CHAPTER V

FIXED POINT THEOREMS IN PSEUDO-COMPACT TICHONOV SPACES

5.1 In the present chapter some results on fixed point and common fixed point have been obtained, which in turn generalize the interesting results, those of Fisher[48], Harinath[68] on compact metric spaces & of Jain and Dixit[82] on pseudo-compact Tichonov spaces.

A topological space \( X \) is said to be pseudo-compact iff every real-valued continuous function on \( X \) is bounded. It is known that every compact space is pseudo-compact, but the converse need not be true (Engleking[44], Example 5, page 150). However in a metric space the notions: compact and pseudo-compact coincide. By Tichonov space we mean a completely regular Hausdorff space. It is observed that the product of two Tichonov spaces is again a Tichonov space, whereas the product of two pseudo-compact spaces need not be pseudo-compact.

The main results in the existence of fixed points for certain continuous functions over pseudo-compact Tichonov space are as follows:
THEOREM (A) (Harinath, Th.3[68]) : Let $P$ be a pseudo-compact Tichonov space and $\mu$ be a non-negative real valued continuous function over $P \times P$ ($P \times P$ is Tichonov but need not be pseudo-compact). If $T:P \rightarrow P$ is a continuous map satisfying

\begin{equation}
\mu(Tx,Ty) < a_1 \mu(x,y) + a_2 \mu(Tx,x) + a_3 \mu(Ty,y)
\end{equation}

for all distinct $x, y$ in $P$, where $a_2 < 1$, $a_1 + a_2 + a_3 \leq 1$. Then $T$ has a fixed point.

THEOREM (B) (Harinath Th.5[68]) : Let $P$ be pseudo-compact Tichonov space and $\mu$ be a non-negative real valued continuous function over $P \times P$ ($P \times P$ is Tichonov, but need not be pseudo-compact) satisfying

\begin{align}
(5.1.2) \quad & \mu(x,x) = 0, \quad \text{for all } x \text{ in } P \\
(5.1.3) \quad & \mu(x,y) \leq \mu(x,z) + \mu(z,y) \quad \text{for all } x, y, z \text{ in } P.
\end{align}

Let $T:P \rightarrow P$ be a continuous map satisfying

\begin{equation}
\mu(Tx,Ty) < a_1 \mu(x,y) + a_2 \mu(Tx,x) + a_3 \mu(Tx,y) \\
+ a_4 \mu(x,Ty) + a_5 \mu(Ty,y),
\end{equation}

for all distinct $x,y$ in $P$, where $a_3 \geq 0$, $a_2 + a_3 < 1$, $a_1 + a_2 + 2a_3 + a_5 \leq 1$. Then $T$ has a fixed point in $P$, which is unique whenever $a_1 + a_3 + a_4 < 1$. 

THEOREM (C) (Jain and Dixit, Th.1[82]) : Let $P$ be pseudo-compact Tichonov space and $\mu$ be a non-negative real valued continuous function over $P \times P$ ($P \times P$ is Tichonov, but need not be pseudocompact). Suppose that $\mu$ also satisfies (5.1.2) and (5.1.3).

If $T : P \to P$ is a continuous map satisfying

\begin{equation}
\mu(Tx,Ty) < a_1 \mu(x,y) + a_2 \mu(Tx,x) + a_3 \mu(Tx,y)
+ a_4 \mu(x,Ty) + a_5 \mu(Ty,y)
+ a_6 \frac{\mu(Tx,x) \mu(Ty,y)}{\mu(x,y)}
+ a_7 \frac{\mu(x,Ty) \mu(Tx,y)}{\mu(x,y)}
\end{equation}

for all distinct $x,y$ in $P$, where $a_3 > 0, a_2 + a_3 + a_6 < 1, a_1 + a_2 + 2a_3 + a_5 + a_6 < 1$. Then $T$ has a fixed point in $P$, which is unique whenever $a_1 + a_3 + a_4 + a_7 < 1$.

