CHAPTER VII

FIXED POINT THEOREMS ON COMPACT AND PSEUDOCOMPACT

TICHONOV SPACES

In the present chapter, some fixed point theorems are obtained in compact and Pseudocompact Tichonov space. The chapter, divide into two sections viz. (i) compact space and (ii) Pseudo-compact Tichonov space.

7.1 There are many generalization of the classical contraction mapping theorem of Banach [4]. First time Edelstein[23] obtained a fixed point theorem in compact metric space, which is generalization of Banach. In the past few years a number of authors such as Iseki[48], Fisher[29],[30],[31], Chatterji[18], Bajaj[9] etc., have established a number of interesting results on compact space.

The aim of in this section, we obtain some fixed point theorems in compact metric space.

Our first theorem is to extend the Edelstein[23].

In fact, we prove
**THEOREM 1**: [121] If $T$ is a continuous mapping of a compact metric space $X$ into itself satisfying:

\[
(7.1.1) \quad d(Tx, Ty) < \alpha d(x, y) + \beta [d(x, Tx) + d(Tx, T^2 x)] + \gamma [d(y, Tx) + d(Ty, T^2 x)]
\]

for all $x, y$ in $X$ ($x \neq y$) where $\alpha + 2 \beta + 2\gamma = 1$, $\alpha, \beta, \gamma > 0$. Then $T$ has a unique fixed point.

**Proof:** Taking a real valued function $f$ on $X$, defined by $f(x) = d(x, Tx)$. Since, $d$ and $T$ are continuous functions, it follows that $f$ is a continuous function on $X$. Since, $X$ is compact, it achieves its minimum value and so there exists a point $z$ in $X$, such that

\[ f(z) = \inf \{ f(x): x \in X \}. \]

Now we will suppose that $Tz \neq z$, then by hypothesis (7.1.1), we have

\[ f(Tz) = d(Tz, T^2 z) \]

\[ < \alpha d(z, Tz) + \beta [d(z, Tz) + d(Tz, T^2 z)] + \gamma [d(Tz, Tz) + d(T^2 z, T^2 z)] \]

i.e.

\[ d(Tz, T^2 z) < (\frac{\alpha + \beta}{1 - \beta}) d(z, Tz) \]

since $(\alpha + 2 \beta) = 1$. Therefore $f(Tz) < f(z)$.

This contradicts, the definition of $z$ so that we must have $Tz = z$ and $z$ is then a fixed point of $T$. 
We will now prove that \( z \) is unique.

Suppose that \( z' \) is a second fixed point with \( z \neq z' \).

We have,

\[
\begin{align*}
    d(z, z') &= d(Tz, Tz') \\
    &< \alpha d(z, z') + \beta d(Tz, Tz') \\
    &+ \gamma d(Tz', Tz') \\
    &= \left( \alpha + 2\gamma \right) d(z, z') \quad \text{as } \alpha + 2\gamma = 1.
\end{align*}
\]

Therefore \( d(z, z') < d(z, z') \) giving a contradiction and so proves the uniqueness of \( z \). This completes the proof of the theorem.

Remark: In theorem -1, if we put \( \alpha = 1 \) and \( \beta = \gamma = 0 \) then we get a result of Edelstin [23].

Next we generalize and extend the result of Jaggi [53].

In fact, we prove

**THEOREM-2.** [122] If \( F \) be a continuous mapping of a compact metric space \((X, d)\) into itself satisfying the condition:

\[
(7.1.2) \quad d(F(x), F(y)) \leq \frac{\alpha d(x, F(x))d(y, F(y))}{d(x, y)} \\
+ \beta \frac{d(y, F(y))}{1+d(x, F(x))} \frac{d(x, F(x))}{1+d(y, F(y))} \\
+ \gamma d(x, y)
\]

for all distinct \( x, y \in X, \quad \alpha, \beta \) and \( \gamma > 0, \alpha + \beta + \gamma < 1 \), then
F has a unique fixed point.

Proof: First we define a function T on X as follows:

$$T(x) = d(x, F(x)), \text{ for all } x \in X.$$ 

Since d and F are continuous on X, T is also continuous on X.

