5.1. There are many generalizations of the classical contraction mapping theorem of S. Banach.

Suppose $X$ denote a Banach space with the norm $||.||$ and let $C$ be a closed subset of $X$. The transformation $F : C \rightarrow C$ is called contraction if there exists a constant $0 \leq \alpha < 1$ such that for arbitrary $x, y \in C$, the inequality

$$ (5.1.1) \quad ||F(x) - F(y)|| \leq \alpha ||x - y||,$$

is true. It is called nonexpansive if the same condition with $\alpha = 1$ holds. By Banach contraction principle each contraction of $C$ has exactly one fixed point. The same is true if we assume that only some powers of $F$ are contractions, but it is not true for expansive mappings. However, Browder[1] has proved that every nonexpansive mapping of a closed bounded convex subset of a uniformly convex Banach space has at least one fixed point. Kirk [1] proved similar theorems in the space with normal structure. Goebel [1] has given a simple proof of the above result of Browder and Kirk.

It is natural to raise the question whether these results can be extended to the case of transformation with a nonexpansive iteration. The answer is in general negative.
However, Klee [1] showed that even in Hilbert space some convex sets admit continuous transformation without fixed points and even such that \( F^2 = I \) where \( I \) denote identity mapping. With this object in view Goebel and Zlotkiewicz[1] has proved a theorem which generalizes the above mentioned result of Browder. In fact they proved:

**Theorem A.** If \( C \) is a closed and convex subset of \( B \) and if \( F : C \to C \) satisfies

\[
(5.1.2) \quad F^2 = I, \quad \text{where} \quad I \quad \text{is the identity mapping.}
\]

\[
(5.1.3) \quad \| F(x) - F(y) \| \leq k \| x - y \| \quad \forall \quad x, y \in C,
\]

where \( 0 \leq k < 2 \), then \( F \) has at least one fixed point.

Further Khan [1] generalizes the above result for two mapping by proving the following theorem:

**Theorem B.** Let \( K \) be a closed and convex subset of a Banach space \( X \). Let \( F : K \to K \), \( G : K \to K \) satisfy the following conditions:

(1) \( F \) and \( G \) commute,

(11) \( F^2 = I, \ G^2 = I \), where \( I \) denotes the identity mapping

(111) \( \| F(x) - F(y) \| \leq \alpha \| G(x) - G(y) \| \),

for every \( x, y \in K \) and \( 0 \leq \alpha < 2 \). Then there exists at least one point \( x_0 \in K \) such that \( F(x_0) = G(x_0) \). Further, if \( \alpha \leq \alpha < 1 \), then \( x_0 \) is unique and \( x_0 = F(x_0) = G(x_0) \).
5.2. We shall prove:

**Theorem 1**: Let $F$ be a mapping of a Banach space $X$ into itself. If $F$ satisfies conditions:

1. $F^2 = I$, where $I$ is the identity mapping.
2. $||F(x) - F(y)|| \leq \alpha ||x - y|| + \beta \frac{||x - F(x)|| \cdot ||y - F(y)||}{||x - y||}$

for every $x, y \in X$, where $0 \leq \alpha + 4\beta < 2$, then $F$ has at least one fixed point.

**Proof**: Let $x \in X$ and let

$$y = \frac{1}{2} (F + I)(x), z = F(y), 4 = 2y - z.$$ 

Then by condition (1) and (2), we have

$$||z - x|| = ||F(y) - x|| = ||F(y) - F^2(x)||$$

$$\leq \alpha ||y - F(x)|| + \beta \frac{||y - F(y)|| \cdot ||F(x) - F^2(x)||}{||x - y||}$$

$$\leq \alpha ||y - F(x)|| + \beta \frac{||y - F(y)|| \cdot ||x - F(x)||}{||x - y||}$$

$$\leq \alpha \left| \frac{1}{2} (F + I)(x) - F(x) \right|$$

$$+ \beta \frac{||y - F(y)|| \cdot ||x - F(x)||}{||x - \frac{1}{2} (F + I)(x)||}$$

