CHAPTER IX

ON A SEQUENCE OF FOURIER COEFFICIENTS

9.1. Let \( f(t) \) be a function which is integrable in the sense of Lebesgue over the interval \( (-\pi, \pi) \) and is defined outside this interval by periodicity. Let the Fourier series of \( f(x) \) be

\[
\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(x);
\]

then the conjugate series of \( (9.1.1) \) is

\[
\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x).
\]

Fejér ¹ has shown that if \( l = \int f(x+\epsilon) - f(x-\epsilon) \)
exists and is finite, the sequence \( \{ n B_n(x) \} \) is summable (C, r), \( r > 1 \), to the value \( l/\pi \); and if \( f \) is of bounded variation, the theorem holds true for \( r > 0 \). It has also been proved that if \( l \) exists and is finite, the sequence \( \{ n B_n(x) \} \) is summable by the first logarithmic mean to the same value. ²

Obrechkoff ³ has shown that if \( f \) is integrable (L) and if \( t^{-1} \int |f(x+t) - f(x-t) - l| \) is integrable near \( t=0 \),
then \( \pi^{-1} \sum_{n=1}^{\infty} t B_n(x) \to \pi^{-1}l. \)

³) N. Obrechkoff [1].

* Published in Proc. Amer. Math. Soc. 7 (1956), 796–803.
Mohanty and Nanda have proved that if
\[ f(x+t) - f(x-t) - l = o\left\{ (\log \frac{1}{t})^{-1} \right\} \]
as \( t \to 0 \)
and \( a_n \) and \( b_n \) are \( O(n^{-\delta}) \), \( 0 < \delta < 1 \), then the sequence \( \{ n B_n (x) \} \) is summable \((C,1)\) to the value \( l/\pi \). From this result they have deduced the Hardy and Littlewood's test for convergence of the conjugate series \((9.1.2)\) by applying a Tauberian theorem of Hardy and Littlewood.

We write
\[ \psi(t) = f(x+t) - f(x-t) - l. \]

In this chapter we prove the following theorem:

**Theorem 1.** If
\begin{equation}
(9.1.3) \quad \Psi(t) = \int_0^t \psi(u) \, du = o(t)
\end{equation}
and
\begin{equation}
(9.1.4) \quad \int_{\varepsilon}^{\delta} \frac{\left| \psi(t+\varepsilon) - \Psi(t) \right|}{t} \, dt \to 0,
\end{equation}
for some fixed \( \delta \), when \( \varepsilon \to 0 \), then the sequence \( \{ n B_n (x) \} \) is summable \((C,1)\) to the value \( l/\pi \).

From Theorem 1, we deduce Lebesgue's test for convergence of the conjugate series \((9.1.2)\) by applying Tauber's Second Theorem.

---

4) R. Mohanty and M. Nanda [1].
It is to be noted that Obrechkoff has proved the summability \((C,1)\) of the sequence \(\{n B_n(x)\}\) under Dini's criterion for the convergence of Fourier series while from Fejér's result it is obvious that the sequence is also summable \((C,1)\) under Jordan's criterion. In Theorem 1, we prove the same result under Lebesgue's criterion. As it is well known that Lebesgue's test includes both Dini's and Jordan's tests, Theorem 1 includes the above mentioned results of Obrechkoff and Fejér.

9.2. We shall require the following lemmas:

**LEMMA 1.** If

\[
I = \int_{\pi/n}^{\delta} \frac{\psi(t) \sin nt}{t^2} \, dt, \quad I' = \int_{\pi/n}^{\delta} \frac{\psi(t) \sin nt}{t (t + \pi/n)} \, dt
\]

and \(\psi(t)\) satisfies (9.1.3), then

\[
I - I' = o(n), \quad \text{as} \quad n \to \infty.
\]

We have

\[
I - I' = \frac{\pi}{n} \int_{\pi/n}^{\delta} \frac{\psi(t)}{t^2 (t + \pi/n)} \sin nt \, dt
\]

\[
= \frac{\pi}{n} \left[ \frac{\psi(\delta) \sin n\delta}{\delta^2 (\delta + \pi/n)} - \int_{\pi/n}^{\delta} \psi(t) \frac{d}{dt} \left\{ \frac{\sin nt}{t^2 (t + \pi/n)} \right\} \right]
\]

\[
= \frac{\pi}{n} \left[ O(1) + o \left( \int_{\pi/n}^{\delta} \frac{t (\frac{n}{t^3} + \frac{1}{t^4}) \, dt} \right) \right]
\]

