CHAPTER IV

NON-ARCHIMEDEAN COMPACTIFICATION OF A TOPOLOGICAL SEMIGROUP

4.1 Non-archimedian Uniform Structure on a Semigroup

Definition 4.1.1 : A uniform structure is said to be non-archimedian, if there exists a base $\mathcal{B}$ for the uniformity, such that, for $U \in \mathcal{B}$, $U \circ U \subseteq U$, where $U \circ U$ denotes the rational product of $U$ with itself (i.e. $U \circ U = \{(x,y) : \text{there exists an element } z \text{ such that } (x,z), (z,y) \in U\}$.

Sometimes we call the base $\mathcal{B}$ itself as a non-archimedian base. We need the following lemma:

Lemma 4.1.2 : A uniform structure $\mathcal{U}$ on a semigroup $S$ is compatible with the semigroup structure if and only if it possesses a symmetric invariant base.

The proof of this lemma is straightforward as

Alfsen and Holm 2).

Theorem 4.1.3 : A uniform structure $\mathcal{U}$ on a semigroup $S$ is

1) Monna, A.F. (1)
2) Alfsen, E.M. and Holm, P. (1)
non-archimedean and compatible, with the semigroup structure if and only if there exists a neighbourhood base $\mathcal{N}$ at the identity $e$ of $S$, for the topology defined by $\mathcal{U}$, consisting of normal sub-semigroups of $S$.

Proof: Since the operation $(x,y) \rightarrow xy$ is uniformly continuous, it easily follows that $\mathcal{U}$ admits an invariant base of surroundings. This implies that $\mathcal{U} = \mathcal{U}_L = \mathcal{U}_R$, where $\mathcal{U}_L$ and $\mathcal{U}_R$ denote the left and right uniform structures of $S$. Hence $\mathcal{U}$ consists of invariant surroundings. Further, since $\mathcal{U}$ is also non-archimedean, there exists a base $\mathfrak{B}$ for $\mathcal{U}$, which is symmetric, invariant and non-archimedean. The family $\mathcal{N} = \{U[e] \mid U \in \mathfrak{B}\}$ forms a neighbourhood base at the identity element $e$ of $S$ (for ref. see Kelley\(^3\)).

Further

$$(x,y) \in U[e] \implies (e,y), (e,x) \in U$$

$$\implies (e,y), (y,xy) \in U \text{ (by left invariance of } \mathfrak{B}).$$

$$\implies (e,xy) \in U \text{ (since } \mathfrak{B} \text{ is non-archimedean base)}$$

$$\implies xy \in U[e].$$

\(^3\) Kelley, J.L. (1)
Also

\[ x \in U[e] \implies (e,x) \in U \]

\[ \implies (e, ax) \in U \]

\[ \implies (e, axa) \in U \]

\[ \implies axa \in U[e]. \]

This proves the necessary part. To prove the sufficiency part, let \( \mathcal{N} \) be a base, for the topology defined by \( \mathcal{U} \), at the identity element \( e \) where each \( N \in \mathcal{N} \) is a normal subsemigroup. Define \( U_N \) as \( U_N = \{ (x,y) : x, y \in N \} \). Then it is easily verified that \( \mathcal{B} = \{ U_N \}_{N \in \mathcal{N}} \) is a base for the uniform structure \( \mathcal{U} \) on \( S \). Also it is clear that this base is symmetric and invariant. Hence by the Lemma 4.1.2 it follows that \( \mathcal{U} \) is compatible with the semigroup structure of \( S \). Further, \( U_N \cdot U_N \subseteq U_N \), for \( N \in \mathcal{N} \), that is the base \( \mathcal{B} \) is non-archimedean. Hence the uniform structure \( \mathcal{U} \) is non-archimedean. This completes the proof.

As an immediate consequence of this theorem, we have the following:

Corollary 4.1.4: \( S \) is totally bounded, with respect to the

3) Kelley, J.L. (1)
4) Kelley, J.L. (1) pp. 198
uniform structure \( \mathcal{U} \) if and only if there exists a
neighbourhood base \( \mathcal{N} \) at \( e \) consisting of the normal sub-
semigroups of finite index.