$$\leq \frac{\alpha}{2} ||x - F(x)|| + 2\beta ||y - F(y)||$$
and

\[ ||u-x|| = ||2y-F(y)-x|| \]

\[ = ||(F+I)(x) - F(y) + x|| \]

\[ = ||F(x) - F(y)|| \]

\[ \leq \alpha ||x-y|| + \beta \frac{||x-F(x)|| \cdot ||y-F(y)||}{||x-y||} \]

\[ \leq \frac{\alpha}{2} ||x-F(x)|| + \beta \frac{||x-F(x)|| \cdot ||y-F(y)||}{||x-F(x)||} \]

\[ \leq \frac{\alpha}{2} ||x-F(x)|| + 2\beta ||y-F(y)||. \]

Hence

\[ ||z-u|| \leq \alpha ||x-F(x)|| + 4\beta ||y-F(y)||. \]

On the other hand, we have

\[ ||z-u|| = 2||y-F(y)||. \]

Therefore,

\[ 2||y-F(y)|| \leq \alpha ||x-F(x)|| + 4\beta ||y-F(y)|| \]

which implies

\[ (2-4\beta)||y-F(y)|| \leq \alpha ||x-F(x)|| \]

Hence,

\[ ||y-F(y)|| \leq \frac{\alpha}{2-4\beta} ||x-F(x)||. \]
Let

\[ G = \frac{1}{2} (F + I), \text{ then for any } x \in X \]

\[ ||G^2(x) - G(x)|| = ||G(y) - y|| \]

\[ = ||\frac{1}{2} (F + I)(y) - y|| \]

\[ = \frac{1}{2} ||y - F(y)|| \]

\[ \leq \frac{1}{2} \cdot \frac{\alpha}{(2 - 4\beta)} ||x - F(x)|| \]

\[ \leq \frac{1}{2} \cdot \frac{\alpha}{(2 - 4\beta)} ||x - 2G(x) - x|| \]

\[ \leq \frac{\alpha}{(2 - 4\beta)} ||G(x) - x||. \]

By the hypothesis, we have \( 0 \leq \frac{\alpha}{2 - 4\beta} < 1 \).

Therefore,

\( G^n(x) \) is a Cauchy sequence in \( X \). By the completeness, \( G^n(x) \) converges to some element \( x_0 \) in \( X \). This implies \( G(x_0) = x_0 \).

Hence \( F(x_0) = x_0 \), which completes the proof of the theorem.

**Remark:**

Taking \( \beta = 0 \), we obtain theorem \( [A] \) of Goebel and Zbiokiewicz \( [1] \).
THEOREM [2]: Let \( K \) be a closed and convex subset of a Banach space \( X \). Let \( F : K \to K \), \( G : K \to K \) satisfy the following conditions:

1. \( F \) and \( G \) commute,

2. \( F^2 = I \) and \( G^2 = I \), where \( I \) denotes the identity mapping,

3. \[
\|F(x) - F(y)\| \leq \alpha \|G(x) - G(y)\| + \frac{\|G(x) - F(x)\| \cdot \|G(y) - F(y)\|}{\|G(x) - G(y)\|},
\]

for every \( x, y \in K \) and \( 0 \leq \alpha + 4\beta < 2 \). Then there exists at least one point \( x_0 \in K \) such that \( F(x_0) = G(x_0) \). Further if \( 0 \leq \alpha < 1 \), then \( x_0 \) is unique fixed point of \( F \) and \( G \).

PROOF: From (1) and (2) it follows that \( (FG)^2 = I \) and by (2) and (3), we have

\[
\|FG.G(x) - FG.G(y)\| \leq \alpha \|G_2(x) - G_2(y)\| + \beta \frac{\|G_2(x) - FG^2(x)\| \cdot \|G_2(y) - FG^2(y)\|}{\|G_2(x) - G_2(y)\|},
\]

\[
\leq \alpha \|G(x) - G(y)\| + \beta \frac{\|G_2(x) - FG(x)\| \cdot \|G(y) - FG(y)\|}{\|G(x) - G(y)\|}.
\]
Now if we put \( G(x) = z \) and \( G(y) = w \), we have

\[
\frac{||z - FG(z)|| \cdot ||w - FG(w)||}{||z - w||} \leq \alpha \frac{||z - FG(z)||}{||z - w||} + \beta \frac{||w - FG(w)||}{||z - w||},
\]

when \((FG)^2 = I\) and \(\alpha + \beta < 2\). So by theorem of Iseki [4] \(FG\) has at least one fixed point, say \(x_0 \in K\).

