CHAPTER I

INTRODUCTION

1.1 The present thesis is devoted to the study of "SOME PROBLEMS OF FIXED POINT THEOREM". In this introductory chapter we give a resume of the important researches done so far which forms the background of the subject matter of the thesis.

Fixed point theory is an important branch of Functional Analysis as well as of Topology. This theory has an extensive application in proving the existence and uniqueness of the solution of differential equations, partial differential equations, integral equations and random differential equations. The study of this theory has also given several fruitful applications in Eigen-value problems, boundary value problems, theory of games and theory of non-linear oscillations etc. For these applications we refer to Kolmogorov and Fomin [1], Smart [1], Szegedely [1] and Swaminathan [1].

1.2 PRELIMINARY CONCEPTS

[a] METRIC SPACE-

DEFINITION [1] Let $X$ be a nonempty set and $d$ be a function from $XX$ into $R^+$ such that for every $x,y,z \in X$, we have
(i) \( d(x, y) \geq 0 \)
(ii) \( d(x, y) = 0 \) iff \( x = y \)
(iii) \( d(x, y) = d(y, x) \) and
(iv) \( d(x, z) \leq d(x, y) + d(y, z) \).

Then \( d \) is called a metric or the distance function and the pair \( (X, d) \) is called the metric space.

**Definition [2]** - A sequence \( \{x_n\} \) of points of a metric space is said to converge to a point \( z \) if for given \( \varepsilon > 0 \), there exists a positive integer \( N_\varepsilon \) such that

\[
d(x_n, z) < \varepsilon \text{ for } n \geq N_\varepsilon.
\]

**Definition [3]** - A sequence \( \{x_n\} \) is said to be a Cauchy sequence if for each \( \varepsilon > 0 \), there exists a positive integer \( N_\varepsilon \) such that

\[
d(x_n, x_m) < \varepsilon \text{ for all } n, m \geq N_\varepsilon.
\]

A convergent sequence is always Cauchy but the converse is not necessarily true.

**Definition [4]** - A metric space \( (X, d) \) is said to be complete if every Cauchy sequence is convergent in \( X \).

**Definition [5]** - A metric space \( (X, d) \) is said to be compact if every open cover of \( X \) contains a finite sub cover.

[b] **Topological Spaces**

**Definition [1]** - Let \( X \) be a nonempty set and let \( T \) be a
collection of subsets of X. The collection \( \mathcal{T} \) is said
to be a topology for X if \( \mathcal{T} \) satisfies each of the following three conditions:

(i) \( X \in \mathcal{T} \) and \( \emptyset \in \mathcal{T} \).

(ii) If \( G_\alpha \in \mathcal{T} \) for \( \alpha \in \Lambda \), where \( \Lambda \) is an index set.

Then

\[ \bigcup \{ G_\alpha : \alpha \in \Gamma \} \in \mathcal{T} \quad \text{and} \quad \bigcap \{ G_\alpha : \alpha \in \Gamma \} \in \mathcal{T} \]

(iii) If \( G_1, G_2, \ldots, G_n \) are members of \( \mathcal{T} \),

then \( \bigcap\{ G_i : i = 1,2, \ldots, n \} \in \mathcal{T} \).

The pair \((X, \mathcal{T})\) is called a topological space.

**Definition** [2] - Let \((X, \mathcal{T}_1)\) and \((Y, \mathcal{T}_2)\) be two topological spaces and let \( \mathcal{E} \) be a mapping of \( X \) into \( Y \). The mapping \( \mathcal{E} \) is said to be **continuous** if \( \mathcal{E}^{-1}(G) \) is \( \mathcal{T}_1 \)-open whenever \( G \) is \( \mathcal{T}_2 \)-open.

**Definition** [3] - A topological space \((X, \mathcal{T})\) is said to be **compact** if every open covering of \( X \) has a finite sub cover.

