
\leq a_1 d(x_{n-1}, x_n) + a_2 d(x_n, x_{n+1}) + a_3 d(x_{n-1}, x_{n+1}) \\
+ a_4 d(x_n, x_n) + a_5 d(x_{n-1}, x_{n+1}) + a_6 d(x_n, x_{n+1}) \\
+ a_7 d(x_n, x_{n+1}) + a_8 d(x_{n+1}, x_n) + a_9 d(x_{n-1}, x_n)
\end{align*}
\]

i.e. \( \min \{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \)

(inequality continued on next page)
\[ \leq (a_1 + a_3 + a_5 + a_9) \ d(x_{n-1}, x_n) \\
+ (a_2 + a_3 + a_5 + a_6 + a_7) \ d(x_n, x_{n+1}) \]

or, \[ d(x_n, x_{n+1}) \leq \frac{a_1 + a_3 + a_5 + a_9}{1 - a_2 - a_3 - a_5 - a_6 - a_7} \ d(x_{n-1}, x_n) \]

Thus, we have

\[(4.2.3) \ d(x_n, x_{n+1}) \leq h \ d(x_{n-1}, x_n) \]

where \[ \frac{a_1 + a_3 + a_5 + a_9}{1 - a_2 - a_3 - a_5 - a_6 - a_7} = h \in (0, 1). \] Since if

\[ \min \{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} = d(x_{n-1}, x_n) \]

then \[ d(x_{n-1}, x_n) \leq (a_1 + a_2 + 2a_3 + 2a_5 + a_6 + a_7 + a_9) d(x_{n-1}, x_n), \]

which is impossible, as \( a_1 + a_2 + 2a_3 + 2a_5 + a_6 + a_7 + a_9 < 1. \)

From (4.2.3), we have

\[ d(x_{n-1}, x_n) \leq h \ d(x_{n-2}, x_{n-1}) \]

and so,

\[ d(x_n, x_{n+1}) \leq h^2 d(x_{n-2}, x_{n-1}) \]

Proceeding in the same manner, we get

\[ d(x_n, x_{n+1}) \leq h^n d(x, Tx) \]

Hence for any \( p \) in \( N \), we have

\[ d(x_n, x_{n+p}) \leq \sum_{k=n}^{n+p-1} d(x_k, x_{k+1}) \]
or, \[ d(x_n, x_{n+p}) \leq \left( \sum_{k=n}^{n+p-1} h^k \right) d(x, Tx) \leq \frac{h^n}{1-h} d(x, Tx) \]

Since \( \lim_{n \to \infty} h^n = 0 \), it follows that (4.2.2) is a Cauchy sequence. Since \( M \) is orbitally continuous, there exists \( u \) in \( M \) such that \( u = \lim_{n \to \infty} T^n x \), \( x \) in \( M \). By orbital continuity of \( T \), we get \( Tu = \lim_{n \to \infty} TT^n x = u \) and hence \( u \) is a fixed point of \( T \).

This completes the proof of the theorem.

Remark 1: In the above Theorem if we let

(i) \( a, a_i \) are positive constants,

(ii) \( a_3 + a_4 + a_6 + a_8 + a_9 < a \),

and the same conditions as in the above theorem then \( T \) has a unique fixed point in \( M \).

Proof: If \( T \) doesn't possess unique fixed point in \( M \) then \( T \) has two distinct fixed point \( u \) and \( v \). Now by (4.2.1), we have

\[
\min \{d(Tu, Tv), d(u, Tu), d(v, Tv)\} + a \min \{d(u, Tv), d(v, Tu)\} \\
\leq a_1 d(u, Tu) + a_2 d(v, Tv) + a_3 d(u, Tu) + a_4 d(v, Tu) + a_5 d(u, T^2 u) \\
+ a_6 d(v, T^2 u) + a_7 d(Tu, T^2 u) + a_8 d(Tv, T^2 u) + a_9 d(u, v)
\]
or, \[ d(u,v) \leq (a_3 + a_4 + a_6 + a_8 + a_9) \ d(u,v) \]

or \[ d(u,v) \leq ((a_3 + a_4 + a_6 + a_8) / a) \ d(u,v) < d(u,v), \]

which is impossible and hence \( u = v \). Thus \( T \) has unique fixed point in \( M \).

