CHAPTER I
INTRODUCTION

1.1. Ultraspheirical polynomials:

The ultraspherical polynomials or, as they are often termed Gegenbauer polynomials, \( P_n^{(\lambda)}(x) \) are defined by the generating function

\[
(1 - 2x \zeta + \zeta^2)^{-\lambda} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} P_n^{(\lambda)}(x), \quad \lambda > 0.
\]

These polynomials exhibit the property of orthogonality under a weight function. We have in fact

\[
\int_{-1}^{+1} (1 - x^2)^{\lambda - \frac{1}{2}} P_n^{(\lambda)}(x) P_m^{(\lambda)}(x) dx = \delta_{n,m}, \quad n \neq m;
\]

\[
\int_{-1}^{+1} (1 - x^2)^{\lambda - \frac{1}{2}} \left\{ \frac{P_n^{(\lambda)}(x)}{P_m^{(\lambda)}(x)} \right\} dx = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2} + \lambda\right) \Gamma\left(\frac{n+2\lambda}{2}\right)}{(n+\lambda) \Gamma\left(\lambda\right) \Gamma\left(\frac{1}{2} + \lambda\right) \Gamma\left(n+1\right)}.
\]

For the case \( \lambda = \frac{1}{2} \), these ultraspherical polynomials reduce to the well known Legendre polynomials.

On the other hand, the trigonometric system

\[\{1, \cos \theta, \cos 2\theta, \ldots, \cos n\theta, \ldots\}\]

1) Sansone (51)
which is orthogonal in \((0, \infty)\), is only the limiting case of the ultraspherical system

\[
\begin{align*}
\mathbf{P}_0^{(\lambda)}(x), \quad \mathbf{P}_1^{(\lambda)}(x), \quad \cdots \cdots \cdots \cdots \cdots, \quad \mathbf{P}_n^{(\lambda)}(x), \quad \cdots \cdots
\end{align*}
\]

for \(\lambda = 0\), since

\[
\lim_{\lambda \to 0} \frac{1}{\lambda} \mathbf{P}_n^{(\lambda)}(\cos \theta) = \frac{n}{n} \cos \theta, \quad n > 1.
\]

We also have

\[
\mathbf{P}_n^{(1)}(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}.
\]

1.2. Ultraspherical series:— If it be assumed that the series

\[
\alpha_0 \mathbf{P}_0^{(\lambda)}(x) + \alpha_1 \mathbf{P}_1^{(\lambda)}(x) + \cdots + \alpha_n \mathbf{P}_n^{(\lambda)}(x) + \cdots
\]

converges uniformly in the interval \((-1, 1)\) to a function \(f(x)\), we can multiply the terms of the series by

\[
(1 - x^2)^{-\frac{1}{2} - \frac{1}{2}} \mathbf{P}_n^{(\lambda)}(x)
\]

and integrate, term by term, over the interval \((-1, 1)\). Using the orthogonality relations (1.1.2), we have then

\[
\alpha_n = \frac{(n+\lambda) \Gamma(\lambda) \Gamma'(n+1) \Gamma(2\lambda)}{\Gamma(\frac{1}{2} + \lambda) \Gamma'(\frac{1}{2}) \Gamma'(n+2\lambda)}
\]

\[
\int_{-1}^{1} f(t) (1-t^2)^{-\frac{1}{2} - \frac{1}{2}} \mathbf{P}_n^{(\lambda)}(t) \, dt,
\]
provided the integral in (1.2.2) exists. There is, however, no priori reason for supposing that a given function can be expanded in a uniformly convergent series of ultraspherical polynomials. So, instead of starting with the series, and assuming that it has a certain property, we start from the function, and define the coefficients by the formula (1.2.2).

1) The series (1.2.1) is spoken of as the ultraspherical series associated with the function \( f(x) \) where the coefficients \( a_n^\prime \) are defined by the relation (1.2.2).

The above series (1.2.1) is a generalisation of Legendre series for a function \( f(x) \) and reduces to it for the value \( \frac{1}{2} \) of the parameter \( \lambda \) i.e. the Legendre series is

\[
(1.2.3) \quad f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x),
\]

where

\[
(1.2.4) \quad a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x') P_n(x') \, dx'.
\]

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1) See also Mac Robert (43), p. 280. He has given an expansion in terms of Legendre's associated functions \( T_n^m(x) \) which are related to ultraspherical polynomials by the formula \( T_n^m(x) = \text{const} \, (1-x^2)^{m/2} P_{n-m}^{(m+1/2)}(x) \).