Thus

(4) \hspace{1cm} FG(x_0) = x_0

and so

(5) \hspace{1cm} FFG(x_0) = F(x_0) \text{ or } G(x_0) = F(x_0).

Now

\[
||F(x_0) - x_0|| = ||F(x_0) - F^2(x_0)|| = ||F(x_0) - F(F(x_0))||
\]

\[
\leq \alpha ||G(x_0) - GF(x_0)||
\]

\[
+ \beta \frac{||G(x_0) - F(x_0)|| \cdot ||GF(x_0) - FF(x_0)||}{||G(x_0) - GF(x_0)||}
\]

\[
\leq \alpha \frac{||G(x_0) - x_0||}{||G(x_0) - GF(x_0)||}
\]

\[
+ \beta \frac{||G(x_0) - F(x_0)|| \cdot ||GF(x_0) - FF(x_0)||}{||G(x_0) - GF(x_0)||} < \alpha \frac{||F(x_0) - x_0||}{||G(x_0) - GF(x_0)||}
\]
Since $a < 1$, it follows $F(x_0) = x_0$, i.e., $x_0$ is the fixed point of $F$. But $F(x_0) = G(x_0)$ and so we have $G(x_0) = x_0$. Hence $x_0$ is the common fixed point of $F$ and $G$.

Now to show that $x_0$ is unique common fixed point of $F$ and $G$, let us consider $y_0$ be another common fixed point of $F$ and $G$. By using (1), (2), (3), (4) and (5), we have

$$||x_0 - y_0|| = ||F^2(x_0) - F^2(y_0)|| = ||FF(x_0) - FF(y_0)||$$

$$\leq \alpha ||GF(x_0) - GF(y_0)||$$

$$+ \beta \frac{||GF(x_0) - FF(x_0)|| \cdot ||GF(y_0) - FF(y_0)||}{||GF(x_0) - GF(y_0)||}$$

$$\leq \alpha ||x_0 - y_0|| .$$

Since $a < 1$, it follows $x_0 = y_0$. Hence $x_0$ is the unique common fixed point of $F$ and $G$. This completes the proof of the theorem.

**Remarks**

[1] Taking $G = I$, we obtain theorem [1].

[2] Taking $G = I$ and $\beta = 0$, we obtain theorem [A].

[3] Taking $\beta = 0$, we obtain theorem [B].
THEOREM [3]: Let $K$ be a closed and convex subset of a Banach space $X$. Let $F$, $G$, and $H$ are three mappings of $K$ into itself such that:

1. $FG = GF$, $GH = HG$ and $FH = HF$,
2. $F^2 = I$, $G^2 = I$ and $H^2 = I$, where $I$ denotes the identity mapping,
3. $||F(x) - F(y)|| \leq \alpha ||GH(x) - GH(y)||$

\[ + \beta \frac{||GH(x) - F(x)|| \cdot ||GH(y) - F(y)||}{||GH(x) - GH(y)||} \]

for every $x, y \in K$ and $0 \leq \alpha, \beta \leq 2$. Then there exists at least one fixed point $x_0 \in K$ such that $F(x_0) = GH(x_0) = F(x_0) = H(x_0)$. Further if $0 \leq \alpha < 1$, then $x_0$ is the unique common fixed point of $F$, $G$ and $H$.