Now let us consider a topological semigroup \((S, \tau)\).
The family \( \mathcal{K} \) of all closed normal sub-semigroups of finite
index in \( S \) defines a topology \( \tau_\mathcal{K} \) on \( S \), coarser than \( \tau \nolinebreak \)
under which \( S \) is again a topological semigroup (see theorem
1.1.16). Let \( \mathcal{U}_\mathcal{K} \) be the uniform structure defined by the
neighbourhood base \( \mathcal{K} \) at the identity element \( e \) of \( S \)(the
left and right uniform structures coincide). Then by
corollary 4.1.4, it follows that \( \mathcal{U}_\mathcal{K} \) is totally bounded. We
now prove:

**Theorem 4.1.5.** Let \((S, \tau)\) be a topological semigroup and
\( \mathcal{U}_\mathcal{K} \) the uniform structure defined as above, then \( \mathcal{U}_\mathcal{K} \) is the
finest uniform structure on \( S \), with the following properties:

(a) \( \mathcal{U}_\mathcal{K} \) is totally bounded and non-archimedean

(b) \( \mathcal{U}_\mathcal{K} \) is compatible with the semigroup structure

and

(c) the topology \( \tau_\mathcal{K} \) given by \( \mathcal{U}_\mathcal{K} \) is coarser than \( \tau \).

**Proof:** (a) and (b) immediately follow from the corollary
4.1.4 and the property (c) follows from theorem 1.1.16.
Further suppose that \( \mathcal{U} \) is a uniform structure on \( S \) with the properties (a), (b) and (c). Then by corollary 4.1.4 it follows that there exists a neighbourhood base \( \mathcal{N} \) at the identity \( e \) of \( S \) for the topology of \( \mathcal{U} \), consisting of the normal subsemigroups of finite index. Now, since the topology defined by \( \mathcal{U} \) is coarser than the topology \( \mathcal{T} \), these sub-semigroups are also open in \( \mathcal{T} \) (and thus closed) and hence belong to the family \( \mathcal{H} \). Hence the uniform structure \( \mathcal{U} \) is coarser than \( \mathcal{U}_\mathcal{H} \).

Remark: The topology \( \mathcal{T}_\mathcal{H} \) as well as the uniform structure \( \mathcal{U}_\mathcal{H} \), are not necessarily Hausdorff.

Theorem 4.1.6.: Let \( S \) be a topological semigroup and \( \mathcal{U} \) a non-archimedean Hausdorff uniform structure of \( S \). Then \( S \) can be imbeded as a dense sub-semigroup of a complete totally disconnected semigroup \( \hat{S} \). Also, the left and right uniform structures on \( \hat{S} \) are coincident and this uniform structure is again non-archimedean.

Proof: It follows from theorem 4.1.3 that there exists a neighbourhood base at the identity \( e \) of \( S \) consisting of the normal sub-semigroups. Hence by the remark following theorem 1.1.10, \( S \) has a unique completion \( \hat{S} \), which has also a neighbourhood base at the identity \( e \), consisting of normal sub-semigroups in \( \hat{S} \). Therefore by theorem 4.1.3, the uniform
structure of $\hat{S}$ (left and right uniform structure coincident) is non-archimedean. Finally, since $\hat{S}$ has a neighbourhood base consisting of sub-semigroups, it is totally disconnected.

This completes the proof.

**Corollary 4.1.7.** : Let $S$ be a semigroup and $\mathcal{U}$ a totally bounded non-archimedean uniform structure compatible with the semigroup structure of $S$. Then the Hausdorff completion $\hat{S}$ of $S$ is a compact totally disconnected topological semigroup.

**Proof :** It is straightforward from corollary 4.1.4, that $S$ has a neighbourhood base $\mathcal{N}$, consisting of the normal sub-semigroups of finite index. Hence $\bigcap_{N \in \mathcal{N}} N = N$ is a normal sub-semigroup of $S$. Now the semigroup $S/N$ has a neighbourhood base at the identity $e$ consisting of normal sub-semigroups of finite index. Hence the uniform structure defined by this base (left and right uniform structure coincident) is a totally bounded, non-archimedean, Hausdorff uniform structure. Further by theorem 4.1.6 the semigroup $S/N$ can be imbedded as a dense sub-semigroup of a totally disconnected and complete semigroup $\hat{S}$, i.e. the Hausdorff completion of $S$. Finally, since $\hat{S}$ is totally bounded and complete, it is compact. This completes the proof.
4.2. Bohr Compactification

A continuous representation \( \sigma \) of a topological semigroup \( S \) on to a dense sub-semigroup of a compact totally disconnected semigroup \( H \) is called a non-archimedean compact representation of \( S \) and let us denote it by \((\sigma, H)\). A non-archimedean compact representation \((\hat{\sigma}, \hat{S})\) of \( S \) will be called maximal, if every non-archimedean compact representation \((\sigma, H)\) of \( S \) admits a unique decomposition \( \sigma = \sigma' \circ H \) where \( \sigma' \) is some continuous representation of \( \hat{S} \) on to \( H \). It is easily seen that a non-archimedean compact representation is determined up to algebraic and topological isomorphism.