2) Sansone (51).
Also, by putting $\kappa = \cos \theta$ and making $\lambda$ tend to zero, the series (1.2.1) reduces to the trigonometric series of $f(\cos \theta)$ in the interval $(0, \pi)$.

1.3. Ultraspheical series on a sphere: If instead of taking the expansion in the linear interval $(-1, +1)$, we take it upon a sphere, we obtain the following ultraspheical series corresponding to a function $f(\theta, \phi)$ which is defined for $0 < \theta < \pi, 0 < \phi < 2\pi$, and is assumed to be integrable $(L)$ over the whole surface of the sphere $S$:

\[(1.3.1) \quad F(\theta, \phi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} \left( \frac{\pi}{\sin \gamma} \right)^{1/2} \int_{S} \int_{S} \frac{p_{n}(\cos \gamma) F(\theta', \phi') d\sigma'}{\left[ \sin^{2} \theta' \sin^{2} (\phi - \phi') \right]^{1/2}},\]

where $\gamma$ is the spherical distance between the points $(\theta, \phi)$ and $(\theta', \phi')$, i.e.

\[\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi'),\]

and

\[d\sigma' = \sin \theta' d\theta' d\phi'.\]

If we put

\[F(\theta, \phi) \equiv f(\cos \theta) = f(\kappa),\]

the series (1.3.1) reduces to the series (1.2.1) at the end.

1) Kogbetliantz (33), p. 627.
points of the linear interval \((-1, 1\)). This is on account of the relation

\[
\frac{\Gamma\left(\frac{1}{2} + \lambda\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma(\lambda)} \int_0^{\pi} \left( \sin \omega \right)^{2\lambda - 1} P_n^{(\lambda)}(\cos \omega) \, d\omega
\]

\[
= \frac{\Gamma(n + 1) \Gamma(2\lambda)}{\Gamma(n + 2\lambda)} P_n^{(\lambda)}(\cos \theta) P_n^{(\lambda)}(\cos \theta'),
\]

where

\[\omega = \phi - \phi'.\]

1.4. In addition to the above-mentioned expansions associated with a given function \(f(\theta)\), there is another type of expansion in a series of ultraspherical polynomials. It can be written as

\[(1.4.1) \quad f(\theta) \sim \frac{\Gamma(\lambda) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \lambda\right)} \sum_{n=0}^{\infty} \frac{n + \lambda}{2\pi} \]

\[
= \int_0^{\pi} f(\phi) P_n^{(\lambda)}(\cos(\phi - \theta)) \left[ \sin^2(\phi - \theta) \right]^{\lambda} d\phi,
\]

\[0 \leq \theta \leq 2\pi.
\]

This expansion is obtained by the help of Angelesco's

\[(6)\]

\[
\int (\pi, \theta) = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(\lambda)}{\Gamma\left(\frac{1}{2} + \lambda\right)} \cdot \frac{\lambda(1 - \eta^2)}{2\pi} \cdot \int_0^{2\pi} f(\phi) \left[ \sin^2(\theta - \phi) \right]^\lambda \frac{d\phi}{\left[1 - 2\eta \cos(\theta - \phi) + \eta^2\right]^{\lambda+1}},
\]

which is a generalisation of Poisson's integral in trigonometric series.

The series (1.4.1) was also termed by Kogbetliantz as an ultraspherical series, but it is essentially different from the series (1.2.1) and the series (1.3.1).

Gegenbauer was the first to have studied the properties of ultraspherical polynomials. He established the property of their orthogonality, their addition theorems and recurrence relations. Szegö has recently given a detailed account of the various properties of the ultra-

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1) Kogbetliantz (34), p. 226.

2) Kogbetliantz (38).

3) Gegenbauer (11), (12), (13), (14), (15). See also Whittaker-Watson (65), pp. 329, 330, 335. The function \( C_\nu^\lambda(\kappa) \) of these authors is identical with \( P_\nu^\lambda(\kappa) \) in our notation.

4) Szegö (60).
spherical polynomials, their asymptotic expansions and inequalities satisfied by them. Some of these inequalities were previously obtained by Heine.