PROOF: From (1) and (2) it follows that $(FGH)^2 = I$ and by (2) and (3), we have

\[ ||FGH \cdot G(x) - FGH \cdot G(y)|| \leq \alpha ||(GH)^2 \cdot G(x) - (GH)^2 \cdot G(y)|| \]

\[ + \beta \frac{||(GH)^2 \cdot G(x) - FGH \cdot G(x)|| \cdot ||(GH)^2 \cdot G(y) - FGH \cdot G(y)||}{||GH^2 \cdot G(x) - (GH)^2 \cdot G(y)||} \]
Now if we put $G(x) = z$ and $G(y) = w$, we have

$$
||F G H(z) - F G H(w)|| \leq \alpha \frac{||z - F G H(z)|| + \beta}{||z - w||}
$$

when $(F G H)^2 = 1$ and $\alpha + 4\beta < 2$. So by theorem of Iseki[1] FGH has at least one fixed point, say $x_0$ in $K$.

Thus,

(4) $F G H(x_0) = x_0$.

and so

(5) $G H(F G H)(x_0) = G H(x_0)$ or $F(x_0) = G H(x_0)$.

also,

(6) $H(F G H)(x_0) = H(x_0)$ or $F G(x_0) = H(x_0)$.

Now using (1), (2), (3), (4), (5) and (6), we have

$$
||H(x_0) - x_0|| = ||F G(x_0) - F^2(x_0)|| = ||F(G(x_0)) - F(F(x_0))||
$$

$$
\leq \alpha ||G H G(x_0) - G H F(x_0)||
$$

$$
+ \beta \frac{||G H G(x_0) - F G(x_0)|| + ||G H F(x_0) - F F(x_0)||}{||G H G(x_0) - G H F(x_0)||}
$$

$$
\leq \alpha ||H(x_0) - x_0|| + \beta \frac{||H(x_0) - H(x_0)|| + ||x_0 - x_0||}{||H(x_0) - x_0||}
$$

$$
\leq \alpha ||H(x_0) - x_0||.
$$
Since $\alpha < 1$, it follows $H(x_0) = x_0$. Hence $x_0$ is the fixed point of $H$. Thus we have by (5) that

$$G(x_0) = F(x_0)$$

Again,

$$||F(x_0) - x_0|| = ||F(x_o) - F^2(x_o)|| = ||F(x_o) - F(F(x_o))||$$

$$\leq \alpha ||GH(x_0) - GH_F(x_0)||$$

$$+ \beta \frac{||GH(x_0) - F(x_0)|| + ||F(x_0) - F^2(x_0)||}{||x_0 - F(x_0)||}$$

$$\leq \alpha ||F(x_0) - x_0|| + \beta \frac{||F(x_0) - F(x_0)|| + ||F(x_0) - x_0||}{||x_0 - F(x_0)||}$$

$$\leq \alpha ||F(x_0) - x_0|| + \beta \frac{||F(x_0) - x_0||}{||x_0 - F(x_0)||}$$

This leads to a contradiction, since $\alpha < 1$. Hence it follows that $F(x_0) = x_0$. But $F(x_0) = G(x_0)$. So we have $F(x_0) = G(x_0) = H(x_0) = x_0$. Hence $x_0$ is the common fixed point of $F$, $G$ and $H$.

In order to prove the uniqueness of $x_0$, let us consider $y_0$ be another common fixed point of $F$, $G$ and $H$. Now by using (1), (2), (3), (4) and (5), we have
\[ \| x_o - y_o \| \leq \| F^2(x_o) - F^2(y_o) \| \leq \| F \circ F(x_o) \circ F(y_o) \| \]

\[ \leq \alpha \| GH \circ F(x_o) - GH \circ F(y_o) \| \]

\[ + \beta \frac{\| GH \circ F(x_o) - F \circ F(x_o) \| \cdot \| GH \circ F(y_o) - F \circ F(y_o) \|}{\| GH \circ F(x_o) - GH \circ F(y_o) \|} \]

\[ \leq \alpha \| x_o - y_o \| + \beta \frac{\| x_o - x_o \| \cdot \| y_o - y_o \|}{\| x_o - y_o \|} \]

\[ \leq \alpha \| x_o - y_o \| \]

Since \( \alpha < 1 \), it follows \( x_o = y_o \), proving the uniqueness of \( x_o \). This completes the proof of the theorem.

**Remarks**

1. Taking \( H = I, \beta = 0 \), we obtain the theorem [B].

2. Taking \( H = G = I \), we obtain the theorem [1].

3. Taking \( H = I \), we obtain theorem [2].