We now prove:

**Theorem 4.2.1.** A topological semigroup \( S \) admits a maximal compact representation iff there exists a finest uniform structure \( \mathcal{U} \) on \( S \) with the following properties:

a) \( \mathcal{U} \) is totally bounded and non-archimedean,

b) \( \mathcal{U} \) is compatible with the semigroup structure,

c) \( \mathcal{U} \) defines a topology coarser than the initial topology of \( S \).

**Proof:** Suppose the topological semigroup \( S \) admits a

* By continuous representation, we mean a continuous homomorphism.
uniform structure \( \mathcal{U} \) with the properties (a)-(c). Then by corollary 4.1.7 it is clear that the Hausdorff compactification \( \hat{S} \) of \( S \), with respect to the uniform structure, is a compact totally disconnected semigroup. Also, it is easy to see that, it is a maximal non-archimedean compact representation of \( S \).

Conversely, let \( S \) admits a maximal non-archimedean compact representation \( (f, \hat{S}) \). The uniform structure on \( \hat{S} \) induces a uniform structure; viz. the coarsest uniform structure such that \( f \) is uniformly continuous from \( S \) into \( \hat{S} \). This uniform structure so defined is the finest uniform structure and satisfies the properties (a)-(c).

**Corollary 4.2.2.** Every topological semigroup \( S \) admits a maximal non-archimedean compact representation \( \hat{S} \).

**Proof:** The proof is straightforward from the theorems 4.2.1 and 4.1.5.

**Definition 4.2.3.** The compact totally disconnected semigroup \( \hat{S} \) is called the non-archimedean Bohr compactification of \( S \).

**Theorem 4.2.4.** A continuous function \( f \) on \( S \) is almost periodic iff uniformly continuous with respect to \( \mathcal{U}_S \).
Proof: If $f$ is a continuous periodic function on the semigroup $S$, its $\varepsilon$-kernel $H$ is a closed normal sub-semigroup of finite index by theorem 2.3.11 and also belongs to $\mathcal{H}$. Further, since the cosets of $H$ form an $(f, a_0, b_0, \varepsilon)$-partition then for any $x, y \in H$, $|f(x) - f(y)| < \varepsilon$. Hence $f$ is uniformly continuous with respect to $\mathcal{U}_{\varepsilon_0}$.

Conversely, suppose that $f$ is uniformly continuous with respect to $\mathcal{U}_{\varepsilon_0}$, then there exists a closed normal sub-semigroup $H \in \mathcal{H}$ of finite index, such that $x, y \in H$ implies $|f(x) - f(y)| < \varepsilon$. Further it is very easy to show that the cosets of $H$ form an $(f, a_0, b_0, \varepsilon)$-partition and hence $f$ is almost periodic.

**Definition 4.2.5:** We call the topological semigroup $S$:

(i) **Maximally almost periodic**, if, for every pair $x, y \in S$, $x \neq y$, there exists a continuous almost periodic function $f$ such that $f(x) \neq f(y)$.

(ii) **Minimally almost periodic**, if the constant functions are only continuous almost periodic functions on $S$.

We now prove:

**Theorem 4.2.6.** A topological semigroup $(S, \tau)$ is
(a) maximally almost periodic iff the topology $\tau_0$ defined on $S$ is Hausdorff, equivalently, iff for each element $x \in S$, $x \neq e$, there exists a closed normal sub-semigroup of finite index in $(S, \tau)$ not containing $x$.

(b) minimally almost periodic if and only if it has no proper closed normal sub-semigroup of finite index.

Proof: (a) Let $(S, \tau)$ be a maximally almost periodic topological semigroup, then for given $x \neq e$, there exists a continuous almost periodic function $f$ such that $f(x) \neq f(e)$. Let $|f(x) - f(e)| = \delta$, let us choose an $\varepsilon$ such that $0 < \varepsilon < \delta$.

Now it is easy to see that the $\varepsilon$-kernel $H$ of $f$, is a closed normal sub-semigroup of finite index not containing $x$. It clearly follows that $\tau_0$ is Hausdorff.

Conversely, if $x, y \in S$, $x \neq y$, then $xy \neq e$.

Hence there exists a closed normal sub-semigroup $H$ of finite index in $S$ not containing $x \sim y$. Then by corollary 2.3.13 the characteristic function $\chi_H$ is a continuous almost periodic function. Hence the function $f$ defined as $f(z) = \chi_H(y, z)$, where $\chi_H(y, z)$ is a function of two variables similar to $\phi(x, y)$ defined in § 3.1 for $z \in S$, is also continuous and almost periodic by theorem 2.2.7. It follows that $f(x) \neq f(y)$, i.e. $S$ is maximally almost periodic.
(b) If $S$ contains no proper closed normal sub-semigroup of finite index, then there does not exist non-constant continuous function on $S$. Otherwise if $f$ is such a function, then it contains no proper sub-semigroup of finite index which is closed and normal, as an $\varepsilon$-kernel for a suitable $\varepsilon$. This contradicts our assumption.