(9.2.1) \(= o(n)\) \quad \text{by (9.1.3)},

which proves the lemma.
Lemma 2. If \( \psi(t) \) satisfies (9.1.3), then
\[
\int_{\pi/n}^{2\pi/n} \frac{\psi(t)}{t} \cos nt \, dt = o(1) \quad \text{as} \quad n \to \infty.
\]

We have, by integration by parts,
\[
\int_{\pi/n}^{2\pi/n} \frac{\psi(t)}{t} \cos nt \, dt = \left[ \frac{\psi(t)}{t} \cos nt \right]_{\pi/n}^{2\pi/n} - \int_{\pi/n}^{2\pi/n} \frac{\psi(t)}{t} \frac{d}{dt} \left( \frac{\cos nt}{t} \right) \, dt
\]
\[
= o(1) + o \left( \int_{\pi/n}^{2\pi/n} \frac{\cos nt}{n + 1/t} \, dt \right)
\]
\[
= o(1) + o \left( \left[ \log t \right]_{\pi/n}^{2\pi/n} \right)
\]
\[
= o(1).
\]

Corollary. If
\[
\pi/n \leq \xi_1 < \xi_2 \leq 2\pi/n,
\]
then
\[
(9.2.2) \quad \int_{\xi_1}^{\xi_2} \frac{\psi(t)}{t} \cos nt \, dt = o(1),
\]
and similarly
\[
\int_{\xi_1}^{\xi_2} \frac{\psi(t)}{t} \sin nt \, dt = o(1).
\]

Lemma 3. 5) If \( \psi(t) \) satisfies (9.1.3) and (9.1.4), then
\[
(9.2.3) \quad \mathcal{J} = \int_{\pi/n}^{\pi} \frac{\psi(t)}{t (t + \pi/n)} e^{int} \, dt
\]
\[
= o(n).
\]

5) The proof of this lemma is on the lines of Hardy and Rogosinsky [1, p.44] and is given only for the sake of completeness.
We have

\[(9.2.4) \quad J = \left( \int_{\pi/n}^{2\pi/n} + \int_{\pi/n}^{\delta} \right) \frac{\psi(t)}{t(t + \pi/n)} e^{nit} \, dt \]

\[= J_1 + J_2 ,\]

say. Now by (9.2.2)

\[(9.2.5) \quad J_1 = \frac{n}{2\pi} \int_{\pi/n}^{\xi} \frac{\psi(t)}{t} e^{nit} \, dt , \quad \frac{\pi}{n} < \xi < \frac{2\pi}{n} ,\]

\[= o(n).\]

Also

\[J_2 = \int_{\pi/n}^{\delta} \frac{\psi(t)}{t(t + \pi/n)} e^{nit} \, dt = -\int_{\pi/n}^{\delta - \pi/n} \frac{\psi(t + \pi/n)}{(t + \pi/n)(t + 2\pi/n)} e^{nit} \, dt \]

\[= -\int_{\pi/n}^{\delta} \frac{\psi(t + \pi/n)}{(t + \pi/n)(t + 2\pi/n)} e^{nit} \, dt \]

\[+ \int_{\delta - \pi/n}^{\delta} \frac{\psi(t + \pi/n)}{(t + \pi/n)(t + 2\pi/n)} e^{nit} \, dt\]

\[(9.2.6) \quad = -\int_{\pi/n}^{\delta} \frac{\psi(t + \pi/n)}{(t + \pi/n)(t + 2\pi/n)} e^{nit} \, dt + o(n).\]

From (9.2.3), (9.2.4), (9.2.5) and (9.2.6) we have

\[J = \frac{1}{2} \int_{\pi/n}^{\delta} \left\{ \frac{\psi(t)}{t(t + \pi/n)} - \frac{\psi(t + \pi/n)}{(t + \pi/n)(t + 2\pi/n)} \right\} e^{nit} \, dt + o(n)\]

\[= \frac{1}{2} \int_{\pi/n}^{\delta} \frac{\psi(t) - \psi(t + \pi/n)}{(t + \pi/n)(t + 2\pi/n)} e^{nit} \, dt \]

\[+ \frac{\pi}{n} \int_{\pi/n}^{\delta} \frac{\psi(t)}{t(t + \pi/n)(t + 2\pi/n)} e^{nit} \, dt + o(n)\]

\[= J_3 + J_4 + o(n) , \text{ say.}\]
Now,

\[ |J_3| \leq \frac{n}{\delta \pi} \int_{\pi/n}^{\delta} \frac{\psi(t + \pi/n) - \psi(t)}{t} \, dt = o(n), \]

since \( \psi(t) \) satisfies (9.1.4).

