CHAPTER I

PRELIMINARIES

In this chapter we discuss and state some well known definitions and results required in connection with the subject matter of this thesis. The proofs of the results may be found in the relevant works referred to.

1.1 Uniform Structure on Semigroups

Definition 1.1.1. Let $S$ be a semigroup. A topological semigroup is a semigroup $S$ endowed with the Hausdorff topology for which the mapping $g(x,y) \rightarrow x \cdot y$ of $S \times S$ into $S$ is continuous in both the variables jointly.

We consider our semigroup with an identity element. This does not affect the general validity of our results, since an identity element may always be algebraically adjoined. We can then ensure that it is topologically irrelevant by making it an isolated point.

Let $S$ be a topological semigroup, then under certain restrictions we have the following property:

(a) $U$ is open in $S \Rightarrow x \cup U$ is open in $S$, for some $x \in S$. 
However, given any topological semigroup $S$ with the identity element $e$, we can find a stronger topology under which $S$ is a topological semigroup satisfying (a) and such that the neighbourhoods at $e$ are same under these two topologies, e.g. a topological semigroup not satisfying (a) is $[0,1]$ with the usual multiplication and topology, since $0, [0,1] = \{0\}$.

To prove the second assertion, let $\tau$ denotes the given topology of $S$ and let $U$ be a basis of open sets at $e$. We define a new topology $\tau'$ on $S$ by requiring that $\{xU : U \in U\}$ be a neighbourhood basis at $x$, for each $x \in S$. Denote this open basis by $B_x$, we must verify that $B_x$ is really a basis for a topology $\tau$. This will be the case, if given $aU, bV \in B_x$, where $U, V \in U$ and $c \in aU \cap bV$, there exists a $W \in U$, such that $cW \subset aU \cap bV$. We have $c = au = bv$, where $u \in U$ and $v \in V$. By continuity of the mapping $x \rightarrow ux$ at $e$, there is a $U_1 \in U$ such that $uU_1 \subset U$, similarly $vV_1 \subset V$ for some $V_1 \in U$. Choosing $W \subset U_1 \cap V_1$, we find that

$$cW \subset (cU_1) \cap (cV_1) = (auU_1) \cap (bvV_1) \subset (aU) \cap (bV).$$

To show that the multiplication is continuous in this topology $\tau'$, let $a, b \in S$ and $U \in U$. If $V \in U$ and $V^2 \subset U$, then $aV^2 = abV^2 \subset abU$. Obviously $\tau'$ is the topology of $S$ satisfying (a), Finally we see that $\tau'$ is stronger than $\tau$, because if $W \in \tau$ and $a \in W$, then the continuity of $x \rightarrow ax$ at $e$ implies that $aU \subset W$ for some $U \in U$.  

The idea of almost periodic functions on a group given by Maak\(^1\) applies to semigroup. In accordance with the definition:

**Definition 1.1.2.**\(^2\) A complex valued function \(\psi\) on a semigroup \(S\) with the identity element \(e\) is called almost periodic, for each \(\varepsilon > 0\) a finite covering \(\{U_i\}_{i=1}^{n}\) of \(S\) exists, such that

1) \[ c' x d', c'yd' \in U_i, \quad x, y, c', d' \in S \]
   for appropriate \(i\)

2) \[ |\psi(cxd) - \psi(cyd)| < \varepsilon, \]
   for all \(c, d \in S\).

It is known that the almost periodic function on this semigroup \(S\) forms a relative uniformly convergent complete algebra with the identity element. With \(\psi(x), \psi(axb), a, b \in S\), is also almost periodic function.

In the following discussion reflexion is important, i.e. to every almost periodic function \(\psi(x)\) on \(S\) in specific method, another function \(\psi(xy)\), \(x, y \in S\), can be constructed, with the properties:

a) \(\psi(x, y)\) is almost periodic in \(X\), for all fixed \(y; x, y \in S\)
b) \(\psi(x, e) = \psi(x)\)
c) \(\psi(xa, ya) = \psi(x, y)\).

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1) Maak, W. (1,2)
2) Kultze, R. (1)
These are equivalent properties for the function $\psi(xy^{-1})$ on groups.

Let $S$ be a topological semigroup with identity $e$ and the complex valued almost periodic function $\psi$ on $S$ with the properties

\( \alpha)\) $\psi(x) \in \mathfrak{B} \implies \overline{\psi(x)} \in \mathfrak{B}$, where bar denotes the conjugate,

\( \beta)\) $e \in \mathfrak{B}$

\( \gamma)\) $\mathfrak{B}$ is complete w.r.t. uniform convergence

\( \delta)\) $\psi(x) \in \mathfrak{B} \implies \psi(axb) \in \mathfrak{B}$,

then $\mathfrak{B}$ defines an algebra of almost periodic functions on $S$.