Conversely, if $H$ has a proper closed normal sub-semigroup of finite index, then the characteristic function of $H$ is a non-constant continuous almost periodic function on $S$. Hence, it follows that $S$ is not minimally almost periodic.

**Theorem 4.2.7.(a)** Every continuous almost periodic function on a topological semigroup $(S, \tau)$ is bounded and uniformly continuous in the initial left and right uniform structures of $(S, \tau)$.

(b) The set of collections of continuous almost periodic functions on $S$ is closed with respect to (pointwise)addition and multiplication, multiplication by scalars from the non-archimedian field $\mathbb{Q}$ and passage to uniform limits.

Proof: (a) From theorems 4.2.4 we know that any continuous almost periodic function is $\mathcal{U}_\omega$-uniformly continuous. Also since $\tau$ is finer than $\tau_\omega$, it is uniformly continuous in the left and right uniform structure of $(S, \tau)$. Finally, the boundedness follows from the compactness of $\hat{S}$. 
(b) The proof immediately follows from theorem 4.2.4.

**Theorem 4.2.8.** Every continuous function \( f \) on a totally disconnected semigroup is almost periodic.

**Proof.** It follows from theorem 1.1.15 that \( S \) itself is totally disconnected and the topology \( \tau_S \) coincides with . Further, since any continuous function on a compact semigroup is uniformly continuous, the result immediately follows from theorem 4.2.4.

**Theorem 4.2.9.** A continuous function on a topological semigroup is almost periodic if and only if there exists a continuous function \( \hat{f} \) on the semigroup \( \hat{S} \), of the maximal non-archimedian compact representation \( (\hat{f}, \hat{S}) \) such that for all \( x \in S \), we have

\[
f(x) = \hat{f}(\hat{f}(x)).
\]

**Proof:** It is clear from theorem 4.2.4 that if \( f \) is a continuous almost periodic function, then it is \( \mathcal{U}_S \)-uniformly continuous and hence it can be extended to a continuous function \( \hat{f} \) on \( \hat{S} \). Hence \( f(x) = \hat{f}(\hat{f}(x)) \), for every \( x \in S \). Conversely, if \( \hat{f} \) is a continuous function on \( \hat{S} \), then it is almost periodic, and hence \( f \) is almost periodic.

**Theorem 4.2.10.** (a) The semigroup \( S \) is maximally almost periodic iff the representation \( \hat{f} \) is one-to-one.
(b) $S$ is minimally almost periodic, iff $\hat{S}$ reduces to a single element semigroup.

Proof: (a) From theorem 4.2.6(a) it follows that $S$ is maximally almost periodic iff $\tau_H$ is Hausdorff, i.e. iff $\beta$ is one-to-one.

(b) From theorem 4.2.6(b), it follows that $S$ is minimally almost periodic iff $S$ has no proper closed normal sub-semigroup of finite index. This clearly shows that it is only possible when $\hat{S}$ is a single element semigroup. This completes the proof.

**Theorem 4.2.11.** The non-archimedean Bohr-compactification of a maximal almost periodic topological semigroup $S$ is the projective limit of finite (discrete) semigroups.

Proof: Let $S$ be a maximally almost periodic semigroup and $\mathcal{F} = \{H_\alpha\}$, the family of closed normal sub-semigroups of finite index. Also suppose that $S_\alpha = S/H_\alpha$, then $S_\alpha$ is a finite (discrete) semigroup. By putting $\alpha < \beta$, whenever $H_\alpha \supset H_\beta$, the indices can be made into a discrete set. A homomorphism $f_{\alpha\beta}$ between $S_\beta$ and $S_\alpha$ is determined by the relation

$$S/H_\alpha \to S/H_\beta \quad \text{and} \quad H_\alpha / H_\beta$$
and the projective limit defined in terms of these homomorphisms \( f_{\alpha \beta} \). Further, since \( S \) is maximally almost periodic, then by theorem 4.2.8(a) the topology \( \tau_f \) is Hausdorff. Also the projective limit is the completion of \((S, \tau_f)\). Again, since \( H^\alpha \)'s are of finite indices, it is easy to see that this completion is a compact semigroup. Hence the theorem follows immediately, as the completion of the topological semigroup \((S, \tau_f)\) is the Bohr-compactification of \( S \) by the Definition 4.2.3.

