CHAPTER III

FIXED POINT THEOREMS FOR MAPPINGS WHICH ARE NOT NECESSARILY CONTINUOUS

3.1 In the present chapter some sufficient conditions have been obtained for self-mappings of a complete metric space which ensure a unique fixed point.

Let \((X,d)\) be a complete metric space. A self-mapping \(T\) of \(X\) is said to be contraction if

\[
(3.1.1) \quad d(Tx,Ty) \leq a \, d(x,y)
\]

for all \(x,y\) in \(X\) and \(0 \leq a < 1\). Then \(T\) has a unique fixed point in \(X\). Contraction mapping (3.1.1) is necessarily continuous and yet the fixed or common fixed points have been obtained for such type of mappings. For example Kannan[83] obtained a unique fixed point for a self-mapping \(T\) of a complete metric space \((X,d)\) satisfying

\[
(3.1.2) \quad d(Tx,Ty) \leq a \,[d(x,Tx) + d(y,Ty)]
\]

for all \(x,y\) in \(X\), where \(0 \leq a < \frac{1}{2}\). The mapping \(T\) satisfying (3.1.2) is not necessarily continuous.

In 1975, Gupta and Ranganathan[61] obtained a unique fixed point theorem for a self mapping \(T\) of a
complete metric space \((X,d)\) which is not necessarily continuous for \(p=0\). Gupta and Ranganathan have proved the following theorem.

**THEOREM (A)**: Let \(T\) be a self-mapping of a complete metric space \((X,d)\) such that

\[(3.1.3)\quad d(T^{p+1}x, T^{p+2}y) \leq a_1 d(T^p x, T^{p+1} x) + a_2 d(T^{p+1} y, T^{p+2} y) + a_3 d(T^p x, T^{p+1} y),\]

for all \(x,y\) in \(X\), \(p\) a non-negative integer and \(a_1, a_2, a_3\) are constants such that \(a_1, a_2, a_3 \geq 0\) with \(a_1 + a_2 + a_3 < 1\). Then \(T\) has a unique fixed point.

Recently in 1980, Fisher[50] has proved the following theorem:

**THEOREM (B)**: If \(S\) and \(T\) are mappings of a complete metric space \((X,d)\) into itself satisfying the inequality

\[(3.1.4)\quad [d(Sx,Ty)]^2 \leq b d(x,Sx) d(x,Ty) + c d(y,Sx)d(y,Ty),\]

for all \(x,y\) in \(X\), where \(b,c > 0\) and

\[(3.1.5)\quad [b+(b^2+4b)^{\frac{1}{2}}][c+(c^2+4c)^{\frac{1}{2}}] < 4.\]

Then \(S\) and \(T\) have a unique common fixed point.

Further, Fisher[50] obtained the particular case
of the above theorem by taking \( S = T \) and \( b = c \), given as follows:

**THEOREM (C):** If \( T \) be a self-mapping of a complete metric space \((X, d)\) satisfying the inequality

\[
(3.1.6) \quad [d(Tx, Ty)]^2 \leq c[d(x, Tx)d(x, Ty) + d(y, Tx)d(y, Ty)]
\]

for all \( x, y \) in \( X \), where \( 0 \leq c < \frac{1}{2} \). Then \( T \) has a unique fixed point.

3.2 Now we prove the following theorem:

**THEOREM 1:** Let \( T \) be a selfmap on a complete metric space \((X, d)\) such that

\[
(3.2.1) \quad d(T^{p+1}x, T^{p+2}y) \leq a_1 \frac{d(T^{p+1}y, T^{p+1}x) \{c + d(T^p x, T^{p+1}y)\}}{\{c + d(T^p x, T^{p+1}y)\} + a_2 \{d(T^p x, T^{p+1}x) + d(T^{p+1}y, T^{p+2}y)\} + a_3 \{d(T^p x, T^{p+2}y) + d(T^{p+1}y, T^{p+1}x)\} + a_4 d(T^p x, T^{p+1}y)}
\]

for all \( x, y \) in \( X \), \( p \) a non-negative integer and \( a_i \) (\( i = 1, 2, 3, 4 \)) are non-negative constants such that \( 0 \leq a_1 + 2a_2 + 2a_3 + a_4 < 1 \) and \( c \neq 0 \). Then \( T \) has a unique fixed point in \( X \).
Proof: We prove the theorem for \( p = 0 \), then the proof in the general case follows on similar lines.