**Definition 1.1.3\(^3\)** Let $S$ be a topological semigroup with identity $e$, the family $\mathcal{U}$ of the subsets of $S$ defined by

$$V(\psi_1, \psi_2, \ldots, \psi_n; \varepsilon) = \left\{ (x, y) : |\psi_\nu(x) - \psi_\nu(y)| \leq \varepsilon, \psi_\nu \in \mathfrak{B}, \nu = 1, 2, \ldots, n \right\},$$

defines the neighbourhood system of $S$.

This neighbourhood system is called the uniform structure $\mathcal{U}_\mathfrak{B}$ with respect to the uniform continuity.

**Definition 1.1.4\(^4\)** Let $V_r$ be a family of subsets of $S$, defined by

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3) Kultze, R. (1)
4) Kultze, R. (1)
\[ V_{\varepsilon}(\psi_1, \psi_2, \ldots, \psi_n; \varepsilon) = \{(x, y) : |\psi_v(x, y) - \psi_v(\varepsilon)| < \varepsilon, \psi_v \in \Phi, \nu = 1, 2, \ldots, n\} \]

this family forms a right invariant neighbourhood basis of
the uniform structure and is called right uniform structure.

Similarly a left uniform structure may also be
defined.

**Definition 1.1.5.** A real or complex valued function \( f \)
defined on a semigroup \( S \) is called uniformly continuous, if
corresponding to an \( \varepsilon > 0 \), there exists a neighbourhood \( U \)
of the identity element \( e \in S \), such that

\[ |f(x) - f(y)| < \varepsilon \text{ for all } (x, y) \in U, \]

where \(||\) denotes the modulus of a real or complex number.

**Remark.** If we consider \( f \) as a function on \( S \), with
values in a valued field \( Q \), then for the definition of
uniform continuity of \( f \), we replace the modulus in the above
definition by the valuation of the field \( Q \).

**Definition 1.1.6.** A topological semigroup \( S \) is said to
be complete if its left and right uniform structures are the
structures of the complete space.

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5) Hofmann, K.H. and Mostert, P.S. (1)
It is sufficient for a semigroup to be complete, if one of the two uniform structures is the structure of the complete space. 6)

The complete semigroup \( \hat{S} \) is called the completion of \( S \) and is unique to an uniform isomorphism.

Remark. In particular if the left and right uniform structures of a topological semigroup are equivalent then it has a unique completion.

Theorem 1.1.7. 7) (i) Let \( E \) be a set, and let \( \mathcal{U}_o \) be a filter base of subsets of \( E \times E \) which contains the diagonal elements. Define a class \( \mathcal{U} \) by the condition that \( U \in \mathcal{U} \) iff there exists a sequence \( \{ V_n \} \) of symmetric sets in \( \mathcal{U}_o \) so that \( V_n \circ V_n \subset V_{n-1} \subset U \), for all \( n \). Then \( \mathcal{U} \) is the finest uniform structure on \( E \) whose filter base of vicinities is coarser than \( \mathcal{U}_o \).

(ii) If \( F \) is another set, \( \mathcal{V} \) is a uniform structure on \( F \), and \( f \) is a mapping of \( E \) into \( F \) with the property that \( (f \times f)^{-1}(V) \) contains a set of \( \mathcal{U}_o \), then \( f \) is uniformly continuous from \( (E, \mathcal{U}) \) to \( (F, \mathcal{V}) \).

6) Hofmann, K.H. and Mostert, P.S. (1)
7) Pym, J.S. (1)
For a given family $\mathcal{F}$ of sets, we write

$$V_\mathcal{F} = \bigcup_{\mathcal{A} \in \mathcal{F}} \mathcal{A}.$$

**Theorem 1.1.3.**

(i) Let $(E, \tau)$ be a topological space, and $\mathcal{U}_o = \{V_\mathcal{F} : \mathcal{F} \text{ is a finite open covering of } E\}$. Then $\mathcal{U}_o$ is the filter base on $E \times E$. The finest uniform structure $\mathcal{U}$ coarser than $\mathcal{U}_o$ is totally bounded, and the topology induced on $E$ by $\mathcal{U}$ is coarser than $\tau$.

(ii) If $f$ is a continuous mapping of $(E, \tau)$ into any totally bounded Hausdorff space $(F, \mathcal{V})$, then $f : (E, \mathcal{U}) \to (F, \mathcal{V})$ is uniformly continuous.