4.3. **Non-archimedian Metric on a Semigroup**

**Definition 4.3.1.** A non-archimedian metric on a topological semigroup \( S \) is a metric \( d \) on \( S \) with the properties:

(i) \( d(x,y) = 0 \), iff \( x = y \);

(ii) \( d(x,y) = d(y,x) \)

(iii) \( d(x,y) \leq \max \{ d(x,z), d(z,y) \} \)

(iv) either \( d(xa,ya) = d(x,y) \) (right invariant)

or \( d(ax, ay) = d(x,y) \) (left invariance),

and the topology defined by \( d \) coincides with the topology of \( S \).

We now prove the existence of non-archimedian metric on \( S \).
Theorem 4.3.2. : The topological semigroup $S$ can be metrized by a non-archimedean metric iff there exists a countable neighbourhood base at the identity $e$ of $S$ consisting of the sub-semigroups of the semigroup $S$.

Proof : Let $S$ be a non-archimedean metrizable semigroup, metrized by a right invariant non-archimedean metric $d$. Let $U_n = \{ x : d(x,e) < \frac{1}{n} \}$, then $U_n$'s form a neighbourhood base at $e$ for $S$. Now, since $d$ is right invariant non-archimedean metric, then by (iii) we have

$$d(xy,e) \leq \max \{ d(xy,y), d(y,e) \}$$

$$= \max \{ d(x,e) , d(y,e) \}$$

$$< \frac{1}{n} \quad \text{, if } x,y \in U_n ,$$

i.e. $x,y \in U_n \implies xy \in U_n$. Hence $U_n$ is a sub-semigroup.

This proves the necessity part.

To prove the sufficiency part, let us suppose that $S$ is a topological semigroup, with a neighbourhood base $\{ H_n \}$ of sub-semigroups of $S$. We can suppose without any loss of generality that $H_{n+1} \subseteq H_n$ \(^{5)}\). We define

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5) Kelley, J.L. (1)
\[ \phi_n (x,y) = 0, \quad \text{if } xy \in H_n, \]
\[ = 1, \quad \text{if } xy \notin H_n. \]

Then

a) \[ \phi_n (x,y) = 0, \quad \text{if } x = y, \]

b) \[ \phi_n (x,y) = \phi_n (y,x), \]

c) \[ \phi_n (x,y) \leq \max \{ \phi_n (x,z), \phi_n (z,y) \}, \]

d) \[ \phi_n (xz, yz) = \phi_n (x, y), \quad \text{for } z \in S. \]

Now (a), (b), and (d) are obvious. If \( \phi_n (x,y) = 0, \) then \( \phi_n (x,z) \) and \( \phi_n (z,y) \) are either both zero or one, and hence (c) is an immediate consequence. But, if \( \phi_n (x,y) = 1, \) then \( xy \notin H_n \) i.e. either \( \phi_n (x,z) = 1 \) or \( \phi_n (z,y) = 1 \) and therefore (c) is true. Hence \( \phi_n \) defines a non-archimedean pseudo-metric \(^6\), i.e. \( \phi_n (x,y) = 0 \) may not imply \( x = y. \)

Further let

\[
d (x,y) = \sup_n \frac{1}{2^n} \cdot \frac{\phi_n (x,y)}{1 + \phi_n (x,y)}
\]

\[= \sup_n \frac{1}{2^n} \cdot \psi_n (x,y).\]

Then,

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6) Kelley, J.L. (1) p.119
\[
\max \left\{ \psi_n(x, z), \psi_n(z, y) \right\} \\
\geq \max \left\{ \frac{\phi_n(x, z)}{1 + \max \left\{ \phi_n(x, z), \phi_n(z, y) \right\}} \right\} \\
\geq \frac{\phi_n(z, y)}{1 + \max \left\{ \phi_n(x, z), \phi_n(z, y) \right\}} \\
\geq \max \left\{ \frac{\phi_n(x, z), \phi_n(z, y)}{1 + \max \left\{ \phi_n(x, z), \phi_n(z, y) \right\}} \right\} \\
\geq \frac{1}{1 + \frac{1}{\phi_n(x, y)}} \\
> \psi_n(x, y).
\]

Thus if each \( \psi_n(x, y) \) being a non-archimedean pseudo-metric implies that \( \frac{1}{2^n} \cdot \psi_n(x, y) \) is also non-archimedean pseudo-metric. Thus

\[
d(x, y) = \sup_n \frac{1}{2^n} \cdot \psi_n(x, y)
\]

\[
\leq \sup_n \max \left\{ \frac{1}{2^n} \psi_n(x, z), \frac{1}{2^n} \psi_n(z, y) \right\}
\]

\[
\leq \max \left\{ \sup_n \frac{1}{2^n} \psi_n(x, z), \sup_n \frac{1}{2^n} \psi_n(z, y) \right\}
\]

\[
\leq \max \left\{ d(x, z), d(z, y) \right\}
\]
i.e. \( d \) is right invariant.

