CHAPTER-III

Semiperfect ring which is Extending for Simple Modules

Any right R-module M is called a CS-module (or extending) if every submodule of M is essential in a direct summand of M, equivalently, if every closed submodule of M is a direct summand of M. If M has finite uniform dimension, then M is CS if and only if every uniform closed submodule of M is a direct summand of M ([35], Corollary 7.8). Any ring R is said to be CS-ring if R as a right R-module is CS [14].

CS-modules and rings have been of considerable interest to may authors including Harada, Huynh, Oshiro, Osofsky, Smith and Wisbauer.

The class of right CS-rings includes all right self-injective rings and all noetherian serial rings.

The CS-conditions can be applied to modules in natural ways; an R-module M is a CS-module if and only if each compliment submodule of M is a direct summand of M. The class of CS-modules contains all injective modules, semi-simple modules and uniform modules. Let M be a CS-module, then arbitrary sub modules and factor modules of M need not be CS-modules, but it follows from proposition 2.2 of [28] that if C is a complement submodule of M then C is also a CS-module.
For any two right R-modules M and N, a submodule S of M is said to be essential in M denoted by $S \subseteq M$, if for any non-zero submodule L of M, $S \cap L \neq 0$. R is said to be semiperfect if it has a complete set $\{e_i\}_{i=1}^{n}$ of primitive orthogonal idempotents such that each $e_iRe_i$ is a local ring.

In this chapter we study semiperfect ring in which each simple right R-module is essential in a direct summand of R. We call such ring as a extending for simple R-module. Here we find that for such rings, every simple R-module is weakly injective if and only if R is weakly injective if and only if R is self-injective if and only if R is weakly-semisimple. Examples are constructed for which simple R-module is essential in a direct summand.

Generalising the idea of CS-module, we have.

**Definition : 3.1.** We say that M is extending for simple module if for each simple submodule S of M there is a direct summand $M'$ of M such that S is essential in $M'$[95].

**Definition : 3.2.** R is said to be extending for simple R-module if R as a right R-module is extending for simple R-module.

**Definition : 3.3.** For any right R-module M, we take a direct decomposition $M = \sum \oplus M_i$. For a submodule $N_i$ of $M_i$, we call $\sum \oplus N_i$ a standard submodule.
of $M$ with respect to this decomposition $\Sigma \oplus M_i$. Thus a standard submodule means a standard submodule with respect to decomposition into indecomposable modules. For any right $R$-module $M$, we note that $J(M)$ and $\text{Soc}(M)$ are always standard submodules with respect to any decompositions of $M$ [12].

**Definition : 3.4.** Let $M$ and $N$ be two right $R$-modules. We say that $M$ is weakly $N$-injective if and only if every map $\phi : N \to E(M)$ from $N$ into the injective hull $E(M)$ of $M$ may be written as a composition $\sigma \circ \hat{\phi}$, where $\hat{\phi} : N \to M$ and $\sigma : M \to E(M)$ is a monomorphism. We say that $M$ is weakly injective if and only if it is weakly $N$-injective for every finitely generated module $N$.

**Definition : 3.5.** A ring $R$ is said to be right weakly-semisimple if every right $R$-module $M$ is weakly-injective.

**Lemma : 3.6.** Let $S$ be any simple submodule of $M$ which is essential in $M$ then $M$ is an indecomposable module.

**Proof :** Let $M = M_1 \oplus M_2$ where $M_1$ and $M_2$ are submodules of $M$. Given that $S$ is essential in $M$. Therefore $S \cap M_1 \neq 0$ and $S \cap M_2 \neq 0$. Since $S$ is simple implies that $S \subseteq M_1$ and $S \subseteq M_2$. This implies that $S \subseteq M_1 \cap M_2$ which is contradiction.
Lemma : 3.7. If any right R-module M has essential simple submodules S, then \( \text{Soc} \ (M) = S \).