Now for \( p = 0 \), we have from (3.2.1),

\[
\begin{align*}
    d(Tx, T^2 y) &\leq a_1 \frac{d(Ty, T^2 y) + c + d(x, Tx)}{c + d(x, Ty)} \\
                 &+ a_2 \{ d(x, Tx) + d(Ty, T^2 y) \} \\
                 &+ a_3 \{ d(x, T^2 y) + d(Ty, Tx) \} + a_4 d(x, Ty).
\end{align*}
\]

For an arbitrary \( x_0 \) in \( X \), we define a sequence \( \{x_n\} \) of elements of \( X \) such that

\[
x_n = Tx_{n-1} = \ldots = T^n x_0 \ ; \ x_{n+1} = T^{n+1} x_0, \ n = 0, 1, 2, \ldots
\]

Now,

\[
d(x_1, x_2) = d(Tx_0, T^2 x_0) \leq a_1 \frac{d(x_1, x_2) + c + d(x_0, x_1)}{c + d(x_0, x_1)} \]

\[
+ a_2 \{ d(x_0, x_1) + d(x_1, x_2) \} \\
+ a_3 \{ d(x_0, x_2) + d(x_1, x_1) \} + a_4 d(x_0, x_1)
\]

i.e.

\[
d(x_1, x_2) \leq h d(x_0, x_1)
\]

where \( h = \frac{a_2 + a_3 + a_4}{1 - a_1 - a_2 - a_3} \in (0, 1) \).
Similarly, \( d(x_2, x_3) \leq h d(x_1, x_2) \leq h^2 d(x_0, x_1) \)

In general, \( d(x_n, x_{n+1}) \leq h^n d(x_0, x_1) \)

Since \( h < 1 \), it follows that \( \{x_n\} \) is a Cauchy sequence and has a limit \( z \) in \( X \), as \( X \) is complete.

Now we shall show that \( z \) is the fixed point of \( T \).

We have,

\[
d(Tz, x_n) = d(Tz, T^2 x_{n-2})
\]

\[
\leq a \frac{d(x_{n-1}, x_n) \{c + d(z, Tz)\}}{d(z, x_{n-1})} + a_2 [d(z, Tz) + d(x_{n-1}, x_n)] + a_3 [d(z, x_n) + d(x_{n-1}, Tz)] + a_4 d(z, x_{n-1})
\]

On letting \( n \to \infty \), we obtain

\[
d(Tz, z) \leq (a_2 + a_3) d(Tz, z) < d(Tz, z),
\]

leading to a contradiction. Therefore \( z \) is a fixed point of \( T \).

Finally, let \( w \) in \( X \) be such that \( w = Tw \). Then by (3.2.1), we have
\[ d(z, w) = d(Tz, T^2w) \]

\[
d(Tw, T^2w) \{c + d(z, Tz)\} \leq a_1 \frac{\{c + d(z, Tw)\}}{\{c + d(z, Tw)\}} \]
\[+ a_2 \{d(z, Tz) + d(Tw, T^2w)\} \]
\[+ a_3 \{d(z, T^2w) + d(Tw, Tz)\} + a_4 d(z, Tw)\]

i.e. \[d(z, w) \leq (2a_3 + a_4) d(z, w) < d(z, w),\]

again leading to a contradiction. Therefore \(z = w\) follows the uniqueness of \(z\).

This completes the proof of the theorem.