Further,

\[
d(x, y) = 0 \implies \text{each } \phi_n(x, y) = 0
\]

\[
\implies xy \in H_n, \text{ for every } n.
\]

Since the topology defined on the semigroup \( S \) is Hausdorff hence it is only possible in the case that \( x = y \), i.e.

\[
d(x, y) = 0 \implies x = y.
\]

Hence the metric \( d \), so defined is a right invariant non-archimedian metric.

Now we want to show that the topology defined by the metric \( d \) is coincident with that of \( S \). To prove it, suppose

\[
U_n = \left\{ x : d(x, e) < \frac{1}{2^{n+1}} \right\}
\]

then \( U_n \)'s form a neighbourhood base at the identity \( e \) for the metric topology. Also

\[
x \in U_n \implies d(x, e) < \frac{1}{2^{n+1}}
\]

\[
\implies \frac{1}{2^k} \frac{\phi_k(x, e)}{1 + \phi_k(x, e)} < \frac{1}{2^{n+1}}, \text{ for each } k
\]

\[
\implies \frac{1}{2^n} \frac{\phi_n(x, e)}{1 + \phi_n(x, e)} < \frac{1}{2^{n+1}}, \text{ for } k = n
\]
\[ \Rightarrow \frac{\phi_n(x,e)}{1 + \phi_n(x,e)} < \frac{1}{2} \]

\[ \Rightarrow \phi_n(x,e) < 1 \]

\[ \Rightarrow \phi_n(x,e) \neq 0 \]

\[ \Rightarrow x \in H_n. \]

i.e. \( U_n \subset H_n \).

Further, since \( H_{n+1} \subset H_n \subset H_{n-1} \subset \cdots \subset H_1 \), then

\[ \phi_{n+1}(x,e) = \phi_n(x,e) = \cdots = \phi_1(x,e) = 0. \]

Hence \( x \in H_{n+1} \Rightarrow \phi_{n+1}(x,e) = 0. \)

Now

\[ \frac{1}{2^k} \psi(x,e) \leq \frac{1}{2^k} < \frac{1}{2^{n+2}}, \text{ if } k > n + 2 \]

and

\[ 0 = \frac{1}{2^k} \psi_k(x,e) < \frac{1}{2^{n+2}}, \text{ for } k < n + 2. \]

Therefore

\[ d(x,e) = \sup. \frac{1}{2^k} \psi_k(x,e) < \frac{1}{2^{n+2}} < \frac{1}{2^{n+1}}. \]

Also, if \( x \in H_n \) and \( x \notin H_{n+1} \), then \( \phi_k(x,e) = 1, \text{ for } k > n + 1. \)
Thus,
\[ \frac{1}{2^k} \psi(x,e) = \frac{1}{2^{k+1}} \leq \frac{1}{2^{n+2}}, \text{for } k > n+1 \]
and
\[ 0 = \frac{1}{2^k} \psi(x,e) < \frac{1}{2^{n+2}}, \text{for } k < n+1 \]
i.e.
\[ d(x,e) = \sup. \frac{1}{2^k} \psi_k(x,e) \leq \frac{1}{2^{n+2}} < \frac{1}{2^{n+1}}. \]
Hence
\[ x \in H_n \implies d(x,e) < \frac{1}{2^{n+1}} \implies x \in U_n, \]
i.e.
\[ H_n \subset U_n. \text{ Therefore } H_n = U_n. \]

This completes the proof of the theorem.

Remark: The metric is also invariant if and only if there exists a countable neighbourhood base consisting of the normal sub-semigroups.

Theorem: 4.3.3. Let $S$ be a semigroup with an invariant (i.e. left and right invariant) metric, then there exists a non-archimedean metric semigroup $\hat{S}$, which is complete and that $S$ is a dense sub-semigroup of $\hat{S}$.

Proof: Since $S$ is a semigroup with an invariant metric, then it is easy to see that $S$ can be imbedded as a dense subsemigroup of a complete metric semigroup, which also has an invariant metric.
Also, since $S$ is non-archimedian metrizable, therefore by the remark following theorem 4.3.2, there exists a countable base of normal sub-semigroups at the identity element of $S$ and by theorem 1.1.10, the closure of these sub-semigroups in $\hat{S}$, which are normal in $\hat{S}$, constitute a base at the identity. Hence $\hat{S}$ is also non-archimedian metrizable. Further it clearly follows that $\hat{S}$ is complete with respect to this non-archimedian metric. This completes the proof.