Remark 2: If we take \( a_i = -1 \) and \( a_i = 0 \) for \( i = 1, 2, \ldots, 8 \), in (4.2.1), then the mappings specified by this inequality reduce to the mappings given by (4.1.1), considered by Cirić [37].

Remark 3: If we take \( a_i = 0 \) for \( i = 2, 3, \ldots, 8 \), in (4.2.1), then the mappings specified by this inequality reduce to the mappings given by (4.1.2), considered by Dhage [38].

Now if we substitute \( a_i = -1 \) in (4.2.1), then we get the following result as corollary:

COROLLARY 1: Let \( T \) be an orbitally continuous self-map of a \( T \)-orbitally complete metric space \( M \) satisfying the condition

\[
(4.2.4) \ \min \{d(Tx,Ty), d(x,Tx), d(y,Ty)\} - \min \{d(x,Ty), d(y,Tx)\} \\
\leq a_1 d(x,Tx) + a_2 d(y,Ty) + a_3 d(x,Ty) + a_4 d(y,Tx) \\
+ a_5 d(x,T^2 x) + a_6 d(y,T^2 x) + a_7 d(Tx,T^2 x) + a_8 d(Ty,T^2 x) \\
+ a_9 d(x,y),
\]
for all $x,y$ in $M$, then the sequence $\{T^nx\}$ converges to a fixed point of $T$. 

Now we prove the following theorem:

**THEOREM 2:** Let $T$ be an orbitally continuous self-map of a $T$-orbitally complete metric space $(M,d)$ satisfying the condition

$$(4.2.5) \min \{(d(Tx,Ty))^2, d(x,y)d(Tx,Ty), (d(y,Ty))^2\}$$

$$\quad + a \min \{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\}$$

$$\leq a_1d(x,Tx)d(y,Ty) + a_2d(x,Ty)d(y,Tx)$$

$$\quad + a_3d(y,T^2x)d(x,y) + a_4d(x,T^2x)d(Ty,T^2x)$$

$$\quad + a_5d(Tx,T^2x)d(x,y)$$

for all $x,y$ in $M$ and $a,a_i$ $(i=1,\ldots,5)$ are real numbers such that $0 \leq a_1 + a_3 + a_5 < 1$, then for each $x$ in $M$, the sequence $\{T^n x\}$ converges to a fixed point of $T$.

**Proof:** Let $x$ in $M$ be arbitrary, we define a sequence

$$x_0 = x, x_1 = Tx_0, x_2 = Tx_1, \ldots, x_n = Tx_{n-1}.$$ 

Now if for some $n$ in $N$, $x_n = x_{n+1}$, then $\{x_n\}$ is a Cauchy sequence, and the limit of $\{x_n\}$ is a fixed point of $T$. Suppose that $x_n \neq x_{n+1}$ for each $n=0,1,2,\ldots$. 


Now by (4.2.5), for \( x = x_{n-1} \) and \( y = x_n \), we have

\[
\min \{ \|d(x_n, x_{n+1})\|^2, d(x_{n-1}, x_n) d(x_n, x_{n+1}) (d(x_n, x_{n+1})^2) \}
\]

\[+ a_1 d(x_{n-1}, x_n) d(x_n, x_{n+1}) + a_2 d(x_{n-1}, x_n) d(x_n, x_{n+1}) \]

\[+ a_3 d(x_n, x_{n+1}) d(x_{n-1}, x_n) + a_4 d(x_n, x_{n+1}) d(x_{n+1}, x_n) \]

\[+ a_5 d(x_n, x_{n+1}) d(x_{n-1}, x_n) \]

i.e.

\[
\min \{ d(x_n, x_{n+1})^2, d(x_{n-1}, x_n) d(x_n, x_{n+1}) \}
\]

\[\leq (a_1 + a_2 + a_5) d(x_{n-1}, x_n) d(x_n, x_{n+1}) \cdot \]