If we assume \(c = 0\) in Theorem 1, then we[15] obtain the following result:

**THEOREM 2** : Let \( T\) be a selfmap on a complete metric space \((X, d)\) such that

\[
(3.2.2) \quad d(T^{P+1}x, T^{P+2}y) \leq a_1 \frac{d(T^Px, T^{P+1}x) \cdot d(T^{P+1}y, T^{P+2}y)}{d(T^Px, T^{P+1}y)}
\]
\[+ a_2 \{d(T^Px, T^{P+1}x) + d(T^{P+1}y, T^{P+2}y)\} \]
\[+ a_3 \{d(T^Px, T^{P+2}y) + d(T^{P+1}y, T^{P+1}x)\} \]
\[+ a_4 d(T^Px, T^{P+1}y)\]

for all \(x, y\) in \(X\), \(p\) a non-negative integer and \(a_i (i=1, 2, 3, 4)\) are non-negative constants such that \(0 \leq a_1 + 2a_2 + 2a_3 + a_4 < 1\). Then \(T\) has a unique fixed point in \(X\).
Proof: The proof can be obtained by applying the same method as described in the proof of Theorem 1.

Now we furnish an example to discuss the validity of the hypothesis and degree of generality of our Theorems.

Example 1. Let \( X = [0,1] \) with the usual metric and \( T: X \to X \) be defined as

\[
T(x) = \begin{cases} 
1/3 & x \neq 2/3, \\
1 & x = 2/3.
\end{cases}
\]

Clearly, \( T \) is a discontinuous self-mapping of \( X \) and therefore it does not satisfy contraction condition (3.1.1). By taking \( x = 2/3 \) and \( y = 0 \), we see that \( T \) does not satisfy (3.1.3), but it satisfies (3.2.1) and (3.2.2). Thus our Theorems are the improved version of the Gupta and Ranganthan's condition (3.1.3).

3.3 Now we prove the following theorem for a new class of mappings.

**THEOREM 3**: Let \( (X,d) \) be a complete metric space and let \( T: X \to X \) satisfying the inequality

\[
(d(Tx, T^2y))^2 \leq c(d(x,Tx)d(x,T^2x) + d(T^2y, T^2x)d(Ty,T^2x))
\]

(inequality continued on next page)
\[ d(y, Ty) d(y, T^2 y) + d(Tx, T^2 x) d(Ty, T^2 y) + d(x, Ty) d(y, Tx) \]

for all \( x, y \) in \( X \), and \( c \), a non-negative constant such that \( [2c + (3c-2c^2)^{1/2}] < 1 \). Then \( T \) has unique fixed point in \( X \).

**Proof**: We define a sequence of elements \( \{x_n : n=1, 2, \ldots \} \) in \( X \) as follows:

Let \( x_0 \) be an arbitrary element in \( X \), and let \( x_n = T^{n-1} x_0 \) for \( n = 1, 2, \ldots \).

Then by (3.3.1), we have

\[ [d(x_1, x_2)]^2 = [d(Tx_0, T^2 x_0)]^2 \]

\[ \leq c [d(x_0, Tx_0) d(x_0, T^2 x_0) + d(T^2 x_0, T^2 x_0) d(Tx_0, T^2 x_0) + d(x_0, Tx_0) d(x_0, Tx_0)] \]

\[ = c \left[ d(x_0, x_1) d(x_0, x_2) + d(x_0, x_1) d(x_0, x_2) + \right. \]

\[ \left. + (d(x_1, x_2))^2 + (d(x_0, x_1))^2 \right] \]

i.e.

\[ (1-c)[d(x_1, x_2)]^2 \leq c d(x_0, x_1) [d(x_0, x_1) + d(x_0, x_2)] \]
or, \((1-c)[d(x_1, x_2)]^2 \leq cd(x_o, x_1)[3d(x_o, x_1) + 2d(x_1, x_o)]\)

or, \((1-c)[d(x_1, x_2)]^2 \leq 3c[d(x_o, x_1)]^2 + 2cd(x_o, x_1)d(x_1, x_2)\)

By noting that the quadratic equation

\((1-c)x^2 = 3c\alpha^2 + 2c\alpha x, \text{where } \alpha = d(x_o, x_1) \& x = d(x_1, x_2)\)

has the roots

\[x = \frac{c + (3c - 2c^2)^{\frac{1}{2}}}{(1 - c)} \alpha\]

Hence it follows that

\(d(x_1, x_2) \leq kd(x_o, x_1)\)

where \(k = \frac{c + (3c - 2c^2)^{\frac{1}{2}}}{(1 - c)} < 1\).

