CHAPTER VI

FIXED POINT THEOREMS ON 3-METRIC SPACE OVER A TOPOLOGICAL SEMI-FIELD

6.1 In 1960 topological semifield was defined [2][3] and he proved that any topological semifield contains a topological field isomorphic with the real line. These elements are denote by Greek letter $\alpha$ and follows the rules operation on real numbers.

We require some definition:

**Definition:-** We shall call a commutative associative topological semifield i.e. there is isolated in some set $K$ satisfying:

1. $K + \bar{K} \subseteq K$, $K \cdot K \subseteq K$.
2. $K - \bar{K} = E$.
3. The least upper bound and greatest upper bounds exists,
4. For $a, b \in K$, the equation $ax = b$ has at least one solution in $K$.
5. The inter-section $\bar{K} \cap (-\bar{K})$ contains only the zero element of the ring.
6. We denote by $F_\alpha (\forall \in E)$ the totallity of all
elements \( x \in E \) satisfying the condition \( a \in \bar{K} \).

Then the totality of all sets of the form \( \beta + F_a \)
\((a, \beta \in E)\) forms a basis system of closed system of closed
sets of topological space \( E \).

**Remark:** 1. The axiom for a topological semifield are so
chosen that its properties recall those of the field of
real numbers.

2. We assume here that multiplication require its
partial inverlibility. That is why it is called 'semi
field'.

3. An interesting example due to Daktyyarer show that
the requirement \( K + \bar{K} \subseteq K \) in the definition can not be
weakened by \( K + K = K \).

4. It is also known that a commutative
topological ring admits at most one semifield structure.

We shall can elements of the set \( K \) positive
elements of the semifield \( E \) and the elements of the set
\( \bar{K} - K \) will be called boundary elements of the semifield
\( E \). We agree to write the relation \( x - y \in K \), \( x - y \in \bar{K} \) also in
the form \( x > y \), \( x \geq y \) (or in the form \( y < x \), \( y \leq x \)).

In particular, the inequality \( x > 0 \) means \( x \in K \) and \( x \geq 0 \).
means that \( x \in \bar{K} \).

The set \( K \) contains element which are different from zero.

Now we define 3-metric space over topological semifield \( E \).

**Definition:** Let \( E \) be a semifield and \( K \) is the set of all its positive elements. The set \( X \) is called a 3-metric space over the semifield \( E \), if there exists a metric,

\[
P : \text{XXX}xXxX \longrightarrow \bar{K}
\]

for each quadruple point \( x,y,z,w \in X \) such that

(6.1.1(a)) To each non-degenerate 2-simplex \( < x,y,z > \) in \( X \) there is a point \( w \in X \) satisfying \( P(x,y,z,w) \neq 0 \).

(6.1.1(b)) \( P(x,y,z,w) = 0 \) only when the 3-simplex, \( < x,y,z,w > \) is degenerate.

(6.1.2) \( P(x,y,z,w) = P(x,w,z,y) \)

(6.1.3) \( P(x,y,z,w) \ll P(x,y,z,u)+P(x,y,u,w)+P(x,u,z,w)+P(u,y,z,w) \)

**Remark:** A 3-metric space over topological semifield is called bounded, if there exists a constant \( M \) such that

\[
P(x,y,a,b) \ll M, \text{ for all } x,y,a,b \in X.
\]

If \( E \) is a field of real numbers, then we
arrive at the definition of 3-metric space (defined above) also.

If X consists only two points, we get the definition of metric space over topological semifield.  

**Definition:** A sequence \( \{x_n\} \) in a 3-metric space over a topological semifield \( E \), is called a Cauchy sequence, if \( \lim_{m \to \infty} P(x_m, x_n, a, b) \in u \), for all \( a, b \in X \), where \( u \subseteq E \) is the neighbourhood of the origin.

**Definition:** A sequence \( \{x_n\} \) in a 3-metric space over topological semifield \( E \) is called a convergent sequence, if there is a \( x \in X \), such that \( \lim_{n \to \infty} P(x_n, x, a, b) \in u \) for all \( a, b \in X \).