1.6. The problem of summability \((C, S)\) of the series (1.2.1) was posed by Nielsen in 1911 in the preface of his book, 'Théorie des fonctions métasphériques'. Darboux investigated the convergence of the series (1.2.1) in the interior of the interval \((-1, 1)\). He proved that the series would be convergent at an internal point provided the generating function \(f(x)\) satisfies Dirichlet's conditions near the point and its order of infinity at the end points is less than \(\frac{1}{2}(λ+1)\). Darboux also showed that the series diverges everywhere in the interior of the interval \((-1, 1)\) if the order of infinity of \(f(x)\), at an end point is equal to or greater than \(\frac{1}{2}(λ+1)\).

Summability theorems for interior points were investigated first by Adamoff, but it was Kogbetliantz who studied systematically in great details the summability \((C, S)\) of the series (1.2.1) and the series (1.3.1).

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1) Heine (27).
2) Darboux (6).
3) Adamoff (1).
4) Kogbetliantz (32), (33), (34), (36), (37), (38), (40).
1) Kogbetliantz constructed two examples of functions which are continuous in the whole of the interval (-1, 1), but their corresponding ultraspherical series (1.2.1) at a given point \( x \) \((-1 < x < 1)\) are divergent. As a natural consequence, he was led to study summability \((C, \delta)\), \(\delta > 0\), of the series (1.2.1) and he succeeded in extending the earlier results of Haar, Chapman and Gronwall on the \((C, \delta)\) summability of Legendre series. Kogbetliantz proved that the series (1.2.1) is summable \((C, \lambda)\) at an internal point \( x \) of the interval (-1, 1) provided the function \( (1-x^2)^{\lambda+\frac{1}{2}} f(x) \) be integrable \((L)\) in the interval (-1, 1) and the mean value \( \frac{1}{x} \left[ f(x-\delta) + f(x+\delta) \right] \) exists at the point \( x \). In the case of \((C, \delta)\) summability, \(0 < \delta < \lambda\), the first of the above conditions was to be replaced by the integrability of the function \( (1-x^2)^{\frac{1}{2}(\lambda+\delta-1)} f(x) \).

Darboux's theorem on \((C, 0)\) summability of ultraspherical series mentioned above was also generalised by Kogbetliantz for \((C, \delta)\) summability, \(\delta > 0\).

1.7. From his study of the series (1.2.1), Kogbetliantz proceeded to discuss various aspects of the summability problem of the series (1.3.1). He investigated its summability \((C, \delta)\), \(\delta > \lambda\), and established an analogy between the

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1) Kogbetliantz (36).
2) Haar (21).
3) Chapman (5).
4) Gronwall (17).
5) Kogbetliantz (36).
6) Kogbetliantz (37), (38).
7) Kogbetliantz (37)
ultraspherical and trigonometric series from the point of view of their summability by Cesàro means. The results obtained by Kogbetliantz include the theorems of Gronwall, Lukács, Hilb and his own on the Cesàro summability of Laplace series which may be considered to be the particular case of the ultraspherical series (1.3.1) for $\lambda = \frac{1}{2}$.

5) The main result of Kogbetliantz deals with the establishing of the summability $(C, \delta)$, $\delta > \lambda$, of the series (1.3.1) under the condition that the generating function is continuous about the point $(\theta, \varphi)$ and a certain mean value of the function exists at that point. Later, using Riemann-Liouville fractional integrals, Obrechkoff obtained theorems of a more general character than those of Kogbetliantz.

Properties of the ultraspherical expansion, analogous to uniqueness theorems in Riemann's theory of trigonometric series, were also investigated by Kogbetliantz in 1926. Particular cases of his theorems for Legendre series had been already worked out by Dini and Plancherel.

1) Gronwall (16), (17).
2) Lukács (42).
3) Hilb (28), (29).
4) Kogbetliantz (35).
5) Kogbetliantz (38).
6) Obrechkoff (48).
7) Kogbetliantz (40).
8) Dini (7).
9) Plancherel (49).
1.8. We shall first define certain methods of summability and absolute summability which will be used in the subsequent chapters.