Since, \( d(x_{n-1}, x_n) d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n) d(x_n, x_{n+1}) \) is impossible, as \( h = a_1 + a_3 + a_5 < 1 \). Thus, we have

\[d(x_n, x_{n+1}^2) \leq h d(x_{n-1}, x_n) d(x_n, x_{n+1}) \cdot \]

i.e.

\[d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n) \cdot \]

Proceeding in the same way, we get

\[d(x_n, x_{n+1}) \leq h^n d(x, Tx) \cdot \]

Now for any positive integer \( p \), we have

\[d(x_n, x_{n+p}) \leq \sum_{k=n}^{n+p-1} d(x_k, x_{k+1}) \]

\[\leq (\sum_{k=n}^{n+p-1} h^k) d(x, Tx) \cdot \]
or, \[ d(x_n, x_{n+p}) \leq \frac{h^n}{1-h} d(x, Tx) \rightarrow 0, \text{ as } n \rightarrow \infty. \]

It follows that \( \{ x_n \} \) is a Cauchy sequence. Since \( M \) is orbitally complete, there is some \( u \) in \( M \) such that

\[ u = \lim_{n \rightarrow \infty} T^n x. \]

By orbital continuity of \( T \), we obtain

\[ Tu = T \lim_{n \rightarrow \infty} T^n x = \lim_{n \rightarrow \infty} TT^n x = u, \] (Note: \( x_n = Tx_{n-1} = TTx_{n-2} = \ldots = T^nx_0 = T^n x) \]

This shows that the sequence \( \{ T^n x \} \) converges to a fixed point \( u \) of \( T \).

This completes the proof of the theorem.

4.3 Recently Achari[1], Dhage[38] has obtained a localised version of Ciric's fixed point theorem. In the present section our results deals with localised versions of Theorem 1 and 2.

**THEOREM 3(a)**: Let \( B = B(x_0, r) = \{ x \in M : d(x_0, x) \leq r \} \), where \( (M, d) \) is an orbitally complete metric space. Let \( T \) be an orbitally continuous mapping of \( B \) into \( M \) and satisfies (4.2.1) for all \( x, y \) in \( B \) and

\[ (4.3.1) \ d(x_0, Tx_0) \leq (1-h)r. \]
where \[ h = \frac{a_1 + a_3 + a_5 + a_9}{1 - a_2 - a_3 - a_5 - a_6 - a_7} \in (0, 1). \]

Then \( T \) has a fixed point in \( M \).

Proof: By (4.3.1), we have

\[ x_1 = Tx_0 \in \mathcal{B}(x_0, r). \]

For \( x = x_0 \) and \( y = x_1 \), by (4.2.1) we have

\[
\min \{d(x_1, x_2), d(x_0, x_1), d(x_1, x_2)\} + a \min \{d(x_0, x_2), d(x_1, x_1)\}
\leq a_1 d(x_0, x_1) + a_2 d(x_1, x_2) + a_3 d(x_0, x_2) + a_4 d(x_1, x_1)
\ + a_5 d(x_0, x_2) + a_6 d(x_1, x_2) + a_7 d(x_1, x_1) + a_8 d(x_2, x_2)
\ + a_9 d(x_0, x_1),
\]

which implies,

\[
d(x_1, x_2) \leq \frac{a_1 + a_3 + a_5 + a_9}{1 - a_2 - a_3 - a_5 - a_6 - a_7} d(x_0, x_1) = hd(x_0, Tx_0)
\]

i.e. \( d(x_1, x_2) \leq h (1-h) r \)

Now,

\[
d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) \leq (1-h) r + h (1-h) r
\]

i.e. \( d(x_0, x_2) \leq (1+h)(1-h) r \)

and \( d(x_0, x_3) \leq d(x_0, x_2) + d(x_2, x_3) \leq (1+h)(1-h) r + h^2 (1-h) r \)

i.e. \( d(x_0, x_3) \leq (1+h+h^2)(1-h) r \).
Proceeding in the same way, we get
\[ d(x_n, x_{n+1}) \leq (1 + h + h^2 + \ldots + h^{n-1})(1-h)r = (1-h^{n-1})r \]