**Theorem 1.1.9.**

Let $S$ be a semigroup, and $\mathcal{U}$ a uniform structure on $S$. For each $U \in \mathcal{U}$, write

$$U' = \bigcup_{a, b \in S} \{axb, ayb : (x, y) \in U\}$$

and let $\mathcal{U}'$ be the set of all such $U'$. Then $\mathcal{U}'$ is a filter base. The finest uniform structure $\mathcal{U}'$ coarser than $\mathcal{U}'$ is also coarser than $\mathcal{U}$ and the multiplication is jointly uniformly continuous on $\mathcal{U}'$.

(ii) Any uniformly continuous homomorphism of a topological semigroup $(S, \mathcal{U})$ into another topological semigroup $(T, \mathcal{V})$ is continuous.

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8) Pym, J.S. (1)
9) Pym, J.S. (1)
is also uniformly continuous with respect to $\mathcal{U}'$.

**Theorem 1.1.10.** Let $S$ be a Hausdorff topological semigroup admitting a completion $\hat{S}$. The closures in $\hat{S}$ of the members at the identity element in $S$ form a base at the identity element in $\hat{S}$.

The proof of this theorem easily follows from Proposition 7 of Bourbaki\(^{10}\), if we consider the base for $S$ at the identity element $e \in S$ in place of all the neighbourhoods at $e$.

**Remark** Let $S$ be a Hausdorff topological semigroup which has a neighbourhood base at the identity $e$ consisting of normal sub-semigroups of $S$. Then the left and right uniform structures of $S$ are equivalent and hence $S$ has a completion $\hat{S}$. The closures in $\hat{S}$ of these normal sub-semigroups are again normal sub-semigroups in $\hat{S}$. Therefore the left and right uniform structures of $\hat{S}$ are equivalent.

**Theorem 1.1.11.**\(^{11}\) Let $(S, \tau)$ be a topological semigroup where the multiplication is jointly uniform continuous, then there exists a totally bounded Hausdorff topological semigroup with jointly uniform continuous multiplication $(S^*, \tau^*)$ and a mapping

$$i : (S, \tau) \to (S^*, \tau^*)$$

with the property that if there is any continuous homomorphism

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10) Bourbaki, N. (l), p. 30
11) Pym, J.S. (l)
of \((S, \tau)\) into any totally bounded Hausdorff topological semigroup \((E, \nu)\), there is a unique uniform continuous homomorphism

\[ h : (S^*, \tau^*) \to (E, \nu) \]

such that \(f = h \circ i\).

We shall give here the following definition with the help of Alfsen and Holm\(^{12}\).

**Definition 1.1.12.** Let \(S\) be a topological semigroup and \(\mathcal{U}\) a uniform structure on \(S\). We say that \(\mathcal{U}\) is compatible with the semigroup structure when the semigroup operations are uniformly continuous with respect to the uniform structure.

**Definition 1.1.13.** (i) A base \(\mathcal{B}\) of uniform structure \(\mathcal{U}\) is called symmetric if, for every \(U \in \mathcal{B}\), \((x, y) \in U \implies (y, x) \in U\).

(ii) It is called left (right) invariant if for every \(U \in \mathcal{B}\), \((x, y) \in U\), \(a \in S \implies (ax, ay) \in U\), \((xa, ya) \in U\).

A base which is both left and right invariant is called an invariant base.

The following theorem can be easily deduced by using the method of Bourbaki\(^{13}\):

\(^{12}\) Alfsen, E.M. and Holm, P. (1)

\(^{13}\) Bourbaki, N. (1)
Theorem 1.1.14. The right uniform structure on a topological semigroup $S$ is the only structure compatible with the topology of $S$ and which admits a right invariant base.

Theorem 1.1.15. Let $S$ be a totally disconnected (locally compact) semigroup. Then every neighbourhood of the identity element $e$ contains a normal (compact open) subsemigroup.

The proof of this theorem is parallel to Proposition 7.7 of Hewitt and Ross\textsuperscript{14}).

Theorem 1.1.16. Let $(S, \tau)$ be a topological semigroup. Then the family $\mathcal{H}$ of all closed normal subsemigroups of finite index defines a topology $\tau_\mathcal{H}$ coarser than the given topology $\tau$.

$S$ with this topology $\tau_\mathcal{H}$ is also a topological semigroup. $\tau_\mathcal{H}$ is Hausdorff iff for each $x \neq e$, there exists a closed normal sub-semigroup $H$ of finite index not containing $x$.