Proof : Let L and S are two simple submodules of M. Since S is essential in M. Therefore \( S \cap L \neq 0 \). Emplies that \( S \subset L \) or \( L \subset S \) i.e. \( S = L \). Hence \( \text{Soc}(M) = S \).

Lemma : 3.8. Let R be a semiperfect ring and let \( e_1, e_2, \ldots, e_m \) be a basic set of primitive idempotents for R. If \( P \) is projective then there exist sets \( A_1, A_2, \ldots, A_m \) (unique to within cardinality and possibly empty) such that
\[
P \cong (e_1 R)^{(A_1)} \oplus (e_2 R)^{(A_2)} \oplus \cdots \oplus (e_k R)^{(A_k)}
\]

Proof : See [1, Theorem 27.11, Page 306].

Lemma : 3.9. Suppose that \( K_1 \subset M_1 \subset M, K_2 \subset M_2 \subset M \) and \( M = M_1 \oplus M_2 \). Then \( K_1 \oplus K_2 \subset M_1 \oplus M_2 \) if and only if \( K_1 \subset M_1 \) and \( K_2 \subset M_2 \).

Proof : See [1, proposition 5.20(2), Page 75].

Lemma : 3.10. The followings are equivalent,

(i) \( R \) is right perfect (\( R \) is left perfect)

(ii) \( R/J \) is semi-simple Artin and every cyclic left (right) R-module has non-zero socle.

.59.
(iii) $R/J$ is semi-simple Artin and every non-zero right (left) $R$-module has a non-zero simple epimorphic image.

(iv) $R/J$ is semi-simple Artin and if \( \{a_i \mid i = 0, 1, \ldots \} \subseteq J \), there is an $n$ such that $a_n a_{n-1} \ldots a_0 = 0$ (where $a_0 a_1 \ldots a_n = 0$).

**Proof:** See [130, Lemma 9]

**Proposition 3.11.** Let $R$ be any semiperfect ring such that $R_R$ is extending for simple submodule, then

(i) For any projective $R$-module $P$, $\text{Soc}(P)$ is essential in $P$.

(ii) If $Q$ is another projective $R$-module such that $\text{Soc}(Q) \cong \text{Soc}(P)$ then $Q \cong P$.

**Proof:** Since $R$ is semiperfect, we may write $R = e_1 R \oplus \ldots \oplus e_n R$, where $P = \{e_1 R, e_2 R, \ldots, e_k R\} \ (k \leq n)$ is an irredundant complete set of representatives for the projective indecomposable $R$-modules. Let $L = \{S_1, S_2, \ldots, S_k\}$ be an irredundant complete set of representatives for the simple $R$-modules.

Since $R_R$ is extending for simple submodule hence for any simple submodule $S_i$ there exist a direct summand $e_i R$ of $R$ such that $S_i$ is essential in $e_i R$ form Lemma 3.6, $e_i R$ should be indecomposable $R$-module. Therefore $e_i R \cong e_j R$ for some $j \in \{1, 2, \ldots, k\}$. Thus we can define a function $f : L \to P$ by...

0.60.
f (S_i) = e_i R, f must be one-one, hence onto. Also by Lemma 3.7, Soc (e_i R) \cong S_i

i.e. Soc (e_i R) = S_i is the unique essential submodule of e_i R. Thus Soc (P) is
essential in P as proved for indecomposable projective R-module e_i R = P.

Let P be an arbitrary projective R-module. Since R is semiperfect there
exist sets A_i , i = 1, 2, .......... , k such that

P \cong (e_1 R)^{(A_1)} \oplus (e_2 R)^{(A_2)} \oplus .......... \oplus (e_k R)^{(A_k)}

By Lemma 3.8, since Soc(P) is an standard submodule of P. Therefore

Soc(P) \cong (Soc(e_1 R))^{(A_1)} \oplus (Soc(e_2 R))^{(A_2)} \oplus .......... \oplus (Soc(e_k R))^{(A_k)}

using Lemma 3.9, we get

(Soc(e_1 R))^{(A_1)} \oplus (Soc(e_2 R))^{(A_2)} \oplus .......... \oplus (Soc(e_k R))^{(A_k)}

\subseteq (e_1 R)^{(A_1)} \oplus (e_2 R)^{(A_2)} \oplus .......... \oplus (e_k R)^{(A_k)}

i.e. Soc (P) \subseteq P.