**Definition:** A 3-metric space over a topological semifield \( E \) in which every Cauchy sequence converges, is called a complete 3-metric space.

6.2 Now, we obtained the following theorem, which is generalization of Kannan [56].

**Theorem 1.** Let \( X \) be a complete 3-metric space over a topological semifield \( E \), \( f \) be a mapping on \( X \) and for all \( x, y, a, b \in X \) such that
(6.2.1) \( P(f(x), f(y), a, b) \ll \alpha \left[ P(x, f(x), a, b) + P(y, f(y), a, b) \right] \)

where \( \alpha \) is a positive number and \( \alpha < \frac{1}{2} \). Then there is a fixed element \( x' \) of the mapping \( f \) such that \( f(x') = x' \).

**Proof:** Take an element \( x_0 \in X \), then by recursive, we define a sequence \( \{x_n\} \) by

\[ x_{n+1} = f(x_n) \quad (n = 0, 1, 2, - - - ) \]

Then we have

\[
P(x_1, x_2, a, b) = P(f(x_0), f(x_1), a, b)
\]

\[ \ll \alpha \left[ P(x_0, f(x_0), a, b) + P(x_1, f(x_1), a, b) \right] \]

\[ = \alpha \left[ P(x_0, x_1, a, b) + P(x_1, x_2, a, b) \right] \]

\[ \ll \left( \frac{\alpha}{1 - \alpha} \right) P(x_0, x_1, a, b) \]

Again, we get

\[
P(x_2, x_3, a, b) = P(f(x_1), f(x_2), a, b)
\]

\[ \ll \alpha \left[ P(x_1, f(x_1), a, b) + P(x_2, f(x_2), a, b) \right] \]

\[ = \alpha \left[ P(x_1, x_2, a, b) + P(x_2, x_3, a, b) \right] \]

\[ \ll \left( \frac{\alpha}{1 - \alpha} \right)^2 P(x_0, x_1, a, b) \]

In general,

\[
P(x_n, x_{n+1}, a, b) \ll q^n P(x_0, x_1, a, b)
\]

where \( q = \left( \frac{\alpha}{1 - \alpha} \right) \).

For \( n < m \) we have
\[
P(x_n, x_m, a, b) \ll P(x_n, x_m, a, x_{n+1}) + P(x_n, x_m, x_{n+1}, b) \\
+ P(x_n, x_{n+1}, a, b) + P(x_{n+1}, x_m, a, b) \\
+ P(x_n, x_{n+1}, a, b) + P(x_{n+1}, x_m, x_{n+2}, b) \\
+ P(x_{n+1}, x_{n+2}, a, b) + P(x_{n+2}, x_m, a, b) \\
+ P(x_{n+1}, x_m, a, x_{n+2}).
\]

\[
= P(x_n, x_{n+1}, x_m, a) + P(x_{n+1}, x_{n+2}, x_m, a) \\
+ P(x_n, x_{n+1}, x_m, b) + P(x_{n+1}, x_{n+2}, x_m, b) \\
+ P(x_n, x_m, a, b) + P(x_{n+1}, x_{n+2}, a, b) \\
+ P(x_{n+2}, x_m, a, b) \\
\ll \frac{m}{k=n} q^k P(x_o, x_1, x_m, a) + \frac{m}{k=n} q^k P(x_o, x_1, x_m, b) \\
+ \frac{m}{k=n} q^k P(x_o, x_1, a, b)
\]

where \( q = \left( \frac{a}{1-a} \right) \).

\[
\ll 3k \frac{a^n}{(1-q)}
\]

Hence \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete, so

\( \{x_n\} \) converges to a point \( v \).

For this point \( v \).

\[
P(v, f(v), a, b) \ll P(v, f(v), a, b) + P(v, f(v), x_{m+1}, b) \\
+ P(v, x_{m+1}, a, b) + P(x_{m+1}, f(v), a, b) \\
\epsilon u + u + u + u.
\]
Hence \( P(v,f(v),a,b) \in u \). Thus \( v = f(v) \).