Similarly,

\([d(x_2, x_3)]^2 = [d(Tx_1, T^2x_1)]^2\)

\[\leq c[d(x_1, x_2)d(x_1, x_3) + d(x_3, x_3)d(x_2, x_3)\]

\[+ d(x_1, x_2)d(x_1, x_3)^2 + d(x_2, x_3)d(x_2, x_3)^2\]

\[+ d(x_1, x_2)d(x_1, x_2)^2],\]

which gives on further generalization,

\(d(x_2, x_3) \leq kd(x_1, x_2) \leq k^2d(x_o, x_1)\)
So, in general, we have

\[ d(x_n, x_{n+1}) \leq k^n d(x_0, x_1) \]

since \( k < 1 \), it follows that the sequence \( \{x_n\} \) is a Cauchy sequence. Now from the completeness of \( X \), there exists a point \( z \) in \( X \), such that

\[ \lim_{n \to \infty} x_n = z. \]

Now we shall show that \( z \) is a fixed point of \( T \).

\[
[d(Tz, x_n)]^2 = [d(T_2, Tx_{n-1})]^2 = [d(Tz, T^2 x_{n-2})]^2 \\
\leq c [d(z, Tz) d(z, T^2 z) + d(x_n, T^2 z) d(x_{n-1}, T^2 z) \\
+ d(x_{n-2}, x_{n-1}) d(x_{n-2}, x_n) + d(Tz, T^2 z) d(x_{n-1}, x_n) \\
+ d(z, x_{n-1}) d(x_{n-2}, Tz)]
\]

On letting \( n \to \infty \), we get \( [d(Tz, z)]^2 = 0 \), which implies that \( z \) is a fixed point of \( T \).

Now, suppose that \( w \) is another fixed point of \( T \) in \( X \) such that \( w \neq z \), then

\[
[d(z, w)]^2 = [d(Tz, T^2 w)]^2 \\
\leq c [d(z, Tz) d(z, T^2 z) + d(T^2 w, T^2 z) d(Tw, T^2 z) \\
+ d(w, Tw) d(w, T^2 w) + d(T_2, T^2 z) d(Tw, T^2 w) \\
+ d(z, Tw) d(w, Tz)] \\
= 2c [d(z, w)]^2
\]
i.e. \( d(z,w) \leq (2c)^{\frac{1}{2}} d(z,w) \),

leading to a contradiction. Hence it follows that \( z = w \),
and so, \( z \) is the unique fixed point of \( T \).

This completes the proof.

Following is the general case of Theorem 3.

THEOREM 4: Let \((X,d)\) be a complete metric space and
let \( T : X \to X \) satisfying the inequality

\[
(3.3.2) \quad [d(T^{P+1}x,T^{P+2}y)]^2 \leq c[d(T^P x,T^{P+1} x)d(T^P x,T^{P+2} x)
+ d(T^{P+2}y,T^{P+2} x)d(T^{P+1} y,T^{P+2} x)
+ d(T^{P+2}y,T^{P+2} y)d(T^{P+2} y,T^{P+2} y)
+ d(T^{P+1} x,T^{P+2} x)d(T^{P+1} y,T^{P+2} y)
+ d(T^P x,T^{P+1} y)d(T^P y,T^{P+1} x)]
\]

for all \( x,y \) in \( X \), and \( c \) a non-negative constant such
that \( (2c+(3c-2c^2)^{\frac{1}{2}}) < 1 \). Then \( T \) has unique fixed point
in \( X \).