CHAPTER VIII

FIXED POINT THEOREMS IN HAUSDORFF SPACE

8.1 In the present chapter we studies some new results on fixed points in Hausdorff space. Also in the present chapter we generalized the results of Edelstien [1] and Banach [1].

Banach [1] contraction principle has been generalized by many authors in various ways.

Jaggi [1] proved the following theorem which generalized this principle as follows:

THEOREM-A. Let $T$ be a continuous self-map defined on a complete metric space $(X,d)$. Further, let $T$ satisfy the following condition:

$$d(Tx,Ty) \leq \alpha \frac{d(x,Tx) \cdot d(y,Ty)}{d(x,y)} + \beta [d(x,y)],$$

for all $x, y \in X$, $x \neq y$ and for some $\alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$. Then $T$ has a unique fixed point.
In 1962, Edelstein [1] proved the following theorem-

**THEOREM-B.** Let $T$ be a contractive mapping of a metric space $X$ into itself. If there exists a point $x_0 \in X$ such that the sequence of iterates $\{T^n x_0\}$ has a subsequence $\{T^{nk} x_0\}$ which converges to a point $\eta \in X$, is a unique fixed point of $T$.

8.2 In the present section, we established the generalization of the Jaggi [1] results in Hausdorff space.

**THEOREM-1.** Let $T$ be a continuous mapping of a Hausdorff space $X$ into itself and let $F : X \times X$ be a continuous mapping such that $F(x,y) = 0$ for $x = y$ and for each pair of distinct $x,y \in X$, one has,

$$F(x,y) \leq F(x,z) + F(z,y)$$

(8.2.1)  

$$F(Tx,Ty) \leq \alpha F(x,y)$$

(8.2.2)  

$$+ \beta \left[ \frac{F(x,Tx)F(y,Ty)}{F(x,y)} \right]$$

$$+ \gamma \left[ \frac{F(x,Ty)F(x,y)}{F(x,y)+F(y,Ty)} \right],$$
for all distinct \( x, y \in X \) and,

(8.2.3) for some \( x_0 \in X \) the sequence \( \{x_n\} = \{T^n x_0\} \) has a convergent subsequence, then \( T \) has a unique fixed point, if \( \alpha + \beta + \gamma < 1 \) and which is unique whenever \( \alpha + \gamma < 1 \).

**Proof:** From condition (8.2.2), we have,

\[
F(x_1, x_2) = F(Tx_0, Tx_1)
\]

\[
\leq \alpha F(x_0, x_1)
\]

\[
+ \beta \left[ \frac{F(x_0, Tx_0) F(x_1, Tx_1)}{F(x_0, x_1)} \right]
\]

\[
+ \gamma \left[ \frac{F(x_0, Tx_1) F(x_0, x_1)}{(F(x_0, x_1) + F(x_1, Tx_1))} \right]
\]

\[
\leq (\alpha + \gamma) F(x_0, x_1) + \beta F(x_1, x_2)
\]
\[ F(x_1, x_2) \leq \frac{(a + \gamma)}{(1 - \beta)} F(x_0, x_1) \]

\[ F(x_1, x_2) \leq h F(x_0, x_1), \text{ is impossible as} \]
\[ a + b + \gamma = h, \ h < 1, \text{ we have}, \]
\[ F(x_1, x_2) < F(x_0, x_1). \]

Repeating the above process, we have,

\[ F(x_0, x_1) > F(x_1, x_2) > F(x_2, x_3) > \ldots \ldots \ldots . \]

Thus we have a monotone sequence of positive real numbers which must converges with all its subsequence to some real \( u \). Again suppose \( \{ x_n \} \) has a convergent subsequence \( \{ x_{n_k} \} \) in \( X \) which converges to some \( x \) in \( X \). The continuity of \( T \) gives,

\[ T x = T \lim_{n_k} x_{n_k} = \lim_{n_k} T x_{n_k} = \lim_{n_k} x_{n_k + 1} \]

\[ T^2 x = T(\lim_{n_k} x_{n_k + 1}) = \lim_{n_k} T x_{n_k + 1} = \lim_{n_k} x_{n_k + 2}. \]

Now we prove that \( x \) is a fixed point of \( T \), we have,

\[ F(x, T x) = F(\lim_{n_k} x_{n_k}, \lim_{n_k} x_{n_k + 1}) \]

\[ = \lim F(x_{n_k}, x_{n_k + 1}) = u, \]
\begin{align*}
&= \lim_{n_k \to +1} F\left(n_k^{-1}, x_{n_k}^{-2}\right) = F(Tx, T^2x).
\end{align*}