**Definition** [4] - A topological space \((X, \mathcal{T})\) is said to be **pseudo-compact** if every real valued continuous function on \( X \) is bounded.
**DEFINITION [3]** - A topological space \((X, d)\) is said to be a **Hausdorff space** if every pair of distinct points have disjoint neighbourhoods.

It is noted that every metric space is a Hausdorff space [Banu [1]].

**DEFINITION [6]** - A topological space \((X, \mathcal{T})\) is said to be **completely regular** if and only if for every closed subset \(F\) of \(X\) and every point \(x \in X - F\), there exists a continuous mapping \(E\) of \(X\) into the subspace \([0, 1]\) of \(\mathbb{R}\) such that \(E(x) = 0\) and \(E(F) = \{1\}\).

**DEFINITION [7]** - A completely regular Hausdorff space is a **Tichonov space**.

It may be remarked that the product of two Tichonov spaces is again a Tichonov space whereas the product of two pseudo-compact spaces need not be pseudo-compact.

**[C] SOME MAPPINGS**

**DEFINITION [1]** A mapping \(E\) of a metric space \(X\) into itself is called a **contraction** if

\[
d(E(x), E(y)) \leq k \, d(x, y) \quad \text{for all} \quad x, y \in X \quad \text{and} \quad 0 \leq k < 1.
\]

It is noted that a contraction mapping is continuous mapping but continuous mapping is not necessarily contraction.
DEFINITION-[2]- A mapping $E$ of a metric space $X$ into itself is called **contractive** if

$$d(Ex, Ey) < d(x, y) \text{ for } x, y \in X \text{ and } x \neq y.$$ 

It is noted that contractive mapping is continuous. Such mappings are more general than the contraction mappings.

[d] **RANACH SPACES**

DEFINITION-[1] A **linear space** or vector space is an additive abelian group with the property that any scalar $\alpha$ and any vector $x$ can be combined by an operation called scalar multiplication to yield a vector $\alpha x$ in such a manner that

(i) $\alpha(x+y) = \alpha x + \alpha y$
(ii) $(\alpha+\beta)x = \alpha x + \beta x$
(iii) $(\alpha\beta)x = \alpha(\beta x)$
(iv) $1.x = x$

A linear space is called a real linear space or a complex linear space according as the scalars are the real number or the complex numbers.

DEFINITION-[2]- A **normed linear space** is a linear space $X$ in which to each vector $x$ there corresponds a real number, denoted by $||x||$, are called the norm of $x$, in such a manner that

(i) $||x|| \geq 0$ and $||x||=0 \iff x = 0$;
(ii) $||x+y|| \leq ||x|| + ||y||$;
(iii) $||\alpha x|| = |\alpha| ||x||$. 
Every normed linear space $X$ is a metric space with respect to the metric $d$ defined by $d(x,y) = ||x-y||$.

**Definition [3]** A complete normed linear space is called a Banach space.

**Definition [4]** A set $C$ in a normed linear space $X$ is said to be **convex** if

$$ax + (1-a)y \in C$$

whenever $x, y \in C$ and $0 \leq a \leq 1$.

1.3 In 1912 Brouwer's [1] asserts that existence of fixed point whenever a closed unit ball $B$ is in Euclidean $n$-dimensional space $\mathbb{R}^n$ and mapping $E$ is continuous. The Brouwer's fixed point theorem remains true, if we replace $B \subset \mathbb{R}^n$ by any topological space $X$ homeomorphic with $B$.

Brouwer's theorem was extended to infinite dimensional spaces by Schauder [1] in 1927 as every continuous mapping $E$ of a compact convex subset $C$ of Banach space $X$ has at least one fixed point in $C$. Meanwhile in 1922, a Polish Mathematician, Banach [1] proved:

If $E$ is a mapping of a complete metric space $(X,d)$ into itself and satisfies:

$$d(Ex, Ey) \leq kd(x,y)$$

for all $x, y \in X$ and for some $0 \leq k < 1$, then $E$ has a unique fixed point.