5.2 In the present section we generalize the above results of Harinath[68] and Jain and Dixit[82]. First of all we[13] prove:

THEOREM 1 : Let $P$ be a pseudo-compact Tichonov space and $\mu$ be a non-negative real valued continuous function over $P \times P$ ($P \times P$ is Tichonov but need not be pseudocompact). Suppose that $\mu$ also satisfies (5.1.2), (5.1.3) and if $T : P \to P$ be a continuous map satisfying

\begin{equation}
\mu(Tx,Ty) < \max \left\{ \mu(x,y), \frac{1}{2} [\mu(Tx,x) + \mu(Ty,y)], \frac{1}{2} [\mu(x,Ty) + \mu(Tx,y)] \right\},
\end{equation}

(inequality continued on next page)
(1/2) \left[ \frac{\mu(Tx,x) \mu(Ty,y)}{\mu(x,y)} + \frac{\mu(x,Ty) \mu(Tx,y)}{\mu(x,y)} \right],

(1/2) \left[ \frac{\mu(Tx,x)(1+\mu(Ty,y)) + \mu(x,Ty)(1+\mu(Tx,y))}{(1+\mu(x,y))^2} \right]

for all distinct x,y in P. Then T has unique fixed point in P.

Proof: We define a function \( \hat{\phi} : P \to R \) by

\[ \hat{\phi}(p) = \mu(Tp,p) \text{ for all } p \text{ in } P, \text{ where } R \text{ is the set of all real numbers.} \]

Clearly \( \hat{\phi} \) is continuous as it is the product of two continuous functions \( T \) and \( \mu \). Since \( P \) is pseudo-compact Tichonov space, every real valued continuous function over \( P \) is bounded and attains its bounds. Thus there exists a point \( u \) in \( P \) such that

\[ \hat{\phi}(u) = \inf \{ \hat{\phi}(p) : p \text{ in } P \} \quad (\text{Note: } \hat{\phi}(p) \in R). \]

We now affirm that \( u \) is a fixed point of \( T \). If not, let us suppose that \( Tu \neq u \). Then using (5.2.1), we have

\[ \hat{\phi}(Tu) = \mu(T^2u,Tu) \]

\[ < \max \{ \mu(Tu,u), \frac{1}{2} [ \mu(T^2u,Tu) + \mu(Tu,u)] \}, \]

\[ \frac{1}{2} [ \mu(Tu,u) + \mu(T^2u,u)] \]

(inequality continued on next page)
\[
\frac{\mu(T^2 u, Tu) \mu(Tu, u)}{\mu(Tu, u)} + \frac{\mu(Tu, Tu) \mu(T^2 u, u)}{\mu(Tu, u)}
\]

\[
\frac{\mu(T^2 u, Tu)(1 + \mu(Tu, u))}{(1 + \mu(Tu, u))} + \frac{\mu(Tu, Tu)(1 + \mu(T^2 u, u))}{(1 + \mu(Tu, u))}
\]

\[= \max\{ \mu(Tu, u), \frac{1}{2}[\mu(T^2 u, Tu) + \mu(Tu, u)], \frac{1}{2} \mu(T^2 u, Tu) \} \]

\[= \max\{ \mu(Tu, u), \frac{1}{2}[\mu(T^2 u, Tu) + \mu(Tu, u)] \}\]

we may take either \( \mu(Tu, u) \) or \( \frac{1}{2}[\mu(T^2 u, Tu) + \mu(Tu, u)] \) as maximum. In both the cases, we get

\( \mu(T^2 u, Tu) < \mu(Tu, u) \)

i.e. \( \phi(Tu) < \phi(u) \),

leading to a contradiction and hence \( u \) in \( P \) is a fixed point of \( T \).