Similarly, we get
\[ d(x_{n-1}, x_n) \leq h^{n-1}d(x_0, x_1) \leq h^{n-1}(1-h)r , (x_n = Tx_{n-1}) \]

Now, for \( x = x_{n-1} \) and \( y = x_n \), by (4.2.1) we get
\[ d(x_n, x_{n+1}) \leq h^nd(x_0, x_1) \leq h^n(1-h)r . \]
Then
\[ d(x_n, x_{n+1}) \leq d(x_0, x_n) + d(x_n, x_{n+1}) \]
\[ \leq (1 + h + h^2 + \ldots + h^{n-1})(1-h)r + h^n(1-h)r \]
\[ = (1-h^{n+1})r \leq r \]

Thus the sequence \( x_0, \ldots, x_{n+1} = Tx_n, n \geq 0 \) is contained in \( B \). Also for some positive integer \( m > n \), we have
\[ d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \]
\[ \leq (h^n + h^{n+1} + \ldots + h^{m-1})(1-h)r \]
\[ = (h^n - h^m)r \leq h^n r \to 0, \text{as } n \to \infty \]

It follows that \( \{x_n\} \) is a Cauchy sequence. Since \( B \) is also orbitally complete, so \( u = \lim_{n \to \infty} T^n x \), for some \( u \) in \( B \).

By orbital continuity of \( T \), we have
\[ Tu = \lim_{n \to \infty} TT^n x = u. \]

Thus \( u \) is a fixed point of \( T \). This completes the proof of the theorem.
THEOREM 3(b): Let $B = B(x_0, r) = \{ x \in M : d(x_0, x) \leq r \}$, where $(M, d)$ is an orbitally complete metric space. Let $T$ be an orbitally continuous mapping of $B$ into $M$ and satisfies (4.2.5), for $x, y$ in $B$ and (4.3.1). Then $T$ has a fixed point in $M$.

Proof: By (4.3.1), we have

$$T_1 = Tx_0 \in B(x_0, r).$$

Now for $x = x_0$ and $y = x_1$, by (4.2.5) we have

$$\min \{ (d(x_1, x_2))^2, d(x_0, x_1)d(x_1, x_2), (d(x_1, x_2))^2 \}$$

$$+ a \min \{ d(x_0, x_1)d(x_1, x_2), d(x_0, x_2)d(x_1, x_1) \}$$

$$\leq a_1 d(x_0, x_1)d(x_1, x_2) + a_2 d(x_0, x_2)d(x_1, x_1)$$

$$+ a_3 d(x_1, x_2)d(x_0, x_1) + a_4 d(x_0, x_2)d(x_2, x_2)$$

$$+ a_5 d(x_1, x_2)d(x_0, x_1),$$

i.e.

$$\min \{ (d(x_1, x_2))^2, d(x_0, x_1)d(x_1, x_2) \}$$

$$\leq (a_1 + a_3 + a_5)d(x_0, x_1)d(x_1, x_2).$$

Since

$$d(x_0, x_1)d(x_1, x_2) \leq h d(x_0, x_1)d(x_1, x_2)$$

is impossible, as $h = a_1 + a_3 + a_5 < 1$.

Therefore,

$$d(x_1, x_2) \leq h d(x_0, x_1) \leq h(1-h)r.$$
Now as in the proof of Theorem 3(a), we get
\[ d(x_0, x_n) \leq (1-h^n)r \]
and that,
\[ d(x_{n-1}, x_n) \leq h^{n-1}(1-h)r. \]

Then by (4.2.5) and (4.3.1), we get for \( x = x_{n-1} \) and \( y = x_n \),
\[
\begin{align*}
d(x_n, x_{n+1}) & \leq hd(x_{n-1}, x_n) \leq \ldots \leq h^{n-1}d(x_0, Tx_0) \\
& \leq h^{n-1}(1-h)r.
\end{align*}
\]
Therefore,
\[
\begin{align*}
d(x_0, x_{n+1}) & \leq d(x_0, x_1) \ast d(x_1, x_2) + \ldots + d(x_{n-1}, x_n) \\
& \leq (1+h+h^2+\ldots+h^{n-1})(1-h)r = (1-h^n)r \leq r.
\end{align*}
\]