From compactness of X, there exists a point $\xi \in X$, such that

$$T(\xi) = \inf \{ T(x): x \in X \}. \quad (7.1.3)$$

If $T(\xi) \neq 0$, it follows that $\xi \neq F(\xi)$ and so,

$$T(F(\xi)) = d(F(\xi), F^2(\xi))$$

$$\leq a \frac{d(\xi, F(\xi))}{d(\xi, F(\xi))} \frac{d(F(\xi), F^2(\xi))}{d(F(\xi), F^2(\xi))}$$

$$+ \beta \frac{d(F(\xi), F^2(\xi))}{[1 + d(\xi, F(\xi)) + d(\xi, F(\xi))]} \frac{[1 + d(\xi, F(\xi)) + d(\xi, F(\xi))]}{[1 + d(\xi, F(\xi)) + d(\xi, F(\xi))]}$$

$$+ \gamma d(\xi, F(\xi))$$

$$\leq a d(F(\xi), F^2(\xi))$$

$$+ \beta \frac{d(F(\xi), F^2(\xi))}{[1 + d(\xi, F(\xi)) + d(\xi, F(\xi))]} \frac{[1 + d(\xi, F(\xi)) + d(\xi, F(\xi))]}{[1 + d(\xi, F(\xi)) + d(\xi, F(\xi))]}$$

$$+ \gamma d(\xi, F(\xi))$$

$$\leq a d(F(\xi), F^2(\xi)) + \beta d(F(\xi), F^2(\xi))$$

$$+ \gamma d(\xi, F(\xi))$$

$$\leq \left( \frac{\gamma}{1 - a - \beta} \right) d(\xi, F(\xi)) < d(\xi, F(\xi))$$
Since \( \alpha + \beta + \gamma < 1 \), which implies, \( T(F(\xi)) < T(\xi) \), which is contradiction to the condition (7.1.3) and hence \( \xi = F(\xi) \).

Consequently \( \xi \) is a fixed point of \( F \).

Now, we will prove the uniqueness of \( \xi \). Let, if possible \( \eta \neq \xi \) be another fixed point of \( F \).

Now,

\[
\begin{align*}
  d(\xi, \eta) & = d(F(\xi), F(\eta)) \\
  & \leq \alpha \frac{d(\xi, F(\xi)) d(\eta, F(\eta))}{d(\xi, \eta)} \\
  & + \beta \frac{d(\eta, F(\eta))[1 + d(\xi, F(\eta)) + d(\eta, F(\xi))]}{[1 + d(\xi, F(\xi)) + d(\eta, F(\eta))]} \\
  & + \gamma d(\xi, \eta) \\
  & \leq \gamma d(\xi, \eta) < d(\xi, \eta)
\end{align*}
\]

which is a contradiction as \( \gamma < 1 \) then \( \xi = \eta \).

Hence \( \xi \) is a unique fixed point of \( F \).

Hence the theorem.

7.2 A topological space \( X \) is said to be pseudocompact, if every real valued continuous function on \( X \) is bounded. It may be noted that every compact space is pseudocompact, but converse in not true (Engleking [24], Examples P.150). How ever, in a metric space notions
'compact' and 'Pseudocompact' coincide. By Tichonov space we mean a completely regular Hausdorff space. It is observed that the product of two Tichonov spaces is again a Tichonov space where as the product of two Pseudo-compact space need not be Pseudocompact.

Recently, some authors such as Harinath [43], Jain and Dixit [54] Dhage and Dhobhale [20] etc. have obtained results on fixed point theorem in Pseudo-compact Tichonov space.

In this section, we prove the following theorems.

In fact, We prove

**THEOREM-1:** Let \( P \) be a Pseudocompact Tichonov space and \( \mu \) be a non-negative real valued continuous function over \( P \times P \) (\( P \times P \) is Tichonov but not be Pseuocompact)satisfying:

\[
(7.2.1) \quad \mu(x,x) = 0 \text{ for all } x \in P, \text{ and }
\]

\[
\mu(x,y) \leq \mu(x,z) + \mu(z,y) \text{ for all } x,y,z \in P.
\]