In order to prove the uniqueness of \( u \), if possible let \( v \) in \( P \) be another fixed point of \( T \) which is distinct from \( u \). Then using \((5.2.1)\), we have

\[\mu(u, v) = \mu(Tu, Tv)\]

\[< \max \{ \mu(u, v), \frac{1}{2}[\mu(Tu, u) + \mu(Tv, v)] \}, \frac{1}{2}[\mu(u, Tv) + \mu(Tu, v)] \},\]

(inequility continued on next page)
\( (1/2) \left[ \frac{\mu(T_u, u) \mu(T_v, v)}{\mu(u, v)} + \frac{\mu(u, T_v) \mu(T_u, v)}{\mu(u, v)} \right], \)

\( (1/2) \left[ \frac{\mu(T_u, u)(1 + \mu(T_v, v))}{(1 + \mu(u, v))} + \frac{\mu(u, T_v)(1 + \mu(T_u, v))}{(1 + \mu(u, v))} \right] \}

= \mu(u, v),

again leading to a contradiction, which proves that \( u \) is unique. This completes the proof of the theorem.

Since every metric space is a Hausdorff space. Therefore as a particular case of the above theorem we have the following result on a compact metric space.

COROLLARY 1: Let \((X, d)\) be a compact metric space and \(T\) be a continuous selfmap on \(X\) satisfying

\( (5.2.2) \quad d(Tx, Ty) < \max\{d(x, y), \frac{1}{2}[d(Tx, x) + d(Ty, y)], \frac{1}{2}[d(Tx, y) + d(x, Ty)] \}, \)

\( (1/2) \left[ \frac{d(Tx, x)d(Ty, y)}{d(x, y)} + \frac{d(x, Ty)d(Tx, y)}{d(x, y)} \right], \)

\( (1/2) \left[ \frac{d(Tx, x)(1 + d(y, Ty))}{(1 + d(x, y))} + \frac{d(x, Ty)(1 + d(Tx, y))}{(1 + d(x, y))} \right] \}

for all distinct \(x, y\) in \(X\). Then \(T\) has unique fixed point in \(X\).
Proof: The proof is very simple and can be obtained by observing the proof of Theorem 1, with "d" playing the role of \( \mu \). We omit the details.

Now we prove the following theorem

**THEOREM 2**: Let \( P \) be a pseudo-compact Tichonov space and \( \mu \) be a non-negative real valued continuous function over \( P \times P \) (\( P \times P \) is Tichonov but need not be pseudo-compact). Suppose that \( \mu \) also satisfies (5.1.2), (5.1.3) and if \( T: P \rightarrow P \) is a continuous map satisfying,

\[
\mu(Tx,Ty) < a_1 \mu(x,y) + a_2 \mu(Tx,x) + a_3 \mu(Ty,y) + a_4 \mu(x,Ty) + a_5 \mu(Tx,y) + a_6 \frac{\mu(Tx,x) \mu(Ty,y)}{\mu(x,y)} + a_7 \frac{\mu(x,Ty) \mu(Tx,y)}{\mu(x,y)} + a_8 \frac{\mu(Tx,x) (1+ \mu(Ty,y))}{1+ \mu(x,y)} + a_9 \frac{\mu(x,Ty) (1+ \mu(Tx,y))}{1+ \mu(x,y)},
\]

for all distinct \( x, y \) in \( P \), where \( a_i \geq 0 \) are such that \( a_1 + a_2 + a_3 + 2a_5 + a_6 + a_9 < 1 \). Then \( T \) has a unique fixed point in \( P \), which is unique, whenever \( a_1 + a_4 + a_5 + a_7 + a_9 < 1 \).