**Theorem 4.3.4**: If $d$ is a non-archimedian invariant metric defined on a topological semigroup $S$, then $a_n \to a$ and $b_n \to b$ in $S$ imply $d(a_n b_n, ab) \to 0$.

**Proof**: Since $a_n \to a$ and $b_n \to b$ in $S$, then for any arbitrary $\varepsilon > 0$, we have $d(a_n, a) < \varepsilon$ and $d(b_n, b) < \varepsilon$. Further, since $d$ is a non-archimedian invariant metric, we have

\[
d(a_n b_n, ab) < \max \left\{d(a_n b_n, a_n b), d(a_n b, ab)\right\}
\]

\[
= \max \left\{d(b_n, b), d(a_n, a)\right\}
\]

\[
< \varepsilon.
\]

This completes the proof.
4.4 Almost Periodic Functions and Associated Compact Semigroups.

We shall define a relation on a semigroup (not necessarily a topological semigroup) associated with a function on $S$ as follows:

**Definition 4.4.1.** For any $a, b \in S$, $a \sim b$ iff

$$| f(xay) - f(xby) | < \varepsilon,$$

for every $x, y \in S$.

From the definition of the relation, it is easy to see that this relation is reflexive and symmetric. Also from ultrametric inequality it follows that it is transitive. Thus this relation is an equivalence relation. Further it can be easily seen that this relation is in fact a congruence relation (i.e. $a \sim b \implies ac \sim bc$ and $ca \sim cb$ for any $c \in S$). Hence the collection of the elements equivalent to the identity element $e$ of $S$ form a normal sub-semigroup $H(\varepsilon)$. When $f$ is an almost periodic function on $S$, this sub-semigroup $H(\varepsilon)$ coincides with the $\varepsilon$-kernel of the function $f$. Let $H$ be the intersection of all sub-semigroups $H(\varepsilon)$, then

1. $f(a, b) \in H \implies f(xay) = f(xby)$, for every $x, y \in S$.

Now (1) viewed as a relation between $a, b$ may be compared with that defined by Kampen\(^7\) for compact groups. Define

\(^7\) Kampen, Van. E.R. (1)
\( a_f(x, y) = f(xay) \) for \( a \in S \). Now consider the mapping \( \theta = \tilde{a} \rightarrow \alpha_f \) defined in \( S/H \) into the collection \( C_2(S) \) of functions of two variables on \( S \). In the view of (1), the mapping is well defined and is one-to-one. The function \( f \) on the cosets of \( H \) is constant and it can be easily verified that \( H \) is a maximal invariant sub-semigroup of \( S \) such that \( f \) is constant on the cosets of \( H \). We shall identify that \( S/H \) with the subclass of functions \( \alpha_f \) of two variables is defined on \( S \).

In the space \( C_2(S) \) of functions of two variables on \( S \), define a metric \( \delta \) by

\[
\delta(f, g) = \min \left\{ \sup_{x, y \in S} |f(x, y) - g(x, y)|, 1 \right\}.
\]

It is a non-archimedean metric on \( C_2(S) \). This metric induces two sided invariant metric on \( S/H \) as follows:

\[
d(\tilde{a}, \tilde{b}) = \delta(\alpha_f, \beta_f)
\]

\[
= \min \left\{ \sup_{x, y \in S} |f(xay) - f(xby)|, 1 \right\}
\]

\[
d(\tilde{a}c, \tilde{b}c) = \delta(\alpha_{f'}, \beta_{f'})
\]

\[
= \min \left\{ \sup_{x, y \in S} |f(xacy) - f(xbcy)|, 1 \right\}
\]

\[
= \min \left\{ \sup_{x, y \in S} |f(xay') - f(xby')|, 1 \right\}
\]

\[= d(\tilde{a}, \tilde{b}).\]
Similarly
\[ d(\bar{c}a, \bar{c}b) = d(\bar{a}, \bar{b}). \]

As a consequence, the completion \( \hat{S}_f \) of \( S/H \) is a complete non-archimedean metric semigroup (by theorem 4.3.3) uniquely determined by \( f \) and \( S \).

As in the classical case, we can easily show that the almost periodicity of \( f \) is equivalent to the total boundedness of the class of functions \( f(xay) \) (considered as functions of \( x \) and \( y \)) in \( C_2(S) \), which is again equivalent to the compactness of \( \hat{S}_f \). Thus \( f \) is almost periodic iff \( \hat{S}_f \) is a compact semigroup.