A mapping satisfying (1.3.1) is known as contraction mapping. It is easy to see that contraction mapping is continuous.
however, a continuous mapping is not necessarily a contraction. For example, a translation map $T: \mathbb{R} \rightarrow \mathbb{R}$ given by $Tx = x + p$, $p > 0$ is continuous but not contraction. A contraction mapping of a complete metric space to itself always has a unique fixed point.

Banach contraction principle is the foundation stone over which the whole bulk of results on fixed point rests.

In 1965, Chu and Diaz [1] generalized this principle by taking $\mathbb{E}^n$ and not $E$ to be a contraction for some positive integer $n$, then $E$ has a unique fixed point.

The result due to Chu and Diaz [1] has been further extended by Schgal [1] in the following way:

Let $X$ be a complete metric space and $E : X \rightarrow X$ be a continuous mapping satisfying the condition that there exists a number $k < 1$ such that for each $x \in X$ there is a positive integer $n = n(x)$ such that

$$(1.3.2) \quad d(E^n x, E^n y) \leq k d(x, y), \quad \text{for all } y \in X.$$

Then $E$ has a unique fixed point.

A further generalization was given by Holmes [1] in 1969. He proved:

If $E : X \rightarrow X$ is a continuous function on a complete metric space $X$ and if for each $x, y \in X$ there exists $n = n(x, y)$ such that

$$(1.3.3) \quad d(E^n x, E^n y) < kd(x, y).$$
Then $E$ has a unique fixed point, For further generalization along this type of work we refer to Guzman [1], Khazanchi [1], Iseki [1], Ciric [2], Sharma [1] and Ray and Rhoades [1].

Rakotch [1] generalizes Banach contraction principle by replacing the Lipschitz constant $k$ by some real valued function whose values are less than 1. He defined a family $f$ of functions $d(x, y)$ where $d(x, y) = \alpha(d(x, y))$, $0 < \alpha(d) < 1$ for $d > 0$ and $\alpha(d)$ is a monotonically decreasing function of $d$ and proved:

If $E: X \rightarrow X$ is a mapping on a complete metric space $X$ into itself such that

$$(1.3.4) \quad d(Ex, Ey) < \alpha(d(x, y))$$

for all $x, y \in X$, then $E$ has a unique fixed point. The work on this line further extended by Boyd and Wong [1] and Browder[1].

In 1968, Kannan [1] investigated a result for mapping which is not necessarily continuous and yet gives a unique fixed point. He proved:

If $E$ is a mapping of a complete metric space $X$ into itself such that

$$(1.3.5) \quad d(Ex, Ey) \leq \alpha[d(x, Ex) + d(y, Ey)]$$

for all $x, y \in X$, where $0 \leq \alpha < 1/2$, then $E$ has a unique fixed point.
Reich [2] unified the result of Banach contraction theorem and theorem of Kannan [1] by considering a self mapping E of a complete metric space X satisfying the following condition:

\[(1.3.6) \quad d(Ex, Ey) \leq \alpha d(x, Ex) + \beta d(y, Ey) + \gamma d(x, y)\]

Where \(\alpha, \beta, \gamma\) are non-negative constants with \(0 \leq \alpha + \beta + \gamma < 1\) and for all \(x, y\) in \(X\). Then \(E\) has a unique fixed point in \(X\).

Chatterjea [1] studied a mapping \(E : X \rightarrow X\) of a complete metric space \((x, d)\) satisfying:

\[(1.3.7) \quad d(Ex, Ey) \leq \alpha [d(x, Ey) + d(y, Ex)]\]

for all \(x, y\) in \(X\), where \(0 \leq \alpha < 1/2\) and obtained a unique fixed point of \(E\).