Proof: Let \( \phi \) and \( u \) have the same meanings, as we defined in the proof of Theorem 1. Then using (5.2.3) we have
\[ \psi(Tu) = \mu(T^2 u, Tu) \]
\[ < a_1 \mu(Tu, u) + a_2 \mu(T^2 u, Tu) + a_3 \mu(Tu, u) + a_4 \mu(Tu, Tu) \]
\[ + a_5 \mu(T^2 u, u) + a_6 \frac{\mu(T^2 u, Tu)}{\mu(Tu, u)} \]
\[ + a_7 \frac{\mu(Tu, Tu)}{\mu(Tu, u)} + a_8 \frac{\mu(T^2 u, Tu)(1 + \mu(Tu, u))}{1 + \mu(Tu, u)} \]
\[ + a_9 \frac{\mu(Tu, Tu)(1 + \mu(T^2 u, u))}{1 + \mu(Tu, u)} \]
\[ = (a_1 + a_3 + a_5) \mu(Tu, u) + (a_2 + a_5 + a_6 + a_8) \mu(T^2 u, Tu), \]

i.e. \[ \psi(Tu) < h \mu(Tu, u) < \mu(Tu, u) = \phi(u), \]

where \[ h = \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5 - a_6 - a_8} < 1, \]

leading to a contradiction and hence \( u \) in \( P \) is a fixed point of \( T \).

If possible let \( v \) in \( P \) be another fixed point of \( T \), which is distinct from \( u \). Then using (5.2.3), we have

\[ \mu(u, v) = \mu(Tu, Tv) \]
\[ < a_1 \mu(u, v) + a_2 \mu(Tu, u) + a_3 \mu(Tv, v) + a_4 \mu(u, Tv) \]
\[ + a_5 \mu(Tu, v) + a_6 \frac{\mu(Tu, u) \mu(Tv, v)}{\mu(u, v)} \]
\[ + a_7 \frac{\mu(u, Tv) \mu(Tv, v)}{\mu(u, v)} + a_8 \frac{\mu(Tu, u)(1 + \mu(Tv, v))}{1 + \mu(Tu, v)} \]
\[ + a_9 \frac{\mu(u, Tv)(1 + \mu(Tv, v))}{1 + \mu(Tu, v)} \]
or, \[ \mu(u, v) < (a_1 + a_4 + a_5 + a_7 + a_9) \mu(u, v) < \mu(u, v), \]
again leading to a contradiction, as \( a_1 + a_4 + a_5 + a_7 + a_9 < 1 \),
which proves that \( u \) is unique. This completes the proof.

As a particular case of the above Theorem 2, we have the following result on a compact metric space.

**COROLLARY 2:** Let \((X, d)\) be a compact metric space and \( T \) be a continuous self-map on \( X \) satisfying

\[ (5.2.4) \quad d(Tx, Ty) < a_1 d(x, y) + a_2 d(Tx, x) + a_3 d(Ty, y) + a_4 d(Tx, y) + a_5 d(x, Ty) + a_6 \frac{d(Tx, x) d(Ty, y)}{d(x, y)} + a_7 \frac{d(x, Ty) d(Tx, y)}{d(x, y)} + a_8 \frac{d(Tx, x) (1 + d(y, Ty))}{(1 + d(x, y))} + a_9 \frac{d(x, Ty) (1 + d(Tx, y))}{(1 + d(x, y))} \]

for all distinct \( x, y \) in \( X \), where \( a_i > 0 \) are such that
\[ a_1 + a_2 + 2a_5 + a_6 + a_8 < 1. \]
Then \( T \) has a fixed point in \( X \), which is unique, whenever, \( a_1 + a_4 + a_5 + a_7 + a_9 < 1 \).

The proof can be obtained by observing the proof of Theorem 2, with \( d \) playing the role of \( \mu \).

If we take \( a_4 = a_5 = a_7 = a_9 = 0 \) in Theorem 2, then we get the following result under weaker conditions
\( (5.2.1) \) & \( (5.2.2) \) of the function \( \mu \).
THEOREM 3: Let $P$ be a pseudocompact Tichonov space and
$
\mu$
be a non-negative real valued continuous function
over $P \times P$ ($P \times P$ is Tichonov but need not be pseudocompact).
If $T : P \rightarrow P$ is a continuous map satisfying

\[
\text{(5.2.5) } \mu(x,y) < a_1 \mu(u,v) + a_2 \mu((x,x) + a_3 \mu(y,y)
\]

for all distinct $x, y$ in $X$, where $a_i \geq 0$ are such that
$a_1 + a_2 + a_3 + a_6 + a_8 < 1$. Then $T$ has unique fixed point in $P$.