Proof. Let the index of $H \in \mathcal{H}$ in $S$ be denoted by $[S:H]$. Let $H_1, H_2 \in \mathcal{H}$, then, if $x, y \in H_1$, and $xH_2 = yH_2$, then $(x, y) \in H_1 \cap H_2$ and $x(\cap_{\mathcal{H}} H_2) = y(\cap_{\mathcal{H}} H_2)$.

Hence if $x(H_1 \cap H_2)$ and $y(H_1 \cap H_2)$ are distinct cosets of $H_1 \cap H_2$ in $H$, $xH_2$ and $yH_2$ are distinct cosets of $H_2$ in $S$. That is $[S:H_1 \cap H_2] < [S:H_2]$.

It can be easily seen that $[S:H_1 \cap H_2] = [S:H_1][H_1:H_1 \cap H_2]$

\textsuperscript{14} Hewitt, E. and Ross, K.A. (1)
and so $H_1 \cap H_2$ has finite index in $S$. Also $H_1 \cap H_2$ is closed normal sub-semigroup. It can be easily verified that the family $\mathcal{H}$ defines a topology $\tau_{\mathcal{H}}$ under which $S$ is a topological semigroup. Further, since any closed normal sub-semigroup in $\tau$ is also open in $\tau$ and, consequently, is coarser than $\tau$. It is clear that the topology $\tau_{\mathcal{H}}$ is Hausdorff if and only if for $x \neq e$ there exists a $H \in \mathcal{H}$ such that $x \in H$.

This completes the proof.

1.2 Fields with Valuation

Definition 1.2.1. Let $Q$ be a field. A valuation on $Q$ is a function $|\cdot|$ on $Q$ with values in the real number field, such that

(i) $|a| > 0$ and $|a| = 0$ iff $a = 0$

(ii) $|ab| = |a| |b|$

(iii) $|a+b| \leq |a| + |b|$.

$Q$ is called a field of valuation or a valued field.

As a consequence of this definition, we can easily deduce that $|e| = 1$ where $e$ is the identity element of the field, $|a| = |-a|$ and $|a^{-1}| = \frac{1}{|a|}$. The function $|\cdot|$ is a homomorphism of a multiplicative group $Q^*$ into the reals. The valuation $|\cdot|$ is said to be discrete if the image $|Q^*|$ in the reals is an infinite cyclic group. The valuation is called
archimedean, if for every $a \in Q$, there exists an integer $n = n(a)$ (i.e. ne) such that $|a| < |n|^{15}$. A valuation is said to be non-archimedean if it is not archimedean and the field with such a valuation defined on it is called non-archimedean valued field, or in brief, a non-archimedean field.

**Theorem 1.2.2.** In a valuated field $Q$ the following properties are equivalent:

(i) $Q$ is a non-archimedean field.

(ii) For every integer $n \in Q$, $|n| \leq 1$

(iii) For every $a, b \in Q$, $|a + b| \leq \max(|a|, |b|)$.

Hence by a non-archimedean valued field, we mean a valuated field $Q$ with the property for valuation

$$|a + b| \leq \max(|a|, |b|), a, b \in Q.$$ 

The last inequality will be termed, hereafter, the ultrametric inequality.

**Theorem 1.2.3.** If $| \cdot |$ is a non-archimedean valuation on a field $Q$ and $|a| \neq |b|, a, b \in Q$, then $|a + b| \leq \max(|a|, |b|)$.

15) Throughout this thesis, we use $|n|$, for an integer $n$, only in this sense viz $|n| \leq |ne|


17) Bachman, G. (1) pp. 8
Q can be viewed as a metric space with a distance defined by \( d(x,y) = |x-y| \) for \( x,y \in Q \). If in addition, Q is a non-archimedean field, then this distance can be easily seen to satisfy the ultra metric inequality:

\[
d(x,y) \leq \max (d(x,z), d(z,y)).
\]

If \( \{a_n\} \) is a sequence of elements in Q, then it is said to converge the element \( a \in Q \), if for every real number \( \epsilon > 0 \), there exists an integer \( N(\epsilon) \) such that \( |a_n - a| < \epsilon \) for \( n > N(\epsilon) \). In terms of the metric, this means

\[
d(a_n, a) \rightarrow 0, \text{ as } n \rightarrow \infty.
\]

Similarly, the sequence \( \{a_n\} \) is said to be a Cauchy sequence if there exists an \( N(\epsilon) \) such that

\[
|a_n - a_m| < \epsilon, \text{ whenever } n, m > N(\epsilon).
\]

**Definition 1.2.4** 18) The field Q is called complete with respect to the valuation \( |\cdot| \), if every Cauchy sequence in Q, has a limit in Q.