(1.8.1) Harmonic summability

**Definition 1.** An infinite series \( \sum u_n \) with partial sums \( S_n = \sum_{k=0}^{n} u_k \) is said to be harmonic summable to the sum \( s \), if the sequence \( \{ t_n \} \) tends to \( s \) as \( n \to \infty \), where

\[
t_n = \frac{S_0 + \frac{S_1}{1} + \frac{S_2}{1+\frac{1}{2}} + \ldots + \frac{S_n}{1+\ldots+\frac{1}{n}}}.
\]

This method was introduced by M. Riesz in 1924. This is a regular Nörlund method generated by the sequence \( \{ \frac{1}{n+1} \} \). Here we have

\[
\rho_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+1} \sim \log n.
\]

The method is clearly equivalent to another method where \( t_n \) is replaced by

\[
t'_n = \frac{S_0 + \frac{S_1}{1} + \frac{S_2}{1+\ldots+\frac{1}{n}}}{\log n}.
\]

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1) Riesz (50).
1) Riesz also proved that an infinite series which is harmonic summable to the sum \( s \) is also summable \( \langle c, \kappa \rangle \) to the same sum for every positive \( \kappa \).

2) \textbf{Definition 2.} A sequence \( \{s_n\} \) is said to be summable by harmonic means, if

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=0}^{n} \frac{s_{n-k}}{k+1}
\]

exists.

1.8.2. \textbf{Absolute Cesàro summability}

The absolute summability \( \langle c, \alpha \rangle \) or summability \( |c, \alpha| \), of a series was defined by Fekete, for the case \( \alpha \) is an integer, and in general by Kogbetliantz.

We denote the \( n \)th Cesàro means of order \( \alpha \) of the sequences

\[
s_n = a_1 + a_2 + \cdots + a_n.
\]

and

\[
t_n = n a_n
\]

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1) Riesz (50).
2) Siddiqi (52).
3) Fekete (9).
4) Kogbetliantz (39).
by $S_n^\alpha$ and $T_n^\alpha$ respectively.  

2) Definition. The series $\sum A_n$ is said to be summable $|C, \alpha; \alpha > 0$, if the series 

$$\sum |S_n^\alpha - S_{n-1}^\alpha|$$

is convergent.

Thus summability $|C, 0|$ is the same as absolute convergence.

From the identity 

$$n(S_n^\alpha - S_{n-1}^\alpha) = T_n^\alpha,$$

it is plain that the summability $|C, \alpha|$ of $\sum f_n$ is equivalent to the convergence of the series 

$$\sum n^{-1} |T_n^\alpha|.$$ 

1) Thus 

$$S_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} S_\nu,$$

where 

$$A_n^\alpha = \frac{\Gamma(\alpha + n + 1)}{\Gamma(\alpha + 1) \Gamma(n + 1)}, \alpha > -1, A_0^{-1} = 1, A_n^{-1} = 0, n > 0.$$

2) Bosanquet (3).
1.8.3. **Summability by logarithmic means**

The sequence of partial sums \( \{s_n\} \) is said to be summable by Riesz logarithmic means of order one or summable \((R, \log n, 1)\) if

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{s_k}{k}
\]

exists.

1.8.4. **Strong summability**

A series \( \sum_{\nu} u_\nu \) with partial sum \( s_\nu \), is said to be strongly summable by Cesàro means, with index \( q \), or summable \([c, q]\), to sum \( s \), if there exists a finite \( s \) such that

\[
\sum_{\nu=1}^{n} |s_\nu - s|^q = o(n).
\]

It is well known that the strong Cesàro summability implies ordinary summability but the converse is not true.

1.9. This dissertation contains nine chapters. Chapter II and III are on the Cesàro summability of the ultraspherical series. Chapter IV is on the approximation to the generating function by the Cesàro means of its

1) Hardy (22).

2) Hardy and Littlewood (23).
ultraspherical series. In chapter V we consider the absolute Cesàro summability of the ultraspherical series. Chapter VI is on the summability by logarithmic means of the ultraspherical series. In chapter VII we consider a problem on the partial sums of Legendre series. Chapter VIII is entitled "On the harmonic summability of Legendre series", and the last is on strong summability of Legendre series.

1.10. Chapter II is entitled "On the Cesàro summability of the ultraspherical series".

A generalised mean value of \( f(\theta, \phi) \) on the sphere has been defined by Kogbetliantz as follows:

\[
(1.10.1) \quad f(\omega) = \frac{1}{2\pi (\sin \omega)^{2\lambda}} \int_{C\omega} \frac{f(\theta', \phi') d\theta' d\phi'}{\left[ \sin^2 \theta' \sin^2 (\phi' - \phi) \right]^\frac{1}{2} - \lambda}
\]

where the integral is taken along the circle whose centre is \( (\theta, \phi) \) on the sphere and whose curvilinear radius is \( \omega \).