Remark 1: On taking $a_8 = a_9 = 0$ in Theorem 2, we get a
result proved by Jain and Dixit(Th.1[82]). It is also
observed that, in that theorem the conditions imposed on
the constants $a_i$'s seems to be strong. There is no
mention of the type of the chooseon. This shows
incompleteness of that theorem. Also Dhage and
Dhobale[39] check that the non-negative function $\mu$
defined on $P \times P$ can be weakened to lower semicontinuity of
$\mu$.

Remark 2: On taking $a_6 = a_8 = 0$ in Theorem 3, we get
theorem 3 of Harinath[68].

Remark 3: On taking $a_8 = a_9 = 0$ in Corollary 2, we get
the theorem 3 of Jain and Dixit[82].
Remark 4: On taking $a_6 = a_7 = a_8 = a_9 = 0$ in Theorem 2, we get theorem 5 of Harinath[68].

Remark 5: (a) If we choose $a_2 = a_3$, $a_4 = a_5$ and $a_6 = a_7 = a_8 = a_9 = 0$ in Corollary 2, we get theorem 4 of Fisher[48].

(b) If we choose $a_1 = a_2 = a_5 = a_6 = a_7 = a_8 = a_9 = 0$ and $a_3 = a_4 = \frac{1}{2}$ in Corollary 2, we get theorem 3 of Fisher[48].

(c) If we choose $a_1 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = 0$ and $a_2 = a_3 = \frac{1}{2}$ in Corollary 2, we get theorem 2 of Fisher[48].

(d) If we choose $a_1 = 1$ and $a_2 = \ldots = a_9 = 0$ in Corollary 2, we get theorem 1 of Fisher[48].

5.3 In this section we prove a common fixed point theorem for two continuous self-mappings of psudocompact Tichonov space, which generalize the results of Jain and Dixit[82], Harinath[68] & Dhage and Dhabale[39] on pseudocompact Tichonov spaces.

THEOREM 4: Let $P$ be a pseudocompact Tichonov space and $\mu$ be a non-negative real valued continuous function over $PxP$ ($PxP$ is Tichonov but need not be pseudocompact). Suppose that $\mu$ also satisfies (5.1.2) (5.1.3).

If, $S, T : P \rightarrow P$ be two maps satisfying
(5.3.1) \( \mu(Sx, Ty) < a_1 \mu(x, y) + a_2 \mu(x, Sx) + a_3 \mu(y, Ty) \)

\[ + a_4 \mu(x, Ty) + a_5 \mu(y, Sx) + a_6 \frac{\mu(x, Sx) \mu(y, Ty)}{\mu(x, y)} \]

\[ + a_7 \frac{\mu(x, Ty) \mu(y, Sx)}{\mu(x, y)} + a_8 \mu(x, Sx) \mu(y, Ty) \]

\[ + a_9 \frac{\mu(x, Ty) \mu(y, Sx)}{\mu(Sx, Ty)} + a_{10} \mu(y, Sx)(1 + \mu(x, Ty)) \]

\[ + a_{11} \frac{\mu(x, Ty)(1 + \mu(Sx, y))}{\mu(x, y)} + a_{12} \mu(y, Sx)(1 + \mu(x, Ty)) \]

\[ + a_{13} \frac{\mu(x, Ty)(1 + \mu(Sx, y))}{(1 + \mu(Sx, Ty))} \]

for all \( x, y \) in \( P \), with \( Sx \neq Ty \), where the constants \( a_i \geq 0 \) are such that \( a_1 + a_2 + a_3 + a_6 + a_7 + a_8 + a_9 + 2(a_4 + a_5 + a_{10} + a_{11} + a_{12} + a_{13}) < 1 \). If \( S \) or \( T \) is continuous, then \( S \) and \( T \) have a unique common fixed point in \( P \).