(ii) Let Q = (e_1 R)^{(B_1)} \oplus (e_2 R)^{(B_2)} \oplus .......... \oplus (e_k R)^{(B_k)} be any other
projective R-module such that Soc(Q) \cong Soc(P).

Therefore

(Soc(e_1 R))^{(B_1)} \oplus (Soc(e_2 R))^{(B_2)} \oplus .......... \oplus (Soc(e_k R))^{(B_k)}

\cong (Soc(e_1 R))^{(A_1)} \oplus (Soc(e_2 R))^{(A_2)} \oplus .......... \oplus (Soc(e_k R))^{(A_k)}

and so by the Krull-Schmidt theorem there is a bijection between A_i and B_i for,
i = 1, 2, .......... k. Therefore

.61.
\[(e_1 R)^{(b_1)} \oplus (e_2 R)^{(b_2)} \oplus \ldots \ldots \ldots \oplus (e_k R)^{(b_k)}\]

\[\cong (e_1 R)^{(a_1)} \oplus (e_2 R)^{(a_2)} \oplus \ldots \ldots \ldots \oplus (e_k R)^{(a_k)}\]

i.e. \(Q \cong P\).

**Proposition**: 3.12. If \(R\) is semiperfect and extending for simple right \(R\)-module then \(R\) is left perfect.

**Proof**: We shall show that each cyclic \(R\)-module has non-zero socle by Lemma 3.10. For any cyclic \(R\)-module \(xR\), if it is contained in \(e_i R\) then since \(e_i R\) has essential simple submodule \(S_i\). Therefore \(S_i \cap xR \neq 0\). Thus \(S_i \subseteq xR\) i.e. \(\text{Soc} \ (xR) \neq 0\).

On the other hand if \(xR\) contains any \(e_i R\) then obviously \(\text{Soc} \ (e_i R) \subseteq \text{Soc} \ (xR)\) i.e. \(\text{Soc} \ (xR) \neq 0\).

**Theorem**: 3.13. Let \(R\) be any semiperfect and extending for simple right \(R\)-module, then following conditions are equivalent.

(i) Every right simple \(R\)-module is weakly injective.

(ii) \(R\) is weakly injective.

(iii) \(R\) is self-injective ring.

(iv) \(R\) is weakly-semisimple ring.

(v) Every right \(R\)-module is weakly injective.
Proof: (i) \(\Rightarrow\) (ii) Let \(R = e_1R \oplus e_2R \oplus \cdots \oplus e_kR\) (\(k \leq n\)) as \(R\) is semiperfect and \(S = \{S_1, S_2, \ldots, S_k\}\) be the irredundant set of simple \(R\)-modules. Given that \(S_i\) is weakly injective and \(S_i\) is essential in \(e_iR\) as \(R\) is extending. Therefore \(e_iR\) is weakly injective. Also finite direct sum of weakly injective is weakly injective. Therefore \(R = e_1R \oplus e_2R \oplus \cdots \oplus e_kR\) is weakly injective.

(ii) \(\Rightarrow\) (iii) Suppose \(R\) is weakly injective. By proposition 3.12, \(R\) is left perfect. Over left perfect ring \(R\), \(R\) is weakly injective if and only if \(R\) is self injective [95, Lemma 2.8].

(iii) \(\Rightarrow\) (iv) Given that \(R\) is self-injective hence it would be weakly injective. Every direct summand of \(R\) is injective and hence every direct summand of \(R\) is weakly injective. Therefore \(R\) is weakly-semisimple ring [109, Theorem 2.4].