Hardy and Rogers [1] unified the mapping of Reich [1] and Chatterjea [1] and studied a mapping \(E : X \rightarrow X\) of a complete metric space \(X\) satisfying:

\[(1.3.8) \quad d(Ex, Ey) \leq \alpha_1 d(x, Ex) + \alpha_2 d(y, Ey) + \alpha_3 d(x, Ey) + \alpha_4 d(y, Ex) + \alpha_5 d(x, y),\]

for all \(x, y\) in \(X\) with \(\alpha_1 > 0\) for \(i = 1, 2, 3, 4, 5\) and \(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1\). This work was further extended in various direction by C lick [2], Sharma [1], I seki [1], Singh and Meade [1], Sehgal and Hussain [1] and others.
Kannan [2] again investigated mainly under what condition two mappings, each mapping of complete metric space $X$ into itself, have a common fixed point and proved:

If $E$ and $F$ are two mappings of a complete metric space $X$ into itself and satisfying

$$d(Ex, Fy) \leq \alpha [d(x, Ex) + d(y, Fy)]$$

for all $x, y \in X$ and $0 \leq \alpha < \frac{1}{2}$ the $E$ and $F$ have a common fixed point.

In 1973, C.S. Wong [1] proved a common fixed point theorem of two mappings $E$ and $F$ which are not necessarily continuous nor commuting where

$$d(Ex, Ey) \leq \alpha_1 d(x, Ex) + \alpha_2 d(y, Fy) + \alpha_3 d(x, Fy) + \alpha_4 d(y, Ex) + \alpha_5 d(x, y),$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$ are non-negative real numbers such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ and $\alpha_1 = \alpha_2$ or $\alpha_3 = \alpha_4$.

This result generalizes the Banach contraction principle and some results of Hardy and Rogers [1] Kannan [2], Reich [1], Chatterjea [1] and Gupta and Shrivastava [1].

Recently Jaggi [1] established the following result:

Let $E$ be a continuous self map defined on a complete metric space $(X, d)$. Further, let $E$ satisfy the following condition.
(1.3.11) \( d(Ex, Ey) \leq \alpha \frac{d(x, Ex) d(y, Ey)}{d(x, y)} + \beta d(x, y) \)

for all \( x, y \in X \), \( x \neq y \) and for some \( \alpha, \beta \in [0, 1] \) with \( \alpha + \beta < 1 \). Then \( E \) has a unique fixed point in \( X \).

This result was further extended by Dass [1].

Letter on Jaggi [2] obtained some fixed point theorems in orbitally complete metric space which include his previous result. Ciric' wrote a series of papers on orbitally continuous functions. In a paper [3] he proved the following:

(1.3.12) \( \min \{ d(Ex, Ey), d(x, Ex), d(x, Ey) \} \)

\[ \leq \min \{ d(x, Ey), d(y, Ex) \} \leq q d(x, y) \]

for all \( x, y \in X \) and some \( q \in (0, 1) \). Then for each \( x \in X \), the sequence \( \{Ex_n\} \) converges to a fixed point of \( E \).

Taskovic [1] obtained a generalization of the result of Ciric [1]. He replace condition (1.3.12) by

(1.3.13) \( \alpha_1 d(Ex, Ey) + \alpha_2 d(x, Ex) + \alpha_3 d(y, Ey) \)

\[ + \alpha_4 \min \{ d(x, Ey), d(y, Ex) \} \leq \beta d(x, y) \]

for all \( x, y \in X \), where \( \alpha_1 \) and \( \beta \) are real numbers with \( \alpha_1 + \alpha_2 + \alpha_3 > \beta \) and \( \beta - \alpha_2 > 0 \). The fixed point theorems on orbitally continuous mappings have also been studied by
Pechpatte [1], [2], Taskovic [1], Pal and Maity [1], [2], Achari [1], [2], Yeh [1] Jain and Dixit [3], Bajaj [1] and Dhage and Dhobale [1] and others.

Now another important generalization of Banach contraction principle was obtained by Edelstein [1] in 1962, given as.