Thus the sequence \( x_0, \ldots, x_{n+1} = Tx_n, n \geq 0 \) is contained in \( B \). Also
\[
\begin{align*}
d(x_n, x_m) & \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\
& \leq (h^n + h^{n+1} + \ldots + h^{m-1})(1-h)r = (h^n - h^m)r \\
& \leq h^n r \to 0, \text{ as } n \to \infty.
\end{align*}
\]

It follows that, \( \{x_n\} \) is a Cauchy sequence. Since \( B \) is also orbitally complete, so \( u = \lim_{n \to \infty} T^n x \) for some \( u \) in \( B \). By orbital continuity of \( T \) we have,
\[
Tu = \lim_{n \to \infty} TT^n x = u.
\]

Thus \( u \) is a fixed point of \( T \). This completes the proof of the theorem.
4.4 In this section we establish following theorems, initiated by Maia's [94] fixed point theorem.

THEOREM 4(a): Let \( M \) be a metric space with two metrics \( d_1 \) and \( d \) with

\[
(4.4.1) \quad d_1(x, y) \leq d(x, y) \quad \text{for all } x, y \text{ in } M,
\]

\[
(4.4.2) \quad M \text{ is orbitally complete, with respect to } d_1,
\]

\[
(4.4.3) \quad \text{the mapping } T: M \to M \text{ is orbitally continuous with respect to } d_1 \text{ and }
\]

\[
(4.4.4) \quad d \text{ satisfies (4.2.1) for } x, y \text{ in } M \text{ and } a_i, a_i \text{ are real numbers such that } 0 < a_i < 1, \quad \sum_{i=1}^{9} a_i < 1 \text{ and } a_1 + a_2 + 2a_3 + 2a_5 + a_6 + a_7 + a_9 < 1.
\]

Then \( T \) has a fixed point in \( M \).

Proof: On applying the same arguments as in Theorem 1, we obtain

\[
d(x_n, x_{n+p}) \leq \frac{h^n}{1-h} d(x_0, x_1) \to 0, \text{ as } n \to \infty,
\]

where \( p \) is any positive integer. Therefore by (4.4.1), we have

\[
d_1(x_n, x_{n+p}) \leq d_1(x_n, x_{n+p}) \leq \frac{h^n}{1-h} d(x_0, x_1).
\]

Evidently \( \{x_n\} \) is a Cauchy sequence with respect to \( d_1 \). The rest of the proof is easily follows.
THEOREM 4(b): Let \( M \) be a metric space with two metrics \( d_1 \) and \( d \). If \( M \) satisfies the conditions (4.4.1), (4.4.2), (4.4.3) and

(4.2.5) for all \( x, y \) in \( M \) and \( a, a_i \) \( (i=1,2,\ldots,5) \) are real numbers such that \( 0 < a_1 + a_3 + a_5 < 1 \).

Then \( T \) has a fixed point in \( M \).

Proof: On applying the same method as described in the proof of Theorem 2, we obtain

\[
d(x_n, x_{n+p}) \leq \frac{h^n}{1-h} d(x_0, x_1),
\]

where \( p \) is any positive integer. Therefore by (4.4.1), we have

\[
d_1(x_n, x_{n+p}) \leq d(x_n, x_{n+p}) \leq \frac{h^n}{1-h} d(x_0, x_1) \to 0, \text{ as } n \to \infty.
\]

Evidently \( \{x_n\} \) is a Cauchy sequence with respect to \( d_1 \).

The rest of the proof follows by an argument similar to that given in the proof of Theorem 1 and 2 with suitable modifications.

Remark 4: The conclusion of Theorem 1 is even true, if we replace the condition (4.2.1) by
(4.4.5) \[ \min \{d(Tx,Ty), d(x,Tx), d(y,Ty)\} + a_d(y,Tx) \]
\[ \leq a_1 d(x,Tx) + a_2 d(y,Ty) + a_3 d(x,Ty) + a_6 d(y,Tx) \]
\[ + a_5 d(x,T^2x) + a_6 d(y,T^2x) + a_7 d(Tx,T^2x) \]
\[ + a_8 d(Ty,T^2x) + a_9 d(x,y), \]

for all \( x,y \) in \( M \), \( a,a_i \) are real numbers such that
\[ 0 < a_i < 1, \sum_1^9 a_i < 1 \]
and \( a_1 + a_2 + 2a_3 + 2a_5 + a_6 + a_7 + a_9 < 1 \).