Now suppose \( w \) is another fixed point.

Then we have

\[
P(v,w,a,b) = P(f(v),f(w),a,b)
\]

\[
\ll a [P(v,f(v),a,b) + P(w,f(w),a,b)]
\]

\[
\ll a [P(v,v,a,b) + P(w,w,a,b)]
\]

\[
\ll 0
\]

which implies that \( v = w \).

This complete the proof.

Next, we generalized the results of Jaggi [53].

We prove the following theorems:

**THEOREM 2.** Let \( Y \) be a complete \( \gamma \)--metric space over a topological semifield \( E,f \) be a continuous mapping on \( X \) and for all distinct \( x,y,a,b \in X \) such that

\[
(6.2.2) \quad P(f(x),f(y),a,b) < a \frac{P(x,f(x),a,b) + P(y,f(y),a,b)}{P(x,y,a,b)}
\]

\[
+ \beta \ P(x,y,a,b)
\]

where \( a, \beta \) are positive number and \( a + \beta < 1 \). Then there is a fixed element \( x' \) of the mapping \( f \) such that \( f(x') = x' \).

**Proof:** Take an element \( x_0 \in X \), then by recursive we define a sequence \( \{x_n\} \) by

\[
x_{n+1} = f(x_n) \quad (n = 0,1,2, \ldots)
\]
Then, we have

\[ P(x_1, x_2, a, b) = P(f(x_o), f(x_1), a, b) \]

\[ \ll a \frac{P(x_o, x'_1, a, b) P(x_1, x'_2, a, b)}{P(x_o, x_1, a, b)} + \beta P(x_o, x'_1, a, b) \]

\[ \ll (\frac{\beta}{1-\alpha}) P(x_o, x'_1, a, b) \]

Again,

\[ P(x_2, x_3, a, b) = P(f(x_1), f(x_2), a, b) \]

\[ \ll a \frac{P(x_1, x_2, a, b) P(x_2, x_3, a, b)}{P(x_1, x_2, a, b)} + \beta P(x_1, x_2, a, b) \]

\[ \ll (\frac{\beta}{1-\alpha}) P(x_1, x_2, a, b) \]

\[ \ll (\frac{\beta}{1-\alpha})^2 P(x_1, x_2, a, b) \]

In general,

\[ P(x_n, x_{n+1}, a, b) \ll (\frac{\beta}{1-\alpha})^n P(x_o, x_1, a, b) \]

By inequality (6.1.3), we have for \( n \leq m \),

\[ P(x_n, x_m, a, b) \ll P(x_n, x_{n+1}, a, b) + d(x_{n+1}, x_{n+2}, a, b) \]

\[ + \ldots + d(x_{m-1}, x_m, a, b) \]

\[ \ll (q^n + q^{n+1} + \ldots + q^{m-1}) P(x_o, f(x_o), a, b) \]

where \( q = (\beta/1-\alpha) \)
\[ \frac{q^n}{1-q} P(x_0, f(x_0), a, b) \]

\[ \longrightarrow 0 \quad \text{if} \quad m \quad n \quad \longrightarrow \quad \infty \]

Since \( X \) is complete, there exists a \( v \in X \) such that \( x_n \longrightarrow v \). Further, the continuity of \( f \) in \( X \) implies

\[ f(v) = f(\lim_{n \rightarrow \infty} x_n) \]

\[ = \lim_{n \rightarrow \infty} f(x_n) \]

\[ = v. \]

Therefore \( v \) is a fixed point of \( f \) in \( X \). Now, if there exists another point \( v \neq w \) in \( X \) such that \( f(v) = v \), then

\[ P(v, w, a, b) = P(f(v), f(w), a, b) \]

\[ \ll a \frac{P(v, f(v), a, b)P(w, f(w), a, b)}{P(v, w, a, b)} \]

\[ + \beta P(v, w, a, b) \]

\[ = \beta P(v, w, a, b) \]

\[ \ll P(v, w, a, b) \]

a contradiction. Hence \( v \) is unique fixed point.