It is assumed throughout that the function

\[
(1.10.2) \quad f(\theta', \phi') \left[ \sin^2 \theta' \sin^2 (\phi - \phi') \right]^\lambda - \frac{1}{2}
\]

is absolutely integrable \((L)\) over the sphere \( S \). In the case \( \kappa < 2 \lambda \), \( \kappa \) being the order of Cesàro sum, we also assume the integrability \((L)\) on the whole sphere of the function

1) Kogbetliantz (38).
(1.10.3) \[ (\cos \frac{\omega}{2})^{\kappa - 2\lambda} f(\theta', \phi') \left[ \sin^2 \theta' \sin^2 (\phi - \phi') \right]^{\lambda - \frac{1}{2}} \]

We write

\[ \Phi(\omega) = \left[ f(\omega) - \frac{A\Gamma(\lambda)}{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2} + \lambda)} \right] (\sin \omega)^{2\lambda}; \]

\[ \Phi_p(x) = \frac{1}{\Gamma(p)} \int_0^x (x - t)^{p-1} \Phi(t) \, dt; \]

\[ \Phi_0(x) = \Phi(x); \]

\[ \Phi_p(x) = \Gamma(p+1) \bar{x}^p \Phi_p(x), \quad p > 0 \]

and

\[ \Phi_p'(x) = \frac{d}{dx} \Phi_{p+1}(x), \quad -1 < p < 0. \]

1) Recently, Gupta has obtained theorems on ultraspHERIcal series on a sphere which extend Wang's results on Cesàro summability of Fourier series. Izumi and 3) Sunouchi have proved the following theorem for Fourier series.

1) Gupta (19).
2) Wang (63).
3) Izumi and Sunouchi (31).
Theorem A: If

\[ 0 < \beta < \gamma, \quad \beta \leq 1 + \sqrt{-\beta} \quad \text{and} \]

\[ \Phi_\beta(t) = o( t^{\gamma} ) \]

then the Fourier series of \( f(t) \) is summable \( (C, \frac{\beta}{\sqrt{-\beta} + 1}) \) at \( t = \alpha \).

In this chapter we prove the following theorems which are analogous to those of Izumi and Sunouchi.

Theorem 1. If \( 0 < \beta < 1 \) and

\[ \Phi_\beta(t) = o\left( t^{\gamma + \frac{2\lambda(\gamma + 1)}{\lambda + 1}} \right) \]

for \( \gamma > \beta \) and \( 0 < \lambda < 1 \),

then the series \((1.3.1)\) is summable \( (C, \alpha + \lambda) \) at the point \( (\theta, \phi) \) to the sum \( A \), where

\[ \lambda = \frac{\beta}{\sqrt{-\beta} + 1} \]

Theorem 2. If

\[ 0 < \beta < \gamma, \quad \beta \leq 1 + (\sqrt{-\beta}) \quad \text{and} \]

\[ \Phi_\beta(t) = o\left( t^{\gamma \left(1 + \left(\frac{1+\omega}{2} \right)^2 \lambda \right)} \right) \]

where

\[ 2\lambda \omega > 1 \quad \text{and} \quad 0 < \lambda < 1 \],
(17)

then the series (1.3.1) is summable \((C, \lambda + \gamma)\) at the point \((\theta, \phi)\) to the sum \(A\), where

\[
\lambda = \frac{\beta}{\sqrt{\gamma - \beta + 1}}.
\]

1.11. In Chapter III, "On the Cesàro summability of the ultraspHERical series", we prove the following theorems:

**Theorem 1.** If \(\gamma > \beta > 0\) and

\[
\Phi_{\beta}(t) = O\left(t^{\gamma + (1 - \omega)2\lambda}\right)
\]

for \(2\lambda \omega > 1\) and \(0 < \lambda < 1\),

then the series (1.3.1) is summable \((C, \lambda + \gamma)\) at the point \((\theta, \phi)\) to the sum \(A\), where

\[
\lambda = \frac{\sqrt{(m-1)} + \beta}{\sqrt{\gamma + m - \beta}}
\]

and \(m\) is a positive integer such that

\[m > \beta > m-1.\]

1) The content of this chapter has been reviewed by Kogbetliantz, Review (41).
Theorem 2. If \( \beta > 1 \) and
\[
\Phi_p(t) = o\left(t^{\frac{\beta + \lambda}{\beta - p - 1}}\right)
\]
for \( \beta - 1 < \lambda < \beta \) and \( 0 < \lambda < 1 \),
then the series \( (1.3.1) \) is summable \( (C, \lambda + \lambda) \) at the
point \( (\theta, \phi) \) to the sum \( A \).