A self mapping of a metric space \((x, d)\) is said to be contractive if for all \(x, y\) in \(X\), \(x \neq y\), we have

\[
(1.3.14) \quad d(Ex, Ey) < d(x, y)
\]

A contractive mapping is clearly continuous. Such mappings are more general than contraction mapping. Completeness of the space is not enough to ensure the existence of fixed points for such mappings. Hence, we require further restriction on the spaces or extra conditions on the mappings or on its range. Edelstein [1] proved that if \(E\) is contractive mapping of \(X\) into itself such that sequence \(E^{n}(x)\) of iterates of \(E\) at some \(x_0 \in X\) has a limit point \(z\), then \(z\) is the unique fixed point of \(E\). Singh [1] proved analogous result for Kannan type mapping (1.3.5) and showed that this result is independent of (1.3.13). Many more results pertaining to contractive mappings or its further generalizations have been obtained by Bailey [1], Byrant [1], Guseman and Peters [1], Senegal [1], Cirić [4], Jaggi and Sharma [1] and others.
There is yet another class of mappings which is known as the class of non-expansive mappings. A mapping \( E : X \rightarrow X \) is said to be non-expansive if

\[(1.3.15) \quad d(Ex, Ey) \leq d(x, y),\]

for all \( x, y \) in \( X \). It is remarked here that non-expansive mappings are more general than contractive mappings. This class of mappings includes contraction and contractive mappings. It may be observed that non-expansive mappings, mappings, can have more than one fixed point and the existence of fixed point does not assure it's uniqueness. It was Edelstein [2] again, who gave the first result on fixed points for non-expansive mappings in the year 1961. However, the result for a general type of non-expansive mappings in non-compact metric space was proved independently by Browder [1], Gohde [1] and Kirk [1] in 1965. Cheney and Goldstein [1] have proved a result which was further generalized by Singh and Zorziitio [1]. This class of mappings have further studied by Belluce, Kirk and Steiner [1], Browder and Petryshyn [1] and Kirk [2] and many others.

One of the recent direction in which fixed point theorems are being studied is pseudo-compact Tichonov space. Harinath [1], Jain and Dixit [1] and Pathak [1] have obtained some results in this field. These results include some of the well known results obtained in compact metric space by Fisher [1].
1.4 COMMON FIXED POINTS FOR A PAIR OF MAPPINGS.

A point \( z \in X \) is said to be a common fixed point of a pair of self mappings \( E \) and \( F \) if \( Ez = z = Fz \). A number of mathematicians conjectured that two continuous mappings on \([0,1]\) have a common fixed point. In the year 1969, Boyce [1] and Huang [1] showed that their conjecture is false. Many workers tried to seek for the sufficient condition on a pair of mappings which guarantees the existence of a common fixed point and its uniqueness. The problem of finding a sufficient condition for the existence of a unique common fixed point for a pair of mappings, each defined on a complete metric space was first taken by Kannan [2] in 1968.

A great deal of work on common fixed point has been done by Iseki [2], Yen [1] and others. Rhoades [1], Cirić [2] obtained common fixed points for the most general type of self mapping of a metric space \( X \). Maiti and Pal [2], [3] generalized the idea of Cirić [3]. These results are further generalized by Rhoades [2].

1.5 ORBITALLY CONTINUOUS MAPPING

Let \((M, d)\) be a metric space. A mapping \( T \) on \( M \) is said to be orbitally continuous if \( \lim_{n \to \infty} T^n x = u \) implies \( \lim_{n \to \infty} T^n x = u \) for each \( x \in M \). A space \( M \) is said to be
T-orbitally continuous if every Cauchy sequence of the form 
\[ \{T^n x\}_{i=1}^{\infty}, \quad x \in M \] converges in M. Achari [1], Ciric [3], 
Pachpatte [1] and others have done a lot of work on orbitally 
continuous mapping. It is remarked here that the notion of 
orbital continuity is very useful in proving the existence of 
fixed points of mapping T. The orbitally continuous mapping 
in 2-metric space is introduced by Sharma [2].