Proof: Suppose that the mapping \( S \) is continuous and we define a function \( \phi : P \rightarrow R \) on pseudocompact Tichonov space \( P \) by \( \phi(x) = \mu(x, Sx) \), for all \( x \) in \( P \). Clearly \( \phi \) is continuous, being the composite of two continuous functions \( S \) and \( \mu \). Since \( P \) is pseudocompact Tichonov space, every real valued function over \( P \times P \) is bounded and attains its bounds. Thus there exists a point \( p \) in \( P \) such that

\[ \phi(p) = \inf \{ \phi(x) : x \in P \} \]
Now suppose that neither S nor T has a fixed point, then by (5.3.1), we have

\[
\mu(S,Tsp) < a_1 \mu(Tsp,Sp) + a_2 \mu(Tsp,STsp) + a_3 \mu(Stsp, Tsp)
+ a_4 \mu(Tsp,Tsp) + a_5 \mu(Stsp,STsp)
+ a_6 \frac{\mu(Tsp,STsp) \mu(Stsp, Tsp)}{\mu(Tsp,Sp)} + a_7 \frac{\mu(Tsp,STsp) \mu(Stsp,STsp)}{\mu(Tsp,Sp)}
+ a_8 \frac{\mu(Tsp,STsp) \mu(Stsp, Tsp)}{\mu(STsp, Tsp)} + a_9 \frac{\mu(Tsp,STsp) \mu(Stsp,STsp)}{\mu(STsp, Tsp)}
+ a_{10} \frac{\mu(Stsp,STsp)(1+ \mu(Tsp,STsp))}{(1+ \mu(Tsp,Sp))} + a_{11} \frac{\mu(Tsp,STsp)(1+ \mu(STsp,Sp))}{(1+ \mu(Tsp,Sp))}
+ a_{12} \frac{\mu(Stsp,STsp)(1+ \mu(Tsp,STsp))}{(1+ \mu(STsp,STsp))} + a_{13} \frac{\mu(Tsp,STsp)(1+ \mu(STsp,Sp))}{(1+ \mu(STsp,STsp))}
\]

or,

\[
\mu(STsp, Tsp) < h \mu(Tsp,Sp) < \mu(Tsp,Sp),
\]

where \( h = \frac{a_1 + a_3 + a_5 + a_6 + a_{10} + a_{12}}{1 - a_2 - a_5 - a_6 - a_{10} - a_{12}} < 1. \)

Again by (5.3.1), we get

\[
\mu(Sp,Tsp) < a_1 \mu(p,Sp) + a_2 \mu(p,Sp) + a_3 \mu(Sp,Tsp)
+ a_4 \mu(p,Tsp) + a_5 \mu(Sp,Sp)
+ a_6 \frac{\mu(p,Sp) \mu(Sp,Tsp)}{\mu(p,Sp)} + a_7 \frac{\mu(p,Tsp) \mu(Sp,Sp)}{\mu(p,Sp)}
+ a_8 \frac{\mu(p,Sp) \mu(Sp,Tsp)}{\mu(Sp,Tsp)} + a_9 \frac{\mu(p,Tsp) \mu(Sp,Sp)}{\mu(Sp,Tsp)}
\]

(inequality continued on next page)
\[
\frac{\mu(Sp,Sp)(1 + \mu(p,Tsp))}{(1 + \mu(p,Sp))} + a_{10} \frac{\mu(p,Tsp)(1 + \mu(Sp,Sp))}{(1 + \mu(p,Sp))} \\
\frac{\mu(Sp,Sp)(1 + \mu(p,Tsp))}{(1 + \mu(Sp,Tsp))} + a_{12} \frac{\mu(p,Tsp)(1 + \mu(Sp,Sp))}{(1 + \mu(Sp,Tsp))} \\
\frac{\mu(p,Tsp)(1 + \mu(Sp,Sp))}{(1 + \mu(p,Sp))} + a_{11} \frac{\mu(p,Tsp)(1 + \mu(Sp,Sp))}{(1 + \mu(Sp,Tsp))} \\
\] 

or, \( \mu(Sp,Tsp) < k \mu(p,Sp) < \mu(p,Sp) \)

where \( k = \frac{a_1 + a_2 + a_4 + a_8 + a_{11} + a_{13}}{1 - a_3 - a_4 - a_6 - a_{11} - a_{13}} < 1. \)