If \( x \neq Tx \), then by condition (8.2.2), we have

\begin{align*}
F(Tx, T^2x) &\leq a F(x, Tx) \\
&+ \beta \left[ \frac{F(x, Tx) F(Tx, T^2x)}{F(x, Tx)} \right] \\
&+ \gamma \left[ \frac{F(x, T^2x) F(x, Tx)}{(F(x, Tx) + F(Tx, T^2x))} \right]
\end{align*}

\begin{align*}
&\leq (a + \gamma) F(x, Tx) + \beta F(Tx, T^2x)
\end{align*}

\begin{align*}
F(Tx, T^2x) &\leq (a + \gamma)/(1-\beta) F(x, Tx),
\end{align*}

since \((a + \gamma)/(1-\beta) = h, h < 1\), we have,

\begin{align*}
F(Tx, T^2x) &< F(x, Tx),
\end{align*}

which is a contradiction since \( h < 1 \). Hence \( x = Tx \).

To prove the uniqueness, let \( y(\neq x) \) be another fixed point of \( T \). Then from (8.2.2), we have,

\begin{align*}
F(x, y) &= F(Tx, Ty) \\
&\leq a F(x, y) \\
&+ \beta \left[ \frac{F(x, Tx) F(y, Ty)}{F(x, y)} \right]
\end{align*}
which does not holds as \((a+\gamma) < 1\). Hence \(x = y\). This complete the proof of the theorem.

**REMARKS:**

1. Taking \(\gamma = 0\), then we get the result of Jaggi\([1]\).

8.3 In the present section the result of Edelstein \([1]\) have been generalized in topological space which is not necessarily a metric space. We along with Shrivastava \([6]\) proved the following theorem.

**THEOREM**-2. Let \(T_1\) and \(T_2\) be two self mapping of a Hausdorff space and let \(F : X \times X \rightarrow [0, \infty)\) be a continuous symmetric mappings such that \(F(x,y) = 0\) for \(x = y\) and for each pair of distinct \(x, y \in X\) one has-

\[
\begin{align*}
(8.3.1) \quad & \min \{F(T_1x, T_2y), F(y, T_2y) \\
& + \frac{F(x, T_1x) F(y, T_2y)}{F(T_1x, T_2y)} , F(y, T_2y) \} \\
& - \min[F(x, T_2y), F(y, T_1x)]
\end{align*}
\]
< a_1 F(x,y) + a_2 [F(x,T_1 x) + F(y,T_2 y)]

+ a_3 [F(x,T_2 y) + F(y,T_1 x)]

+ a_4 [\frac{F(x,T_1 x) F(y,T_2 y)}{F(x,y)}]

+ a_5 [\frac{F(x,T_1 x) F(y,T_2 y)}{F(T_1 x,T_2 y)}]

whenever the constants are such that a_1 + 2a_2 + 2a_3 + a_4 + a_5 = 1

and,

(8.3.2) for some x_0 \in X the sequence \{x_n\} where,

x_{2n+1} = T_1 x_{2n}, x_{2(n+1)} = T_2 x_{2n}, has a
subsequence converging to a point \eta \in X. If T_1 and
T_2 T_1 or T_2 and T_1 T_2 are continuous at \eta, is a fixed
point of T_1 or T_2.

PROOF:- Let x_0 \in X,

x_{2n+1} = T_1 x_{2n}, x_{2(n+1)} = T_2 x_{2n+1},

n = 0,1,2... we may assume that x_n \neq x_{n+1} for each n.

From (8.3.1), F(x_0,x_1) > F(x_1,x_2) > ..., and so the
sequence C_n = F(x_n,x_{n+1}) tends to a real number r as
n --> \infty. Since \{x_n\} has a subsequence in X which
converges to point \( n \) in \( X \) we may put, \( n = \lim_{k \to \infty} x_{2n_k} \), also,

\[
x_{2n_k+1} = T_1 x_{2n_k} \quad \to \quad T_1 n
\]

and

\[
x_2(n_k+1) = T_2 T_1 x_{2n_k} = T_2 T_1 n , \quad \text{as} \quad k \to \infty ,
\]

since \( T_1 \) and \( T_2 T_1 \) are continuous at \( n \), we have,

\[
r = \lim_{k \to \infty} F(x_{2n_k}, x_{2n_k+1})
\]

\[
= \lim_{k \to \infty} F(x_{2n_k}, T_1 x_{2n_k}) = F(n, T_1 n)
\]

\[
r = \lim_{k \to \infty} F(x_{2n_k+1}, x_2(n_k+1))
\]

\[
= \lim_{k \to \infty} F(T_1 x_{2n_k}, T_2 T_1 x_{2n_k}) = F(T_1 n, T_2 T_1 n)
\]