1.6 COMMON FIXED POINT OF THREE MAPPINGS

If E, F and T are three self mappings of a metric 
space \((X, d)\) for any \(x \in X\) and if

\[ (1.6.1) \quad Ex = Fx = Tx = x, \]

then \(x\) is said to be common fixed point of \(E, F\) and \(T\).

If \(E\) and \(F\) are the commuting mappings of a metric 
space then \(EF = FE\). In 1976, Jungck [1] proved a common fixed 
point theorem for continuous and commuting self mappings of 
a complete metric space. In 1979, Yeh [2] extended the result 
of Jungck [1] and obtained a unique common fixed point of 
these continuous self mappings of a complete metric space. 
Further Singh and Singh [1], Yeh [3] and Khan and Fisher [1] 
proved some more results on this type of work.

1.7 Further Chapter II consists of two sections. In 
section 1, we obtain a theorem for complete metric space and 
mapping \(T\) satisfying the inequality:
\((1.71) \quad [d(Tx, Ty)]^{p+1} \leq \alpha [d(Tx, Ty)]^{p+2} \quad (Tx, Ty) \quad p+1 \quad p+2
\]
\[+ \beta \quad d(Ty, Tx) \quad d(Ty, Ty) \quad p+1 \quad p+2\]

for all \(x, y\) in \(X\), \(p\) be a non-negative integer and \(\alpha, \beta\) are non-negative constants such that \(0 \leq \alpha + \beta < 1\). Then \(T\) has a unique fixed point in \(X\) if either \(p = 0\) or \(T\) is continuous.

In section 2, we prove some fixed point theorems for metric spaces and we remark that the results of Fisher [4] appear as particular case of our theorems.

Chapter III is devoted to the study of fixed point theorems for orbitally continuous mappings in metric spaces. In section 1, following the technique of Tešković [1], we obtain two results. Our first result generalizes a theorem of Bajaj [1] which was proved for Cirić [3] type mapping and second result generalizes a theorem of Pachpatte [1], which was also proved for Cirić [3] type mapping. The results of section 2 deal with localized versions of our previous results of section 1. In section 3 we discuss sequence of orbitally continuous mappings on compact metric space.

In chapter IV, several results on fixed points in pseudo-compact Tichonov space are presented. In section 1,
we obtain a result on the existence of fixed points for certain continuous functions which generalizes some recent results of Harinath [1] over pseudo-compact Tichonov space and results of Fisher [3] over compact metric space. In section 2, we generalize our result of section 1, for two continuous self mappings in pseudo-compact Tichonov space. In the last section we prove the following theorem for quasi contractive mapping in pseudo-compact Tichonov space, which is published in "The Mathematical Education" Vol. XXII, No.1. March 88.

Let $P$ be a pseudo-compact Tichonov space and $\mu$ be a non-negative real valued continuous function over $P \times P$ ($P \times P$ is Tichonov but need not be pseudo-compact). Suppose $\mu$ satisfies

\[
\begin{align*}
\mu(x,x) &= 0, \text{ for all } x \in P \text{ and } \\
\mu(x,y) &\leq \mu(x,z) + \mu(z,y) \text{ for } x, y, z \in P
\end{align*}
\]

and if $E$ be a self continuous quasi contractive mapping of $P$ i.e. $p$ is continuous and satisfied.

\[
\begin{align*}
\mu(Ex, Ey) &< \frac{1}{2} \left[ \mu(x, Ex) + \mu(y, Ey) + \mu(x, Ey) \\
&+ \mu(y, Ex) + \mu(y, Ex) + \mu(x, Ex) \\
&+ \mu(x, Ex) + \mu(Ex, EEx) \\
&+ \mu(Ey, Ex) + \mu(x, y) \right]
\end{align*}
\]

for all distinct $x, y \in P$. Then $E$ has a fixed point in $P$. 