The above theorems are analogous to the results
1) \( 1 \) and 2) \( 2 \) of Izumi and Sunouchi and Wang on the Cesàro summability
of Fourier series.

1.12. Chapter IV is entitled "Approximation to the
generating function by the Cesàro means of its ultraspherical
series".

3) Obrechkoff has proved the following result on
the order of Cesàro means \( \sigma^\lambda \) of the series \( (1.3.1) \):
Theorem: If \( p > 0 \), \( 0 < \lambda < 1 \), such that
\[
\int_0^t |\Phi_p(t)| \, dt = O\left(t^{1 + 2\lambda + \lambda}\right)
\]
\[
\Phi_{p+1}(t) = o\left(t^{2\lambda + \lambda}\right), \text{ as } t \to 0
\]

1) Izumi and Sunouchi (31).
2) Wang (63), (64).
3) Obrechkoff (43). See also Obrechkoff (47) where similar
theorems in Fourier series were established.
then, for each $\lambda$, 

$$p + \lambda + 1 \leq K > p + \lambda + \lambda$$

we have

$$\sigma_{\lambda}^K - \Lambda = O\left(\frac{1}{n^{\lambda}}\right).$$

In this chapter we propose to calculate the degree of approximation to the generating function $f(\theta, \phi)$ by the $C$-means of the series (1.3.1). We shall prove the following theorem:

**Theorem.** If

$$\Phi^*(t) = \int_0^t \frac{\phi_0(t)}{t^{2+2\lambda}} e^{\lambda t} = o\left[(\log \frac{1}{t})^{n+1}\right]$$

and

$$\Phi_{p+1}(t) = O\left[\frac{2^\lambda}{t} (\log \frac{1}{t})^{n+1}\right]$$

as $t \to 0$, for $-1 < n < \infty$ and $p > 0$, then

$$\sigma_{\lambda}^K - \Lambda = O\left[(\log n)^{n+1}\right]$$

where

$$\lambda + \lfloor p \rfloor + 1 \leq K > p + \lambda.$$
1.13. Chapter V is entitled "On the absolute Cesàro summability of the ultraspherical series".

1) Bosanquet proved the following theorem for absolute Cesàro summability of Fourier series.

**Theorem A.** If \( \phi(t) \) is of bounded variation in the interval \((0, \pi)\), then the Fourier series of \( f(t) \) is summable \( \{c, \delta\} \) at the point \( t = \pi \), for every \( \delta > 0 \).

The object in this paper is to obtain an absolute Cesàro summability of the ultraspherical series. We prove the following theorem which is analogous to the above mentioned theorem of Bosanquet on Fourier series.

**Theorem B.** If \( 0 < \lambda < 1 \) and \( \frac{f(\omega)}{(\omega \sin \omega)^{2\lambda}} \) is of bounded variation in the interval \((0, \pi)\), then the ultraspherical series (1.2.1) is summable \( \{c, \delta\} \) at the point \((\theta, \phi)\), for every \( \delta > \lambda \), where

\[
f(\omega) = f(\omega)(\sin \omega)^{2\lambda - 1}.
\]

1.14. In Chapter VI, we discuss "The summability by logarithmic means of the ultraspherical series".

2) Hardy proved the following theorem on \((R, \log n, 1)\) summability of the Fourier series.

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1) Bosanquet (3).
2) Hardy (22).
Theorem A. If

\[ \Phi(t) = \int_0^t |\phi(u)| \, du = o \left( t \log \frac{1}{t} \right) \]

(a condition satisfied whenever \( \phi(t) = o \left( \log \frac{1}{t} \right) \) and in particular when \( f(t) \) is bounded near \( t = \infty \) )
then a necessary and sufficient condition that the series should be summable (R, 1), for \( t = \infty \), to the sum \( s \), is

\[ \Psi(t) = \int_t^\infty \frac{\phi(u)}{u} \, du = o \left( \log \frac{1}{t} \right) \], as \( t \to 0 \).