Suppose \( n \neq T_1 n \), then we have by our condition (8.3.1)

\[
\min\{F(T_1 n, T_2 T_1 n), F(T_1 n, T_2 T_1 n)\}
\]

\[
+ \frac{F(n, T_1 n) F(T_1 n, T_2 T_1 n)}{F(T_1 n, T_2 T_1 n)} , F(T_1 n, T_2 T_1 n)
\]
\[- \min [F(n, T_2 T_1^n), F(T_1^n, T_1^n)]\]

\[< a_1 F(n, T_1^n)\]

\[+ a_2 [F(n, T_1^n) + F(T_1^n, T_2 T_1^n)]\]

\[+ a_3 [F(n, T_2 T_1^n) + F(T_1^n, T_1^n)]\]

\[+ a_4 \left( \frac{F(n, T_1^n) F(T_1^n, T_2 T_1^n)}{F(n, T_1^n)} \right)\]

\[+ a_5 \left( \frac{F(n, T_1^n) F(T_1^n, T_2 T_1^n)}{F(T_1^n, T_2 T_1^n)} \right)\]

i.e. \(\min [F(T_1^n, T_2 T_1^n), (F(T_1^n, T_2 T_1^n) + F(n, T_1^n))]

\[< (a_1 + a_2 + a_3 + a_5) F(n, T_1^n)\]

\[+ (a_2 + a_3 + a_4) F(T_1^n, T_2 T_1^n)\]

since \(a_1 + 2a_2 + 2a_3 + a_4 + a_5 = 1,\)

\(F(T_1^n, T_2 T_1^n) + F(n, T_1^n) < F(n, T_1^n) + F(T_1^n, T_2 T_1^n).\)

does not hold. Therefore we have,

\(F(T_1^n, T_2 T_1^n) < \frac{a_1 + a_2 + a_3 + a_5}{1 - a_2 - a_3 - a_4} F(n, T_1^n)\)

\[= F(n, T_1^n),\]
a contradiction. Therefore we have \( n = T_1^n \). Similarly if we take \( n = \lim_{k \to \infty} x_{2n_k} + 1 \), then one can show that \( n = T_2^n \). This completes the proof of the theorem.

**Theorem 3.** Let \( S \) and \( T \) be continuous self-mappings of a Hausdorff space \( X \). Let \( F \) be a continuous symmetric mapping of \( X \times X \) into \( R^+ \) satisfying the following conditions:

\[
(8.3.3) \quad F(x, x) = 0, \text{ for all } x \in X \text{ and } F(x, y) \neq 0, \\
(8.3.4) \quad F(x, y) \leq F(x, z) + F(z, y), \\
(8.3.5) \quad \text{there exists } p, q \in \mathbb{N} \text{ such that}
\]

\[
F(S^p x, T^q y) < \max \{ F(x, y), F(x, S^p x), \\ F(y, T^q y), \frac{F(x, T^q y)F(x, y)}{F(x, y) + F(y, T^q y)} \}, \\
\frac{1}{2} \{ F(x, T^q y) + F(y, S^p x) \}
\]

for all \( x, y \in X, x \neq y \). If for some \( x_0, y \in X \), the sequence \( \{ x_n \}_{n=1}^{\infty} \) consisting of distinct points \( \{ x_n \} \) has a convergent subsequence where \( S^p x_{2n} = x_{2n+1} \) and \( T^q x_{2n+1} = x_{2n+2} \) for all \( n \in \mathbb{N} \subseteq \mathbb{N} \cup \{ 0 \} \), then \( S \) and \( T \) have a unique common fixed point.
PROOF: Let \( F_n = F(x_n, x_{n+1}) \) for \( n \in \mathbb{N}_0 \). Taking \( x = x_{2n} \) and \( y = x_{2n+1} \) in (8.3.5), we have,

\[
(F_{2n+1}) = F(S^0 x_2n, T^q x_{2n+1}) < \max(F(x_{2n}, x_{2n+1}), F(x_{2n}, x_{2n+1})),
\]

\[
F(x_{2n+1}, x_{2n+2}), \quad \frac{F(x_{2n}, x_{2n+2}) \cdot F(x_{2n}, x_{2n+1})}{\{F(x_{2n}, x_{2n+1}) + F(x_{2n+1}, x_{2n+2})\}}
\]

\[
\frac{1}{2}[F(x_{2n}, x_{2n+2}) + F(x_{2n+1}, x_{2n+1})]
\]

which implies that \( F_{2n+1} < F_{2n} \), for all \( n \in \mathbb{N}_0 \).