**Theorem 3.** Let \( f \) be a selfmaps defined on a complete \( 3 \)-metric space \( (X, P) \) over a topological semifield \( E \), such that (6.2.2) holds. If for some positive integer \( p \), \( f^p \) is continuous, then \( f \) has a unique fixed point.
**Proof:** Define a sequence \( \{x_n\} \) as in Theorem 2. Clearly, it converges to some point \( v \in X \). Therefore its subsequence \( \{x_{n_k}\} \), \( (n_k = k^p) \) also converges to \( v \).

Also

\[
f^p(v) = f^p(\lim_{k \to \infty} x_{n_k}) = \lim_{k \to \infty} x_{n_k + 1} = v.
\]

Therefore \( v \) is a fixed point of \( f^p \). We now show that \( f(v) = v \). Let \( m \) be the smallest positive integer such that \( f^m(v) = v \) but \( f^q(v) \neq v \) (\( q = 1, 2, \ldots, m-1 \)). If \( m > 1 \), then

\[
P(f(v), v, a, b) = P(f(v), f^m(v), a, b)
\]

\[
= \alpha \frac{P(v, f(v), a, b)P(f^{m-1}(v), f^m(v), a, b)}{P(v, f^{m-1}(v), a, b)} + \beta P(v, f^{m-1}(v), a, b)
\]

which implies that

\[
P(f(v), v, a, b) \ll \left( \frac{\beta}{1 - \alpha} \right) P(v, f^{m-1}(v), a, b)
\]

As

\[
P(v, f^{m-1}(v), a, b) = P(f^m(v), f^{m-1}(v), a, b)
\]

\[
= \alpha \frac{P(f^{m-1}(v), f^m(v), a, b)P(f^{m-2}(v), f^{m-1}(v), a, b)}{P(f^{m-1}(v), f^{m-2}(v), a, b)} + \beta P(f^{m-1}(v), f^{m-2}(v), a, b)
\]
it follows that

\[ P(v, f^{m-1}(v), a, b) = P(f^m(v), f^{m-1}(v), a, b) \]

\[ \ll (\frac{\beta}{1-a}) P(f^{m-1}(v), f^{m-2}(v), a, b) \]

\[ \vdots \]

\[ \vdots \]

\[ \ll (\frac{\beta}{1-a})^{m-1} P(f(v), v, a, b) \]

Therefore,

\[ P(f(v), v, a, b) \ll (\frac{\beta}{1-a})^m P(f(v), v, a, b) \]

\[ \ll P(f(v), v, a, b) \]

a contradiction. Hence \( f(v) = v \).

The unicity of \( v \) as a fixed point of \( f \) follows as in our theorem 1.

This complete the proof of the theorem.

**THEOREM 4.** Let \( f \) be a self map defined on a complete 3-metric space \((X, P)\) over a topological semifield \( E \), such that for some positive integer \( m \) \( f \) satisfy the condition

\[ (6.2.3) \quad P(f^m(x), f^m(y), a, b) \ll \frac{\alpha P(x, f^m(x), a, b)P(y, f^m(y), a, b)}{P(x, y, a, b)} + \beta P(x, y, a, b) \]

for all \( x, y, a, b \in X, x \neq y \) and for some \( \alpha, \beta \in [0, 1) \) with \( \alpha + \beta < 1 \). If \( f^m \) is continuous then \( f \) has a unique fixed point.
Proof:- That \( f^m \) has a unique fixed point \( v \) (say) in \( X \) follows from Theorem 2.

Also,

\[
\begin{align*}
f(v) &= f(f^m(v)) \\
&= f^m(f(v))
\end{align*}
\]

which implies that \( f(v) = v \). Further, same a fixed point of \( f \) is also a fixed point of \( f^m \) and \( f^m \) has a unique fixed point \( v \), it follows that \( v \) is a unique fixed point of \( f \).

This completes the proof of the theorem.