Thus we obtain,

\( \phi(Tsp) = \mu(Tsp,STsp) < \mu(Sp,Tsp) < \mu(p,Sp) = \phi(p), \)

which is a contradiction to the definition of \( \phi(p) \).

Therefore either \( S \) or \( T \) has a fixed point. Suppose there is a fixed point \( u \) in \( P \) such that \( u = Su \). Now we show that \( u \) is also a fixed point of \( T \).

Let us suppose \( u \neq Tu \), then by (5.3.1), we get

\[ \mu(u,Tu) = \mu(Su,Tu) < a_1 \mu(u,u) + a_2 \mu(u,Su) + a_3 \mu(u,Tu) + a_4 \mu(u,Tu) \]

\[ + a_5 \mu(u,Su) + a_6 \frac{\mu(u,Su)}{\mu(u,u)} \mu(u,Tu) \]

\[ + a_7 \frac{\mu(u,Tu)}{\mu(u,u)} \mu(u,Su) + a_8 \frac{\mu(u,Su)}{\mu(u,Tu)} \mu(u,Tu) \]

\[ + a_9 \frac{\mu(u,Tu)}{\mu(Su,Tu)} \mu(u,Su) + a_{10} \frac{\mu(u,Su)(1 + \mu(u,Tu))}{\mu(Su,Tu)} \]

(inequality continued on next page)
\[
\begin{align*}
+a_{11} \frac{\mu(u, Tu)(1+ \mu(Su, u))}{(1+ \mu(u, u))} + a_{12} \frac{\mu(u, Su)(1+ \mu(u, Tu))}{(1+ \mu(Su, Tu))} \\
+a_{13} \frac{\mu(u, Tu)(1+ \mu(Su, u))}{(1+ \mu(Su, Tu))}
\end{align*}
\]

or,
\[
\mu(u, Tu) < (a_3 + a_4 + a_5 + a_7 + a_{11} + a_{13}) \mu(u, Tu) < \mu(u, Tu)
\]

leading to a contradiction, as \(a_3 + a_4 + a_5 + a_7 + a_{11} + a_{12} < 1\).

This gives \(u = Tu\). Thus \(S\) and \(T\) always have a common fixed point \(u\).

To prove the uniqueness of \(u\) let \(v \neq u\) be another common fixed point of \(S\) and \(T\). Then by (5.3.1), we get

\[
\mu(u, v) = \mu(Su, Tv)
\]

\[
< a_1 \mu(u, v) + a_2 \mu(u, Su) + a_3 \mu(v, Tv) + a_4 \mu(u, Tv)
\]

\[
+ a_5 \mu(v, Su) + a_6 \frac{\mu(u, Su) \mu(v, Tv)}{\mu(u, v)}
\]

\[
+ a_7 \frac{\mu(u, Tv) \mu(v, Su)}{\mu(u, v)} + a_8 \frac{\mu(v, Su) \mu(v, Tv)}{\mu(Su, Tv)}
\]

\[
+ a_9 \frac{\mu(u, Tv) \mu(v, Su)}{\mu(Su, Tv)} + a_{10} \frac{\mu(v, Su)(1+ \mu(u, Tv))}{(1+ \mu(Su, Tv))}
\]

\[
+ a_{11} \frac{\mu(u, Tv)(1+ \mu(Su, v))}{1+ \mu(u, v)} + a_{12} \frac{\mu(v, Su)(1+ \mu(u, Tv))}{(1+ \mu(Su, Tv))}
\]