Similarly we can prove that \( F_{2n+2} < F_{2n+1} \) for all \( n \in \mathbb{N}_0 \). Thus \( \{F_n\}_{n=0}^\infty \) is decreasing and hence convergence to some real number \( w \in \mathbb{R}_+ \). Since \( \{x_n\}_{n=0}^\infty \) has a convergent subsequence \( \{x_{n_k}\}_{k=0}^\infty \) in \( X \) which converges to some \( x \in X \), we may take \( x = \lim_{k \to \infty} x_{2n_k} \).

Since \( F \) and \( S \) are continuous,

(8.3.6) \[ F(x, S^D z) = F(\lim_{k \to \infty} x_{2n_k}, S^D \lim_{k \to \infty} x_{2n_k}) \]

\[ = F(\lim_{k \to \infty} x_{2n_k}, \lim_{k \to \infty} x_{2n_k+1}) \]
\[
= \lim_{k} F(x_{2n+k}^{2n+k+1})
\]

\[
= w \lim_{k} F(x_{2n+k+1}^{2n+2}) = F(S^p, T^q S^p z).
\]

If \( z \neq S^p z \), then it follows from (8.3.5) and (8.3.6) that,

\[
F(S^p z, T^q S^p z) < \max \{ F(z, S^p z), F(z, S^p z), F(S^p z, T^q S^p z) \}
\]

\[
\frac{F(z, T^q S^p z) - F(z, S^p z)}{F(z, S^p z) + F(S^p z, T^q S^p z)}
\]

\[
\frac{1}{2} [F(z, T^q S^p z) + F(S^p z, S^p z)]
\]

i.e.

\[
F(S^p z, T^q S^p z) < F(S^p z, T^q S^p z),
\]

which is a contradiction. Thus \( z = S^p z \). Similarly \( z = T^q z \).

Let \( z_1 (\neq z) \) be another common fixed point of \( S^p \) and \( T^q \). Then using (8.3.5), we have,

\[
F(z_1, z) = F(S^p z_1, T^q z)
\]

\[
< \max \{ F(z_1, z), F(z_1, S^p z_1), F(z, T^q z) \},
\]
\[
\frac{F(z_1, T^q z) F(z_1, z)}{\{F(z_1, z) + F(z, T^q z)\}} \leq \frac{\{F(z_1, T^q z) + F(z, S^p z_1)\}}{\{F(z_1, T^q z) + F(z, S^p z_1)\}}
\]

i.e. \( F(z_1, z) < F(z_1, z) \),

which is a contradiction. Hence \( z \) is the unique common fixed point of \( S^p \) and \( T^q \).

We now show that \( z \) is a common fixed point of \( S \) and \( T \). Since \( S^p S z = S S^p z = S z, S z \) is also a fixed point of \( S \). By the uniqueness of \( z \), it follows that \( S z = z \). Similarly \( T z = z \). Suppose \( z_2 \neq z \) is another common fixed point of \( S \) and \( T \). Then,

\[
F(z_2, z) = F(S^p z_2, T^q z)
\]

\[
< \max[F(z_2, z), F(z_2, S^p z_2), F(z, T^q z)],
\]

\[
\frac{F(z_2, T^q z) F(z_2, z)}{\{F(z_2, z) + F(z, T^q z)\}} \leq \frac{\{F(z_2, T^q z) + F(z, S^p z_2)\}}{\{F(z_2, T^q z) + F(z, S^p z_2)\}}
\]

i.e. \( F(z_2, z) < F(z_2, z) \),

a contradiction. Hence \( z_2 = z \). This complete the proof.
Since every sequence in a compact Hausdorff space has a convergent subsequence, we have,

**THEOREM-4.** Let \( S \) and \( T \) be continuous self-mappings of a compact Hausdorff space \( X \). Let \( F \) be a continuous symmetric mapping of \( X \times X \) into \( \mathbb{R}_+ \) satisfying the condition \((8.3.3),(8.3.4)\) and \((8.3.5)\). Then \( S \) and \( T \) have a unique common fixed point.