Each almost periodic function \( f \), thus determines a compact non-archimedean (complete) metric semigroup \( \hat{S}_f \). We define a function \( \bar{f} \) on \( S/H \) as \( \bar{f}(\bar{a}) = f(a) \) where \( a \) is the coset containing \( a \). \( \bar{f} \) is a uniformly continuous function on \( S/H \) and therefore extended to a continuous function of \( \hat{S}_f \). Similarly, to any continuous function on this semigroup \( \hat{S}_f \), there corresponds a uniquely defined function on \( S \), which is almost periodic on \( S \). Let \( mf \) denotes a smallest closed invariant sub-algebra containing \( f \), i.e. \( mf \) is the smallest closed subset of almost periodic functions satisfying the following conditions:
1) \( h, g \in mf \) and \( a \in Q \implies ah, h+g \) and \( h, g \in mf \)
2) \( h \in mf \), \( a \in S \impliesah, h_a \in mf \), where \( ah(x) = h(ax) \)
    and \( h_a(x) = h(xa) \) for \( x \in S \). 
3) if \( h_n \) converges uniformly to \( h \) and \( h_n \in mf \), 
    then \( h \in mf \).

Then every function \( g \in mf \) is also constant on the coset of 
\( H \). The function on \( S/H \) induced by \( g \) is uniformly continuous 
and hence can be extended uniquely to a continuous function 
on \( \hat{S}_f \). We denote, by \( \hat{mf} \) the collection of all continuous 
functions on \( \hat{S}_f \) associated with \( mf \). Now we prove:

**Theorem 4.4.1.** If the valuation on the field \( Q \) is of rank one, 
then corresponding to each almost periodic function \( f \) on 
a semigroup \( S \) there exists a compact totally disconnected semi-
group such that the smallest closed subalgebra \( \hat{mf} \) of 
continuous functions on \( \hat{S}_f \) associated with \( mf \) coincides 
with the algebra of all continuous functions on \( \hat{S}_f \).

**Proof.** As in the classical case for groups (e.g. Kampen\(^8\)), 
it is easily seen that \( mf \) distinguishes the points on \( \hat{S}_f \) and 
for each element in \( \hat{S}_f \) there exists a function in \( \hat{mf} \) not 
vanishing at that element. Hence, if the valuation on the 
field \( Q \) is of rank one, then by Stone-Weierstrass theorem 
in the non-archimedean case (corollary 1.2.10) \( \hat{mf} \) is the

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8) Kampen, Van, E.R. (1)
same as the collection of all continuous functions on \( S_f \).

This completes the proof.

**Theorem 4.4.2.** If \( S \) is a maximally almost periodic topological semigroup, and \( A = \mathcal{A}_c(S) \) is a set of all continuous functions on \( S \), then the Bohr compactification of \( S \) coincides with \( \overline{\phi(S)} \subseteq \hat{S} \), where \( \phi \) is the mapping defined by \( \phi(a) = (a_f, \ldots, a_f, \ldots) \) from \( S \) in to \( \hat{S} = \prod S_f \).

\( f \in \mathcal{A}_c(S) \)

**Proof.** For each \( 1 > \epsilon > 0 \), the \( \epsilon \)-kernel \( H \) of \( f \) is an open and closed sub-semigroup of \( S \). For \( ab \in H \) and \( x, y \in S \) we have

\[
| f(xay) - f(xby) | < \epsilon.
\]

Hence

\[
\delta(a_f, b_f) = \min \left\{ \sup_{x, y \in S} | f(xay) - f(xby) |, 1 \right\} < \epsilon.
\]

Hence \( a, b \in H, \delta(a_f, b_f) < \epsilon \). This proves that the mapping \( a \rightarrow a_f \) from \( S \) to \( S_f \) is uniformly continuous.
Hence the mapping $\phi$ itself is continuous. It is clear from the maximal periodicity of $S$, that $\phi$ is an algebraic isomorphism into $S$. Further $\tilde{S} = \mathcal{P} \hat{S}_f$ is a compact totally disconnected semigroup, since each $\hat{S}_f$ is so. Hence $\varphi(S) \subseteq \tilde{S}$ is also compact totally disconnected.

If $(\mathcal{P}, \hat{S})$ is the Bohr compactification of $S$ and $S$ is maximally almost periodic, then by theorem 4.2.10(a) $\mathcal{P}$ is one-to-one and hence can be identified as a dense sub-semigroup of $\hat{S}$. Now, since $\phi$ is a continuous isomorphism of $S \subseteq \hat{S}$ into $\mathcal{P} \hat{S}_f = \tilde{S}$. Hence it can be extended $\mathcal{P} \subseteq A_c(S)$ to a continuous isomorphism of $\hat{S}$ onto $\varphi(S)$. Since $S$ is compact and $\varphi(S)$ is Hausdorff. Hence $\phi$ is a homeomorphism.

This completes the proof.