\[
+ a_{13} \frac{\mu(u, Tv)(1+ \mu(Su, v))}{(1+ \mu(Su, Tv))}
\]

\[
< (a_1 + a_4 + a_5 + a_7 + a_9 + a_{10} + a_{11} + a_{12}) \mu(u, v) < \mu(u, v)
\]

again leading to a contradiction and hence \(u = v\). This completes the proof.
Every metric space is a Hausdorff space. Hence as a particular case of the above theorem, we have a following corollary on compact metric spaces.

**COROLLARY 3:** Let \((X,d)\) be a compact metric space and \(T : X \to X\) be a continuous map satisfying

\[
(5.3.2) \quad d(Sx,Ty) < a_1 d(x,y) + a_2 d(x,Sx) + a_3 d(y,Ty) + a_4 d(x,Ty) + a_5 d(y,Sx) + a_6 \frac{d(x,Sx)d(y,Ty)}{d(x,y)} + a_7 \frac{d(x,Ty)d(y,Sx)}{d(x,y)} + a_8 \frac{d(x,Sx)d(y,Ty)}{d(Sx,Ty)} + a_9 \frac{d(x,Ty)d(y,Sx)}{d(Sx,Ty)} + a_{10} \frac{d(y,Sx)(1+d(x,Ty))}{d(x,y)} + a_{11} \frac{d(x,Ty)(1+d(Sx,y))}{d(y,Sx)(1+d(x,Ty))} + a_{12} \frac{d(y,Sx)(1+d(x,Ty))}{d(x,y)} + a_{13} \frac{d(y,Sx)(1+d(x,Ty))}{d(Sx,Ty)}
\]

for all \(x,y\) in \(X\) with \(Sx \neq Ty\).

If \(S\) or \(T\) is continuous, then \(S\) and \(T\) have a common fixed point in \(X\), whenever the constants \(a_i \geq 0\) are such that, \(a_1 + a_2 + a_3 + a_6 + a_7 + a_8 + a_9 + 2(a_4 + a_5 + a_{10} + a_{11} + a_{12} + a_{13}) < 1\).

If we take \(a_4 = a_5 = a_7 = a_9 = a_{10} = a_{11} = a_{12} = a_{13} = 0\) in Theorem 4, then we get following result under
weaker condition (5.1.2) and (5.1.3) of the function $\mu$.

**COROLLARY 4**: Let $P$ be a pseudocompact Tichonov space and $\mu$ be a non-negative real valued function over $P \times P$ ($P \times P$ is Tichonov but need not be pseudocompact). If $S, T : P \to P$ be two maps satisfying

\[(5.3.3) \quad \mu(Sx, Ty) < a_1 \mu(x, y) + a_2 \mu(x, Sx) + a_3 \mu(y, Ty) + a_6 \frac{\mu(x, Sx) \mu(y, Ty)}{\mu(x, y)} + a_8 \frac{\mu(x, Sx) \mu(y, Ty)}{\mu(Sx, Ty)}\]

for all $x, y$ in $X$, with $Sx \neq Ty$ and $a_1 > 0$ are such that, $a_1 a_2 a_3 a_6 a_8 < 1$. If $S$ or $T$ is continuous then $S$ and $T$ have a unique common fixed point in $P$.

Remark 6: If we take $a_{10} = a_{11} = a_{12} = a_{13} = 0$ in Theorem 4, we get theorem 1 of Dhage and Drobale[39].

Remark 7: If we take $a_{10} = a_{11} = a_{12} = a_{13} = 0$ in Corollary 3, we get theorem 4 of Dhage and Drobale[39].

By taking $S = T$ and some specific values to the constants $a_i$'s as mentioned in the previous section, we see that, our results (Theorem 4, Corollary 3,4) contains several results of Fisher[48], Harinath[68], Jain and Dixit[82].

* * *