In this chapter we shall prove the following theorem:

Theorem. If \( 0 < \lambda < 1 \) and

\[ \int_t^\infty \frac{|\phi(u)|}{u^{1+\lambda}} \, du = o \left( \log \frac{1}{t} \right) \], as \( t \to 0 \)

then the series (1.3.1) is summable (R, log \( n \), 1) to the value zero, where

\[ \phi(\omega) = \frac{\Gamma'(\lambda)}{\Gamma'(\frac{1}{2}) \Gamma'(\frac{1}{2} + \lambda)} f(\omega) (\sin \omega)^{2\lambda - 1}. \]

For \( \lambda = \frac{1}{2} \) , we get a corresponding result for Laplace series.
1.15. Chapter VII is entitled "On the partial sums of Legendre series".

1) Szàsz proved the following theorem:

**Theorem A.** Let \( f(\theta) \) be a real, even and Lebesgue integrable function, let

\[
f(\theta) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta
\]

and

\[
S_n = \frac{1}{2} a_0 + a_1 + \cdots + a_n, \quad n \gg 1,
\]

If

\[
\int_{0}^{\theta} |f(t)| dt = o\left(\frac{\theta}{\log(\frac{1}{\theta})}\right)
\]

then

\[
S_n = o(\log \log n)
\]

We shall prove the following theorem:

**Theorem B.** If \( f(x) \in L \), in the linear interval \((-1, 1)\), then at every point \( x \) within the range \((-1, 1)\) for which

1) Szàsz (59).
(23)

\[ \Phi_x(\hbar) = \int_0^1 |f(x \pm t) - f(x)| \, dt = O\left( \frac{\hbar}{\log \frac{1}{\hbar}} \right) \]

the following relation holds:

\[ |S_n(x) - f(x)| = O(\log \log n) \]

where

\[ S_n(x) = \sum_{m=0}^{n} a_m P_m(x) \]

\[ a_m = \frac{1}{2} (2m+1) \int_{-1}^{1} f(x) P_m(x) \, dx \]

\( P_m(x) \) being the \( m \)th Legendre polynomial.

This extends the theorem of Szász to Legendre series.

1.16. Chapter VIII is entitled "On the harmonic summability of Legendre series".

For the harmonic summability of Fourier series

1) Siddiqi proved the following theorem:

Theorem A: The Fourier series of a function \( f(x) \), integrable in the sense of Lebesgue and periodic with period \( 2\pi \), is summable by harmonic means at a point \( x \) at which

1) Siddiqi (52).
(24)

\[ q_1(t) = \int_{0}^{t} |q(u)| \, du = o\left(\frac{t}{\log \frac{1}{t}}\right) \]

where

\[ q(t) = f(x+t) + f(x-t) - 2f(x) \cdot \]

In this chapter, we generalise the above theorem of Siddiqi for Legendre series. We shall prove the following theorem:

**Theorem B.** If

\[ (1.16.1) \quad \Phi_{\kappa}(h) = \int_{0}^{\frac{h}{\kappa}} |f(x+\pm t) - f(x)| \, dx = o\left(\frac{h}{\log \frac{1}{h}}\right) \]

then the Legendre series (1.2.3) is summable by harmonic means at every interior point \( x \) of the range \((-1,1)\) for which the condition (1.16.1) is satisfied.

1.17. Chapter IX deals with "Strong summability of Legendre series".

For the strong summability of the Fourier series,\(^1\)

Hardy and Littlewood proved the following theorem:

**Theorem A.** If \( f(x) \in L^p, \quad p > 1, \) and if \( \kappa \) is any positive number, then, at every point \( x \) for which

\[ (1.17.1) \quad \Phi_{\kappa,x}(h) = o(h) \rightarrow \]

\[ \]

---

we have
\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \left| S_{n}(x) - f(x) \right|^{k} = 0,
\]
where
\[
\phi(x, h) = \int_{0}^{h} |\phi(t)|^{n} \, dt,
\]
\[
\phi(x, h) = f(x+h) + f(x-h) - 2f(x),
\]
and
\[
S_{n}(x) = a_{0} + \sum_{m=1}^{n} \left( a_{m} \cos mx + b_{m} \sin mx \right).
\]

Following the result of Carleman Sutton proved the assertion of theorem A even when the condition (1.17.1) be replaced by the following two conditions
\[
\int_{0}^{t} |\phi(x)|^{q_{r}} \, dx = o(t), \quad 0 < q_{r} < 1,
\]
\[
\int_{0}^{t} |\phi(x)|^{p} \, dx = \mathcal{O}(t), \quad 1 < p < 2.
\]

Ultimately, Hardy and Littlewood proved the following theorem:

1) Carleman (4).
2) Sutton (58).
3) Hardy and Littlewood (24).
Theorem B. If \( p > 1 \) and

\[
\int_0^t \phi(u)^p \, du = O(t) ,
\]
\[
\int_0^t \phi(u) \, du = o(t) ,
\]

then

\[
\sum_{n=0}^\infty | S_m - s_n |^{q_r} = O(n)
\]

for every positive \( q_r \).