In chapter V, we discuss some fixed point theorems in Banach space. In theorem 1, we generalize the result of Goebel and Zlotkiewicz [1] by proving the following:

Let $F$ be a mapping of a Banach space $X$ into itself.

If $F$ satisfies conditions:

\begin{equation}
F = I, \quad \text{where $I$ is the identity mapping.}
\end{equation}

\begin{equation}
\|F(x) - F(y)\| \leq \alpha \|x - y\| + \beta \frac{\|x - F(x)\| \cdot \|y - F(y)\|}{\|x - y\|}
\end{equation}

for every $x, y \in X$, where $0 \leq \alpha + \beta < 2$, then $F$ has at least one fixed point. Further we generalize our result 1 for two and three commuting mappings and remark that the result of Khan [1] appear as a particular case of our results.

In Chapter VI, we prove several common fixed point theorems for three self mapping of a complete metric space.

The main result of section 1 generalizes the result of Jungck[1], Banach [1] and Iskii, Rajput and Sharma [1], by proving the following:

Let $E$, $F$ and $T$ are three continuous mappings of a complete metric space $(X, d)$ satisfying the following conditions.

\begin{equation}
ET = TE, \quad FT = TF,
\end{equation}

$E(X) \cap T(X) = F(X) \cap T(X)$ and
\[(1.7.7)\quad d(Ex, Fy) \leq \alpha \frac{d(Ty, Fy) [1+d(Tx, Ex)]}{[1+d(Tx, Ty)]} + \beta [d(Tx, Ex) + d(Ty, Fy)] + \gamma [d(Tx, Fy) + d(Ty, Ex)] + \delta d(Tx, Ty)\]

for all \(x, y \in X\) where \(\alpha, \beta, \gamma, \delta \geq 0\) and \(\alpha + 2\beta + 2\gamma + \delta < 1\).

Then \(E, F\) and \(T\) have a unique common fixed point in \(X\). This result is accepted for publication in "BHARATA GANITA PARISHAD".

The result of section 2 generalizes the result of Fisher [2] by proving:

Let \((x, d)\) be a complete metric space.
Let \(S, T\) and \(P : X \rightarrow X\) satisfying the following condition
\[(1.7.8)\quad [d(Spx, Tpy)]^2 \leq \alpha d(Spx, Tpy) d(x, y) + \beta d(x, Tpx) d(y, Tpy) + \gamma d(x, Tpy) d(y, Spx)\]

for all \(x, y \in X\) where \(\alpha, \beta, \gamma \geq 0\) with \((\alpha + \beta + \gamma < 1)\). Further assume that either \(SP = PS\) or \(TP = PT\) then \(S, T\) and \(P\) have a unique common fixed point in \(X\). This result is accepted for publication in "Vijnana Parishad Anushadhan Patrika".

The result of last section generalize the result of Fisher [2] and Rao and Rao [1] by proving the following:
Let \((x, d)\) be a complete metric space. Let \(S, T\) and \(P : X \rightarrow X\) satisfying the following conditions:

\[
(1.7.9) \quad [d(SPx, TPy)]^2 \leq \alpha [d(x, y)]^2 \\
\quad + \beta d(x, SPx) d(y, TPy) \\
\quad + \gamma d(x, TPy) d(y, SPx) \\
\quad + \delta d(SPx, TPy) d(x, y)
\]

for all \(x, y \in X\), where \(\alpha, \beta, \gamma, \delta \geq 0\) with \(\alpha + \beta + \delta < 1\) and \(\alpha + \gamma + \delta < 1\). Further, assume that either \(SP = PS\) or \(TP = PT\), then \(S, T\) and \(P\) have a unique common fixed point is \(X\). This is also accepted for publication in "THE MATHEMATICAL EDUCATION"

Chapter VII consist of two sections. In section 1, we prove the following fixed point theorem for non-contractive mapping on 2- Banach space.