Let \( T: P \rightarrow P \) be a continuous map satisfying

\[
(7.2.2) \quad \mu(Tx,Ty) \leq \max\left\{ \frac{\mu(Tx,x) \mu(Ty,y) + \mu(x,Ty) \mu(Tx,y)}{\mu(x,y)} \right\}
\]

for all distinct \( x,y \in P \). Then \( T \) has a fixed point in \( P \) which is unique.
Proof: We define $\mathcal{G}: P \rightarrow R$ by $\mathcal{G}(p) = \mu(Tp, p)$ for all $p \in P$, where $R$ is the set of real numbers. Clearly, $\mathcal{G}$ is continuous being the composite of two functions $T$ and $\mu$. Since $P$ is Pseudo-compact Tichonov-space, every real valued continuous function over $P$ is bounded, and attains its bounds. Thus there exists a point say $v \in P$ such that $\mathcal{G}(v) = \inf \{\mathcal{G}(p) : p \in P\}$ where 'inf' denotes the infimum or the greatest lower bound in $R$. It may be noted that $\mathcal{G}(p) \in R$. We now affirm that $v$ is a fixed point for $T$. If not, let us suppose that $Tv \neq v$.

Then using (7.2.2) we have

$$\mathcal{G}(Tv) = \mu(T^2v, Tv)$$

$$< \max \left\{ \frac{[\mu(T^2v, Tv) \mu(Tv, v)] + [\mu(Tv, Tv) \mu(T^2v, v)]}{\mu(Tv, v)}, \mu(Tv, v) \right\}$$

$$= \max \{ \mu(T^2v, Tv), \mu(Tv, v) \}$$

if $\mu(T^2v, Tv) < \mu(T^2v, Tv)$ and since $\mu(T^2v, Tv) \geq 0$, which is possible then $\mathcal{G}(Tv) = \mu(T^2v, Tv) < \mu(Tv, v)$

i.e. $\mathcal{G}(Tv) < \mathcal{G}(v)$

leading to a contradiction and therefore $Tv = v$ i.e. $v \in P$ is a fixed point for $T$. 
To prove the uniqueness of $v$, if possible, let $w \in P$ be another fixed point for $T$ i.e. $Tw = w$ and $w = v$.

Then using (7.2.2), we have

$$
\mu(v,w) = \mu(Tv,Tw)
$$

$$
< \max\left\{ \frac{[\mu(Tv,v) \mu(Tw,w)] - [\mu(v,Tw) \mu(Tv,w)]}{\mu(v,w)}, \mu(v,w) \right\}
$$

$$
= \max\left\{ \frac{\mu(v,w) \mu(v,w)}{\mu(v,w)}, \mu(v,w) \right\}
$$

$$
= \max\{ \mu(v,w), \mu(v,w) \}
$$

Then $\mu(v,w) < \mu(v,w)$,

again leading a contradiction. Hence $v \in P$ is unique for $T$ in $P$.

This complete the proof.

Now, next we prove the following theorem which is generalization of our Theorem 1 in this section.

**THEOREM-2.** Let $P$ be a Pseudocompact Tichonov space and $\mu$ be a non-negative real valued continuous function over $P \times P$ ($p \times p$ is Tichonov but need not be Pseudocompact) satisfying (7.2.1) and (7.2.3) if $s$ and $T$ are two continuous self-mappings of $P$ satisfying,

(7.2.4) $ST = TS$ and
(7.2.5) \( \mu(STx, Sy) < \max \left\{ \frac{\mu(Tx, STx)}{\mu(y, Tx)} + \frac{\mu(Ty, Sy)}{\mu(y, Tx)} \right\}, \mu(y, Tx) \)

for all distinct \( x, y \in P \). Then \( S \) and \( T \) have a unique common fixed point.

Proof: We define \( \phi: P \rightarrow R \) by \( \phi(p) = \mu(STp, Tp) \) for all \( p \in P \), where \( R \) is the set of real numbers. Clearly \( \phi \) is continuous being the composite of the continuous function \( S, T \) and \( \mu \). Since \( P \) is pseudocompact Tichonov space, every real valued continuous function over \( P \) is bounded and attains its bounds. Thus there exists a point \( v \in P \) such that \( \phi(v) = \inf \{ \phi(p) : p \in P \} \) where 'inf' denotes the infimum or the greatest lower bound in \( R \). It may be noted that \( \phi(p) \in R \). We now affirm that \( v \) is a fixed point for \( S \). If not, let us suppose \( Sv \neq v \).