**Theorem 1.2.5** 19) Given valuation \( |\cdot| \) on a field Q, there exists a field \( \hat{Q} \) called the completion of Q with respect to \( |\cdot| \).

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18) Bachman, G. (1) pp. 24
19) Bruhat, F. (1) pp. 33
to \( | | \), such that \( \hat{Q} \) is a complete field with respect to a valuation which is an extension of \( | | \). \( Q \) is dense in \( \hat{Q} \).

The extended valuation on \( \hat{Q} \) is archimedean or non-archimedean according as the valuation in \( Q \) is archimedean or non-archimedean.

Lemma 1.2.6. Let \( f_n \) is a sequence of functions defined on a set \( X \), with values in a non-archimedean valued field. If \( f_n \) converges uniformly to a function \( f \) and if \( \inf_{x \in X} |f(x)| > 0 \),

then there exists an integer \( N \) such that for all \( n > N \), and for all \( x \), \( |f_n(x)| = |f(x)| \).

Proof. Since \( f_n \) converges uniformly to \( f \), given \( \epsilon > 0 \), there exists an \( N \) such that for all \( n > N \) and for all \( x \in X \)

\[
|f_n(x) - f(x)| < \epsilon.
\]

Now

\[
|f_n(x)| \leq \max \left\{|f_n(x) - f(x)|, |f(x)|\right\}.
\]

then choosing \( \epsilon \) such that

\[
\epsilon < \inf_{x \in X} |f(x)|,
\]

we see that, for all \( n > N \), and all \( x \)

\[
|f_n(x)| - f(x)| \not= |f_n(x)|.
\]

Hence by theorem 1.2.3.

\[
|f_n(x)| = |f(x)|.
\]

This completes the proof.
We assume that \( Q \) is a non-archimedean field. Let \( V \) be the set of all \( a \in Q \) such that \( |a| < 1 \). Then \( V \) is a ring with identity. Let \( P \) be the set of all \( a \in Q \) such that \( |a| < 1 \). \( P \) is a maximal ideal in \( V \). Hence \( V/P \) is a field. This is called the residue class field of valuation \( | \cdot | \) and is denoted by \( F \). The ring \( V \) is called the valuation ring of the valuation \( | \cdot | \) in \( Q \).

**Theorem 1.2.7.** 20) Let \( Q \) be a field with a proper valuation \( | \cdot | \) then the following conditions are equivalent:

(i) \( Q \) is locally compact

(ii) \( V \) is compact

(iii) \( Q \) is complete, \( | \cdot | \) is a discrete valuation and \( F \) is a finite field.

**Theorem 1.2.8.** If \( A \) is the set of all elements \( x \) such that \( |x| < \alpha \) for some fixed non-negative real number \( \alpha \), in a locally compact, valued field \( Q \), then the set \( A \) is compact (i.e. all the spheres \( S(0,\alpha) = \{ x : |x| < \alpha \} \) are compact).

**Proof.** Given \( \alpha > 0 \), there exists a \( \beta > 0 \) and an element \( x \in Q \) such that \( \alpha < \beta \) and \( |x_0| = \beta \). Since \( Q \) is locally compact, its valuation ring \( V \) is compact by theorem 1.2.7.

20) Bruhat, F. (1) pp.18.
Now the image $Y = \{ x : |x| \leq \beta \}$ of $V$ by the mapping $x \rightarrow x x_0$, is compact. The set of all $x$ such that $|x| \leq \alpha$ is a closed subset of $Y$ and is therefore compact.

This completes the proof.

**Theorem 1.2.9.** Let $F$ be a division ring with a valuation of rank one, $X$ a totally disconnected locally compact Hausdorff space and $C$ be the ring of all continuous functions from $X$ to $F$ vanishing at infinity. Topologize $C$ by uniform convergence. Further let $D$ be a closed subring of $C$, admitting left multiplication by the constant functions and containing for any two points $x, y \in X$, a function vanishing at $x$ but not at $y$. Then $D = C$.

This theorem is due to Kaplansky$^{21}$, for non-archimedean fields, analogous to Stone-Weierstrass theorem.

In the special case of compact spaces we can easily deduce the following corollary:

**Corollary 1.2.10.** Let $Q$ be a non-archimedean field with a valuation of rank one, $X$ a compact totally disconnected space and $C$ be the algebra of all continuous functions from $X$ to $Q$. Topologize $C$ by uniform convergence. Let $D$ be a closed subalgebra of $C$. Then $D = C$, if $D$ distinguishes points on $X$, and if for every $x \in X$, there exists an $f$ in $D$, such that $f(x) \neq 0$.

$^{21}$ Kaplansky, I. (1).