Let \(F\) be a mapping of a 2- Banach space \(X\) into itself. If \(F\) satisfies the conditions.

\[
(1.7.10) \quad F = I, \text{ where } I \text{ is the identity mapping,}
\]

\[
(1.7.11) \quad ||F(x) - F(y)|| \leq \alpha ||x-y||, \quad \alpha||+\beta\frac{||x-F(x),a||+||y-F(y),a||}{||x-y||, a} 
\]

for every \(x, y\) and \(a \in X\), where \(0 \leq \alpha, \beta\) and \(0 \leq \alpha + 4\beta < 2\), then \(F\) has at least one fixed point.
In section 2, we prove the following theorem.

Let $Y$ be a convex, closed subset of a strictly convex 2-Banach space $X$ and $F: Y \rightarrow Y$ be a continuous mapping which satisfies the following condition:

(1.7.12) For every $x$, $y$ and $a$ of $Y$,

$$\|F(x) - F(y), a\| \leq \alpha \|x - y, a\| + \beta \frac{\|x - F(x), a\| \cdot \|y - F(y), a\|}{\|x - y, a\|} + \gamma \frac{\|x - F(y), a\| - \|y - F(x), a\|}{\|x - y, a\|}$$

where $\alpha + \gamma \leq 1$.

(1.7.13) $F(Y)$ is contained in the compact set of $Y$.

Then for every $x$ of $Y$, the iteration of the mapping

$$U^\lambda(x) = F(x) + (1-\lambda) x$$

where $0 < \lambda < 1$, converges to a fixed point of $F$.

In chapter VIII, we shall prove some fixed point theorems. In section 1, we prove some fixed point theorems. Following theorem is the main result of this section.

Let $E$, $F$ and $T$ be three continuous mappings of a complete metric space $X$ into itself satisfying
(1.7.14) \( ET = TE, FT = TF, E(X) \subset T(X) \) and \( F(X) \subset T(X) \).

(1.7.15) \( d(Ex, Fy) \leq \alpha_1 \frac{d(Tx, Ex) \cdot d(Ty, Fy)}{d(Tx, Ty)} \)

\[ + \alpha_2 \frac{d(Tx, Fy) \cdot d(Ty, Fy)}{d(Ex, Fy)} \]

\[ + \alpha_3 [d(Tx, Ex) + d(Ty, Fy)] \]

\[ + \alpha_4 [d(Tx, Fy) + d(Ty, Ex)] \]

\[ + \alpha_5 d(Tx, Ty) \]

for all \( x, y \in X \) with \( Tx \neq Ty \) where \( \alpha_1 \geq 0, \)

\( 0 \leq \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 < 1 \) and \( 2\alpha_4 + \alpha_3 < 1 \). Then \( E, F \) and \( T \) have a common fixed point. This result is published in "Journal of Scientific Research" Vol. 10, No. 1 11-13 (1998). In the last section, we generalize a result of Das [1] for three mappings and we prove the following theorem.

Let \( E, F \) and \( T \) be three continuous mappings of the complete metric space \( (X, d) \) into itself satisfying the following conditions:

(1.7.16) \( ET = TE, FT = TF, F(X) \subset T(X) \) and \( E(X) \subset T(X) \).
\[ (1.7.17) \quad d(Ex, Fy) \leq \alpha \frac{d(Ty, Ex^2) + d(Ty, Fy)}{d(Ex, E^2x)} \]

\[ + \beta [d(Tx, Ex) + d(Ty, Fy)] \]

\[ + \gamma [d(Tx, Fy) + d(Ty, Ex)] \]

\[ + \delta d(Tx, Ty) \]

for all \( x, y \) in \( X \) with \( Tx \neq Ty \) where \( \alpha, \beta, \gamma, \delta \geq 0 \), \( \alpha + 2\beta + 2\gamma + \delta < 1 \) and \( (2\gamma + \delta) < 1 \). Then \( E, F \) and \( I \) have a unique common fixed point in \( X \). This result is published in "The Mathematica students Vol. 56 (1988) Nos 1-4 pp 1-4."