Then using (7.2.4) and (7.2.5), we have

\[
\phi(Sv) = \mu(STSv, Tsv) = \mu(STSv, STv)
\]

\[
< \max\left\{ \frac{\mu(TSv, STSv)}{\mu(Tv, TSv)} + \frac{\mu(TSv, STv)}{\mu(Tv, STv)} \right\}, \mu(Tv, TSv) \}
\]

\[
< \max\{ \mu(TSv, STSv), \mu(Tv, TSv) \}
\]

i.e. \( \mu(STSv, STv) < \mu(Tv, STv) \)

or \( \phi(Sv) < \phi(v) \)

a contradiction because \( \mu(STSv, STv) > 0 \)
Hence \( v \in P \) is a fixed point for \( S \).

i.e. \( S v = v \).

Using (7.2.4), we have

\[(7.2.6) \quad STv = TSv =Tv.\]

Now we shall prove that \( Tv = v \). If possible let \( Tv \neq v \). Then, we have by (7.2.6) and (7.2.5).

\[
\mu(Tv,v) = \mu(STv,Sv)
\]

\[
< \max \left\{ \frac{\mu(Tv,STv) \mu(v,Sv)}{\mu(v,Tv)}, \frac{\mu(Tv,Sv) \mu(v,STv)}{\mu(v,Tv)} \right\},
\]

\[
\mu(v,Tv)
\]

\[
= \max \{ \mu(Tv,v), \mu(v,Tv) \}
\]

\[
< \mu(v,Tv)
\]

leading to a contradiction and hence \( v \in P \) is a fixed point of \( T \) i.e. \( Tv = v \).

To prove the uniqueness of \( v \). If possible, let \( w \) be another fixed point for \( S \) and \( T \) i.e. \( v = Sv = Tv \) and \( w = Sw = Tw \) \((w \neq v)\).

Then using (7.2.5) we have

\[
\mu(v,w) = \mu(STv,Sw)
\]

\[
< \max \left\{ \frac{\mu(Tv,STv) \mu(w,Sw)}{\mu(w,Tv)}, \frac{\mu(Tv,Sw) \mu(w,STv)}{\mu(w,Tv)} \right\},
\]

\[
\mu(w,Tv)
\]

\[
\mu(v,Tv)
\]
\[
= \max \left\{ \frac{\mu(v,v) \mu(w,w)}{\mu(w,v)}, \mu(w,v) \right\},
\]

\[
= \max \left\{ \mu(v,w), \mu(w,v) \right\},
\]

\[
< \mu(w,v)
\]

i.e. \( \mu(v,w) < \mu(w,v) \)

leading to a contradiction, which proves that \( v \in P \) is unique.

This completes the proof of the theorem.

In fact, we prove the following:

**THEOREM-3[123]:** Let \( P \) and \( \nu \) be the same as defined in our theorem 1 in this section and satisfy (7.2.1). Let \( T: P \to P \) be a continuous map satisfying

\[
(7.2.7) \quad \nu(Tx,Ty) < \frac{\left(\nu(x,Ty)\right)^2 + \left(\nu(x,Tx)\right)^2 + \left(\nu(y,Ty)\right)^2}{\left(\nu(x,Ty) + \nu(x,Tx) + \nu(y,Ty)\right)^2}
\]

for all \( x,y \in P \) for which \( \nu(x,Ty) + \nu(x,Tx) + \nu(y,Ty) \neq 0 \). Then \( T \) has a fixed point in \( P \). Further, if \( \nu(x,Ty) + \nu(x,Tx) + \nu(y,Ty) = 0 \) implies \( \nu(Tx,Ty) = 0 \) then the fixed point is unique.

**Proof:** Let \( \emptyset \) and \( v \) as in the proof of our Theorem 1. If \( v \in P \) is not a fixed point of \( T \) then applying (7.2.7), we have

\[
\emptyset(Tv) = \mu(T^2v,Tv)
\]
\[
\frac{[\mu(TvTv)]^2 + [\mu(TvT^2v)]^2 + [\mu(vTv)]^2}{[[\mu(TvTv)] + [\mu(TvT^2v)] + [\mu(vTv)]]}
\]
\[
= \frac{[\phi'(Tv)]^2 + [\phi(v)]^2}{[[\phi(Tv)] + [\phi(v)]]}
\]
i.e. \[
[\phi(Tv)]^2 + \phi(Tv) \phi(v)
\]
\[
< [\phi'(Tv)]^2 + [\phi'(v)]^2
\]
which, since \(\phi'(v) > \phi'(v)\) implies
\[
\phi'(Tv) < \phi'(v),
\]
leading to a contradiction and hence \(Tv = v\). Uniqueness of \(v\) follows from the stated condition.

Completes the proof of theorem.

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