Further work on the strong summability of Fourier series and its conjugate series has been done by U.N. Singh.

2) Foà has established a result on the strong summability of Legendre series which is analogous to theorem A of the Fourier series. Let \( f(x) \) be a function belonging to the class \( L^p \), \( p > 1 \), in the linear interval \((-1, +1)\). The Legendre series associated with the function is

\[
(1.17.2) \quad f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x) ,
\]

where

\[
a_n = \frac{1}{2} (2n+1) \int_{-1}^1 f(u) P_n(u) \, du .
\]


2) Foà (10).
(27)

If, now, \( \bar{S}_n(x) \) be used to denote the \( n \)th partial sum of the series \((1.17.2)\), then Foà's result can be stated in the form of theorem C given below:

**Theorem C.** If

\[
\Phi_{x,p}(\delta) = \int_0^\delta |f(x + t) - f(x)|^p \, dt = o(\delta)
\]

then

\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{\nu=0}^n \big| S_\nu(x) - f(x) \big|^\kappa = o
\]

at every interior point \( x \) of the range \((-1, +1)\), \( \kappa \) being any positive number represented by \( \frac{p}{p-1} \).

1) Gupta generalised the above theorem on the lines of Sutton. He proved the following theorem:

**Theorem:** If

\( f(x) \in L^p \), \( 1 < p < 2 \), and if

\[
\Phi_{x,p}(\delta) = \int_0^\delta |f(x + t) - f(x)|^p \, dt = O(\delta)
\]

and

\[
\Phi_{x,q}(\delta) = \int_0^\delta |f(x + t) - f(x)|^q \, dt = O(\delta), \quad o < q < 1
\]

then

1) Gupta (18).
\[(28)\]
\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{\nu=0}^{n} \left| \overline{f}_\nu(x) - f(x) \right|^p = 0
\]

at an interior point \( x \) of the interval \((-1, +1)\).

In this chapter we shall prove the following theorems:

**Theorem 1.** If

\[ f(x) \in L^p, \quad p > 1, \quad \text{and if} \]
\[
\phi_{x,p}(h) = \int_0^h \left| f(x + t) - f(x) \right|^p \, dt = O(h),
\]

and

\[
\int_0^h \left\{ f(x + t) - f(x) \right\} \, dt = o(h)
\]

then

\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{\nu=0}^{n} \left| \overline{f}_\nu(x) - f(x) \right|^{q_n} = 0
\]

at every interior point \( x \) of the range \((-1, +1)\), and for every positive \( q_n \).

This theorem is analogous to theorem B of Hardy and Littlewood.
Theorem 2. If

\[ f(x) \in L^p, \quad 1 \leq p \leq 2, \quad \text{and if} \]

\[ \Phi_{x,p}(h) = \left[ \int_{0}^{h} \left| f(x + t) - f(x) \right|^p \, dt \right]^{1/p} = O \left( \frac{h^{p-1}}{p} \right), \]

and

\[ \int_{0}^{h} \left\{ f(x + t) - f(x) \right\} \, dt = O \left( \frac{2(p-1)}{p} h \right), \]

then

\[ \sum_{\nu=1}^{n} \frac{\left| \tilde{S}_\nu(x) - f(x) \right|^p}{\nu^{2-p}} = O(n), \]

at every interior point \( x \) of the range \( [-1, +1] \).

Theorem 3. If

\[ f(x) \in L^p, \quad 1 \leq p \leq 2, \quad \text{and if} \]

\[ \Phi_{x,p}(h) = \left[ \int_{0}^{h} \left| f(x + t) - f(x) \right|^p \, dt \right]^{1/p} = O(h), \]

and

\[ \int_{0}^{h} \left\{ f(x + t) - f(x) \right\} \, dt = O(h) \]

then

\[ \sum_{\nu=1}^{n} \frac{\left| \tilde{S}_\nu(x) - f(x) \right|^p}{\nu^{2-p}} = o(n^{p-1}) \]
at every interior point $x$ of the range $(-1, +1)$.

Theorems 2 and 3 are analogous to those of 1)

B.D. Singh for Fourier series.

1) Singh, B.D. (53).