**Theorem 5.** Let \( f_1 \) and \( f_2 \) be two self-maps defined on a complete 3-metric space \((X, P)\) over a topological semifield \( E \), satisfying the condition:

\[ (6.2.4) \text{ For some } a, b \in [0, 1) \text{ with } a + b < 1 \]

\[
P(f_1(x), f_2(y), a, b) \leq a P(x, f_1(x), a, b) P(y, f_2(y), a, b) + b P(x, y, a, b)
\]

for all \( x, y \in X, \ x \neq y \)

\[ (6.2.5) \ \ f_1 f_2 \text{ is continuous on } X. \]

\[ (6.2.6) \text{ there exists an } x_0 \in X \text{ such that in the sequence } \{x_n\}, \text{ where} \]


\[ x_n = \begin{cases} 
  f_1(x_{n-1}), & \text{when } n \text{ is even} \\
  f_2(x_{n-1}), & \text{when } n \text{ is odd} 
\end{cases} \]

\[ x_n \neq x_{n+1} \text{ for all } n. \]

Then \( f_1 \) and \( f_2 \) have a unique common fixed point.

**Proof:** We have

\[
P(x_{2n}, x_{2n+1}, a, b) = P(f_1(x_{2n-1}), f_2(x_{2n}), a, b)
\]

\[
\preceq \alpha \frac{P(x_{2n-1}, f_1(x_{2n-1}), a, b)P(x_{2n}, f_2(x_{2n}), a, b)}{P(x_{2n-1}, x_{2n}, a, b)} + \beta P(x_{2n-1}, x_{2n}, a, b)
\]

which implies that

\[
P(x_{2n}, x_{2n+1}, a, b) \preceq \left( \frac{\beta}{1-\alpha} \right) P(x_{2n-1}, x_{2n}, a, b)
\]

\[
\preceq \left( \frac{\beta}{1-\alpha} \right)^2 P(x_0, x_1, a, b)
\]

Similarly, we can show that

\[
P(x_{2n+1}, x_{2n+2}, a, b) \preceq \left( \frac{\beta}{1-\alpha} \right)^{2n-1} P(x_0, x_1, a, b)
\]

Now it can be easily seen that \( \{x_n\} \) is a Cauchy sequence.

Let \( \{x_n\} \rightarrow v \), then the subsequence \( \{x_{n_k}\} \rightarrow v \), where

\[ n_k = 2k. \]

Now \( f_1f_2(v) = f_1f_2 \left( \lim_{k \to \infty} x_{n_k} \right) \]

\[ = \lim_{k \to \infty} x_{n_k+1} = v. \]
We now show that $f_2(v) = v$. If $f_2(v) \neq v$, then

$$P(f_2(v), v, a, b) = P(f_2(v), f_1f_2(v), a, b)$$

$$\leq \frac{\alpha P(v, f_2(v), a, b) P(f_2(v), f_1f_2(v), a, b)}{P(v, f_2(v), a, b)} + \beta P(v, f_2(v), a, b)$$

$$= (\alpha + \beta) P(v, f_2(v), a, b)$$

a contradiction, since $\alpha + \beta < 1$. Hence $f_2(v) \rightarrow v$. Also

$$P(f_1(v), v, a, b) = P(f_1(f_2(v)), f_2(v), a, b)$$

$$\leq \frac{\alpha P(f_2(v), f_1(f_2(v)), a, b)P(v, f_2(v), a, b)}{P(f_2(v), v, a, b)} + \beta P(f_2(v), v, a, b)$$

$$= 0,$$

which shows that $f_1(v) = v$.

If possible, let $w(\neq v) \in X$ be such that $f_1(w) \rightarrow w$.

Then,

$$P(w, v, a, b) = P(f_1(w), f_2(v), a, b)$$

$$\leq \frac{\alpha P(w, f_1(w), a, b)P(v, f_2(v), a, b)}{P(w, v, a, b)} + \beta P(w, v, a, b)$$

$$= \beta P(w, v, a, b)$$

$$\leq P(w, v, a, b),$$
a contradiction. Hence $v$ is a unique fixed point of $f_1$. It is easy to check that $v$ is also a unique fixed point of $f_2$. 

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