Again,

\[ J_4 = \frac{\pi}{n} \frac{\Psi(\delta) e^{ni\delta}}{\delta (\delta + \pi/n)(\delta + 2\pi/n)} + o(n) \]

\[ - \frac{\pi}{n} \int_{\pi/n}^{\delta} \Psi^*(t) \frac{d}{dt} \left\{ \frac{e^{ni t}}{t (t + \pi/n)(t + 2\pi/n)} \right\} \, dt \]

\[ = O\left(\frac{1}{n}\right) + o(n) + \frac{1}{n} \int_{\pi/n}^{\delta} o(t) \cdot O\left(\frac{n}{t^3} + \frac{1}{t^4}\right) \, dt \]

\[ = o(1) + o(n) = o(n). \]

Thus

(9.2.7) \quad J = o(n),

which proves the lemma.

9.3. PROOF OF THE THEOREM. From Mohanty and Nanda 6), we have

\[ \frac{1}{n} \sum_{i=1}^{n} B_{\alpha}(x) - \frac{1}{\pi} = \frac{1}{\pi} \int_0^\pi \left\{ f(x+t) - f(x-t) - l \right\} g(n, t) \, dt + o(1) \]

\[ = \frac{1}{\pi} \int_0^\pi \psi(t) \, g(n, t) \, dt + o(1) \]

(9.3.1) \quad = \frac{1}{\pi} P + o(1),

say, where

-----------

6) R. Mohanty and M. Nanda [1].
\[ g(n,t) = -\frac{i}{n} \frac{d}{dt} \left\{ \cos t + \cos 2t + \ldots + \cos nt \right\} \]

\[ = -\frac{i}{n} \frac{d}{dt} \left\{ \cos \left(\frac{(n+1)/2}t\right) \frac{\sin nt/2}{\sin t/2} \right\} \]

\[ = -\frac{i}{2n} \frac{d}{dt} \left\{ \frac{\sin nt}{\tan t/2} + \cos nt \right\} - \left\{ \frac{1}{4n} \frac{\sin nt}{\sin^2 t/2} - \frac{1}{2n} \frac{\sin nt}{\tan t/2} \right\} \]

Now

\[ P = \int_0^\pi \psi(t) g(n,t) \, dt \]

\[ = \int_0^\pi \psi(t) \left[ \frac{\sin nt}{4n \sin^2 t/2} - \frac{\cos nt}{2 \tan t/2} \right] \, dt + \frac{1}{2} \int_0^\pi \psi(t) \sin nt \, dt \]

\[ = \int_0^\pi \psi(t) \left[ \frac{\sin nt}{nt^2} - \frac{\cos nt}{t} \right] \, dt + o(1) \]

\[ = \left( \int_0^{\pi/n} + \int_{\pi/n}^{\pi/2} + \int_{\pi/2}^\pi \right) \psi(t) \left[ \frac{\sin nt}{nt^2} - \frac{\cos nt}{t} \right] \, dt + o(1) \]

\[ = \left( \int_0^{\pi/n} + \int_{\pi/n}^{\delta} \right) \psi(t) \sigma(n,t) \, dt + o(1) \]

\[ (9.3.2) = P_1 + P_2 + o(1), \]

say, by Riemann-Lebesgue Theorem, where

\[ \sigma(n,t) = \frac{\sin nt}{nt^2} - \frac{\cos nt}{t} = O\left(n^2 t\right), \quad \text{as } t \to 0. \]

Now,

\[ P_1 = \int_0^{\pi/n} \psi(t) \sigma(n,t) \, dt - \int_0^{\pi/n} \psi(t) \left( \frac{d}{dt} \sigma(n,t) \right) \, dt \]

\[ (9.3.3) = o(1) + o\left(n^2 \int_0^{\pi/n} \frac{1}{t} \, dt\right) = o(1), \]

by (9.1.3).
Again, by applying Lemma 1 and Lemma 2, we have

\[ p_2 = \frac{1}{n} \int_{\pi/n}^{\pi} \frac{\psi(t)}{t^2} \frac{\sin nt}{t} \, dt - \int_{\pi/n}^{\pi} \frac{\psi(t)}{t} \frac{\cos nt}{t} \, dt \]

\[ = \frac{1}{n} \int_{\pi/n}^{\pi} \frac{\psi(t) \sin nt}{t} \, dt - \int_{\pi/n}^{\pi} \frac{\psi(t) \cos nt}{t} \, dt + o(1) \]