Also Theorem 3(a) holds under this modified condition.

Remark 5: The conclusion of Theorem 2 is also true, if we replace the condition (4.2.5) by

(4.4.6) \[ \min \{(d(Tx,Ty))^2, d(x,Tx)d(y,Ty), (d(y,Ty))^2\} \]
\[ + a_d(x,Ty)d(y,Tx) \]
\[ \leq a_1 d(x,Tx)d(y,Ty) + a_2 d(x,Ty)d(y,Tx) \]
\[ + a_3 d(y,T^2x)d(x,y) + a_4 d(x,T^2x)d(Ty,T^2x) \]
\[ + a_5 d(x,T^2x)d(x,y), \]

for all \( x,y \) in \( M \) and \( a,a_i \) \((i=1,2,\ldots,5)\) are real numbers such that \( 0 < a_1 + a_3 + a_5 < 1 \).

Also Theorem 3(b) holds under this modified condition.

4.5 Finally we furnish some examples to discuss the validity of the hypothesis & degree of generality of our Theorems.
Example (4.5.1): Let $R$ denote the set of all real numbers and let $T: R \rightarrow R$ be a mapping defined by $Tx = x^2 + 3x - 8$ for all $x$ in $R$ and $(R, d)$ is the complete metric space with metric $d$ is defined by $d(x, y) = |x - y|$.

For $x = 1$ and $y = -2$, (4.4.1) yields

$$5 + 2a \leq 5a_1 + 8a_2 + 11a_3 + 2a_4 + 5a_5 + 2a_6 + 6a_7 + 3a_8.$$ 

Taking $a = -1$, we observe that the above result is true for every set of $a_i$ $(i = 1, 2, \ldots, 9)$ such that $0 < a_i < 1$, $\sum a_i < 1$ and $a_1 + a_2 + 2a_3 + 2a_4 + a_5 + a_6 + a_7 + a_8 < 1$. In fact, $T$ has two fixed points, namely, -4 and 2.

Example (4.5.2): Let $M = \{0, 1, 2, 3\}$, $d(0, 1) = d(1, 2) = 1/2$, $d(0, 2) = d(0, 3) = d(1, 3) = d(2, 3) = 1$. Define $T: M \rightarrow M$ by $T_0 = T_1 = 1$, $T_3 = 1$, $T_2 = 2$. Then condition (4.2.5) is satisfied for all $x, y$ in $M$. Clearly 1 and 2 are fixed points of $T$.

From the foregoing discussion we see that, our Theorem 1 is the improved version of the Ciric’s (4.1.1) and Dhage’s condition (4.1.2) and while completing this chapter we discuss Example (4.5.1) to show the advantage of our condition (4.2.1) of Theorem 1.

Example (4.5.3): Let $T, R, (R, d)$ be as define in Example (4.5.1). Then for $x = 3$ and $y = 10$, by condition (4.1.2) of Dhage[38], we get
\[ \min \{d(T3, T10), \, d(3, T3), \, d(10, T10)\} \]
\[ + \, \alpha \min \{d(3, T10), \, d(10, T3)\} \leq pd(3, 10) + qd(3, T3), \]

or,
\[ \min \{d(10, 122), \, d(3, 10), \, d(10, 122)\} \]
\[ + \, \alpha \min \{d(3, 122), \, d(10, 10)\} \leq pd(3, 10) + qd(3, 10) \]

or,
\[ 7 \leq 7(p+q), \text{ i.e. } 1 \leq p+q, \]

which is a contradiction, as \( p+q < 1 \). Hence condition (4.1.2) fails to prove the existence of the fixed point of the map \( T \). But this mapping \( T \) is contained in our class of mappings (4.2.1) and (4.2.4) and hence it guarantees the fixed point of \( T \). In fact \( T \) has two fixed points, namely -4 and 2.

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