(iv) \(\Rightarrow\) (v) Since \(R\) is weakly-semisimple, therefore every right \(R\)-module \(M\) will be weakly injective.

(v) \(\Rightarrow\) (i) Obvious.

Example 3.14:

1. Let \(R = \begin{bmatrix} Z & Q \\ 0 & Q \end{bmatrix}\) is a weakly \(R\)-injective. Here simple \(R\)-module \([0, Q] = e_{22}R\) is not weakly \(R\)-injective i.e. \(R\) is not weakly-semisimple ring and \(R\) is also not self-injective ring.
(2) For a Boolean ring $R$, following are equivalents

(i) $R$ is weakly $R$-injective.

(ii) $R$ is weakly-semisimple ring.

(iii) $R$ is self-injective ring.

Proof: For any Boolean ring, its injective hull $E(R)$ and classical quotient ring $Q(R)$ of $R$ are same i.e. $R = E(R) = Q(R)$.

Example 3.15: Let $S = \begin{bmatrix} B & A \\ 0 & A \end{bmatrix}$ where $A = Q(x_1, x_2, \ldots, x_n)$ is a field of rational functions in $n$ indeterminates and $B = (x_1^2, x_2^2, \ldots, x_n^2)$ is a subfield of $A$. Let $f: A \to B$ defined by $f(x_i) = x_i^2$, $f(a) = a \forall a \in Q$.

$\forall i = 1, 2, \ldots, n$ then $B$ is epimorphic image of $A$ [45, Page 338]

or $S = \frac{Z}{p^2Z}$

$S$ has three right ideals $S$, $J = \text{Rad}(S) = xS$ and $(0)$.

Also $J^2 \subset J$. Therefore $J^2 = 0$.

Now let $R = \left\{ \begin{bmatrix} a & t \\ 0 & a \end{bmatrix} | a, t \in S \right\} \subset \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}$

i.e. $R$ is the split extension of the ring $S[48]$. .64.
The lattice of right ideals of $R$ is

\[
\begin{array}{c}
R \\
\text{Rad}(R) = \begin{bmatrix} J & S \\ 0 & J \end{bmatrix} = J_1 \\
J_2 = \begin{bmatrix} J & J \\ 0 & J \end{bmatrix} \\
J_3 = \begin{bmatrix} x & u \\ 0 & x \end{bmatrix} R \\
\text{Soc}(R) = \begin{bmatrix} 0 & J \\ 0 & 0 \end{bmatrix} = (J_1)^2 = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} R \\
0
\end{array}
\]

where $u \not\in J$ in the generator $\begin{bmatrix} x & u \\ 0 & x \end{bmatrix}$ for the cyclic $R$-module $\begin{bmatrix} x & u \\ 0 & x \end{bmatrix} R$.

Since $\text{End } (R_R) \cong R$ is a local ring hence $R_R$ is indecomposable, it is semiperfect. The irredundant complete set of representatives for projective indecomposable $R$-module contains single element namely $R$ only. And hence...
irredundant complete set of representatives for simple $R$-module also contains

only single element namely $\text{Soc}(R) = \begin{bmatrix} 0 & J \\ 0 & 0 \end{bmatrix}$.

Clearly $\frac{R}{\text{Rad}(R)} = \text{Soc}(R) \subset R$. i.e. $R$ is semiperfect and $R_\pi$ is extending for simple $R$-module. However the factor ring $\text{R} = \frac{R}{\text{Soc}(R)}$ is also semiperfect but not extending for simple module as $\frac{J}{\text{Soc}(R)}$, $\frac{K_u}{\text{Soc}(R)}$, $\frac{J_2}{\text{Soc}(R)}$ are three simple $\text{R}$-modules.

Clearly intersection of any two is zero i.e. $\text{R}$ is not extending for the simple $\text{R}$-module $\frac{K_u}{\text{Soc}(R)}$. 

.66.