(9.3.4) \[ = -\int_{\pi/n}^{\pi} \frac{\psi(t) \cos nt}{t} \, dt + o(1), \quad \text{by (9.2.7)} \]

\[ = -\left( \int_{\pi/n}^{\pi} + \int_{\pi/n}^{\pi/2} - \int_{\pi/2}^{\pi} \right) \frac{\psi(t) \cos nt}{t} \, dt + o(1) \]

\[ = -\int_{\pi/n}^{\pi/2} \frac{\psi(t) \cos nt}{t} \, dt + o(1) \]

(9.3.5) \[ = \int_{\pi/n}^{\pi/2} \frac{\psi(t + \pi/n) \cos nt}{t + \pi/n} \, dt + o(1). \]

By (9.3.3) and (9.3.4) we have

\[ |p_2| = \frac{1}{2} \left| \int_{\pi/n}^{\pi/2} \left\{ \frac{\psi(t + \pi/n)}{t + \pi/n} - \frac{\psi(t)}{t} \right\} \cos nt \, dt \right| + o(1) \]

\[ = \frac{1}{2} \left| \int_{\pi/n}^{\pi/2} \frac{\psi(t + \pi/n) - \psi(t)}{t + \pi/n} \cos nt \, dt \right| \]

\[ - \frac{\pi}{2n} \int_{\pi/n}^{\pi/2} \frac{\psi(t)}{t (t + \pi/n)} \cos nt \, dt \right| + o(1) \]

\[ \leq \frac{1}{2} \int_{\pi/n}^{\pi/2} \left| \frac{\psi(t + \pi/n) - \psi(t)}{t} \right| \, dt \]

\[ + \frac{\pi}{2n} \left| \int_{\pi/n}^{\pi/2} \frac{\psi(t)}{t (t + \pi/n)} \cos nt \, dt \right| + o(1) \]

(9.3.6) \[ = o(1) + \frac{\pi}{2n} o(n) = o(1), \]
by (9.2.7) and (9.1.4).

From (9.3.2), (9.3.3) and (9.3.6) we obtain
\[ p = o(1). \]

Hence from (9.3.1) we have
\[ \frac{1}{n} \sum_{1}^{n} \lambda B_{k}(x) - \frac{1}{\pi} = o(1) \quad \text{as} \ n \to \infty, \]
which proves the theorem.

9.4. Lebesgue's criterion for the convergence of the conjugate series is

**Theorem 2.** If
\[ \theta(t) = f(x+t) - f(x-t) \]
satisfies the conditions

(9.4.1)
\[ \int_{0}^{t} \theta(u) \, du = o(t) \]

and

(9.4.2)
\[ \int_{t}^{\delta} \frac{|\theta(t+\epsilon) - \theta(t)|}{t} \, dt \to 0, \]

for some fixed \( \delta \), when \( \epsilon \to 0 \), then the conjugate series \( \sum B_{n}(x) \) converges to the value

(9.4.3)
\[ \frac{1}{\pi} \int_{0}^{\pi} \theta(t) \cot \frac{t}{2} \, dt \]

provided that the integral exists as a Cauchy integral at the origin.

We shall now deduce Theorem 2 as a corollary of Theorem 1 by employing the following
TAUBER'S SECOND THEOREM. If $\Sigma u_n$ is summable (A), then a necessary and sufficient condition that it should be convergent is that the sequence $\{n u_n\}$ is summable (C,1) to the value zero.

PROOF OF THEOREM 2. The existence of the integral (9.4.3) as a "Cauchy Integral" at the origin implies the summability (A) of the conjugate series $\sum B_n(x)$. 7)

By using Theorem 1, we find that conditions (9.4.1) and (9.4.2) of Theorem 2 imply the summability (C,1) of the sequence $\{n B_n(x)\}$ to the value zero. The convergence of the conjugate series then follows from Tauber's Second Theorem.

---