COMMON FIXED POINT THEOREM FOR THREE MAPPINGS IN COMPLETE METRIC SPACE

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Abstract: In this paper we shall establish a unique common fixed point theorem which is the generalization of the results due to Banach [1], Kannan [5], Fisher [4] Chatterjee [2] and others.

The object of this paper is to prove the following theorem:

THEOREM:—Let E, F and T be three continuous mappings of the complete metric space \( (X, d) \) into itself satisfying the following conditions;

\[
ET = TE, FT = TF, F(X) CT(X) \text{ and } E(X) CT(X) \tag{1.1}
\]

\[
d(Ex, Fy) \leq \alpha \frac{d(Tx, Ex) + d(Ty, Fy)}{d(Ex, Ty)} \tag{1.2}
\]

\[
\theta [d(Tx, Ex) + d(Ty, Fy)]
\]

\[
\gamma [d(Tx, Fy) + d(Ty, Ex)] + \delta d(Tx, Ty)
\]

for all \( x, y \) in \( X \) with \( Tx \neq Ty \) where \( \alpha, \beta, \gamma, \delta \geq 0 \), \( \alpha + 2\beta + 2\gamma + \delta < 1 \), \((2\beta + \delta) < 1 \). Then \( E, F \) and \( T \) have a unique common fixed point in \( X \).

Proof: Let \( x_0 \in X \) be an arbitrary element and let \( (Tx_n) \) be defined as

\[
Tx_{n+1} = Ex_n, Tx_{n+2} = Fx_{n+1}, \text{ for } n = 0, 1, 2, \ldots \tag{1.3}
\]

We can do this since \( F(X) CT(X) \) and \( E(X) CT(X) \). Then by (1.2) we have

\[
d(Tx_{n+1}, Tx_{n+2}) = d(Ex_n, Fx_{n+1})
\]

\[
\leq \alpha \frac{d(Tx_{n+1}, Ex_n) + d(Tx_{n+1}, Fx_{n+1})}{d(Ex_n, Ty)}
\]

\[
\theta [d(Tx_n, Ex_n) + d(Tx_{n+1}, Fx_{n+1})]
\]

\[
\gamma [d(Tx_n, Fx_{n+1}) + d(Tx_{n+1}, Ex_n)]
\]

\[
\delta d(Tx_n, Ty)
\]

It follows that

\[
d(Tx_{n+1}, Tx_{n+2}) \leq \frac{\delta + \gamma + \delta}{\frac{1}{\alpha} - \frac{\delta}{\gamma} - \theta} d(Tx_n, Tx_{n+1})
\]
Put $\beta \neq \gamma + \delta \neq \gamma$, \( \lambda \). Thus $h < 1$ and we have

\[ d(Tx_{n+1}, Tx_{n+1}) < \delta d(Tx_n, Tx_{n+1}) \]

Similarly we can see

\[ d(Tx_{n+1}, Tx_{n+1}) < h^{n+1} d(Tx_n, Tx_1) \]

By routine calculation the following inequalities hold for $k > n$

\[ d(Tx_n, Tx_{n+k}) \leq \sum_{i=0}^{k-1} d(Tx_i, Tx_{i+1}) \]

\[ \leq \sum_{i=0}^{k-1} h^{i+1} d(Tx_i, Tx_{i+1}) \]

\[ \leq \frac{h^k}{1 - h} d(Tx_n, Tx_1) \]

\[ \to 0 \text{ as } n \to \infty \]

Hence \( \{Tx_n\} \) is a Cauchy sequence. By the completeness of \( X \), \( \{Tx_n\} \) converges to a point \( u \in X \). It follows from (1.1) that \( \{Ex_n\} \) and \( \{Fx_{n+1}\} \) also converges to \( u \), since \( E \), \( F \) and \( T \) are continuous, we have,

\[ E(Tx_n) = Eu, F(Tx_{n+1}) = Fu. \]

From (1.1) \( T \) commutes with \( E \) and \( F \), therefore

\[ E(Tx_n) = T(Ex_n), F(Tx_{n+1}) = T(Fx_{n+1}) \]

for all $n = 0, 1, 2, \ldots$. Taking $n \to \infty$ we have

\[ Eu = Tu = Fu \] \hspace{1cm} (1.5)

\[ T(Tu) = T(Eu) = E(Tu) = E(Eu) = E(Fu) = T(Fu) \] \hspace{1cm} (1.6)

by (1.2), (1.5) and (1.6). If \( Eu \neq F(Eu) \) we have

\[ d(Eu, F(Eu)) \leq \alpha d(T(Eu), Efu) d(T(Eu), F(Eu)) \]

\[ + \beta d(Tu, Eu) + d(T(Eu), F(Eu)) \]

\[ + \gamma d(Tu, F(Eu)) + d(T(Eu), F(Eu)) + \delta d(Tu, T(Eu)) \]

\[ \leq (2\alpha + \beta) d(Eu, F(Eu)) \]

\[ < d(Eu, F(Eu)) \] \hspace{1cm} \( \forall \ (2\alpha + \beta) \leq 1 \)

leading to a contradiction. Hence

\[ Eu = F(Eu) \] using (1.6) and (1.7) we get

\[ Eu = F(Eu) = T(Eu) = E(Eu) \]
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which shows that \( Ew \) is the common fixed point of \( E, F \) and \( T \).

Let \( Z = W \) \((Z \neq W)\) be two points in \( X \), such that

\[
Ez = Fz = Tz = z \quad \text{and} \quad Ew = Fw = Tw = w
\]

Then by (1.2) we have

\[
d(z, w) = d(Ex, Fw)
\]

\[
\leq \alpha \frac{d(Tw, Ez) d(TW, Fw)}{d(Ez, Ez)} + \beta [d(Tz, Ez) + d(Tw, Fw)]
\]

\[
+ \gamma (d(Tz, Fw) + d(TW, Ez)) + d(Tz, Tw)
\]

\[
\leq (2\gamma + \delta) d(Ex, Fw)
\]

\[
\leq (Ex, Fw) \quad [\gamma (2\gamma + \delta) < 1]
\]

leading to a contradiction. Hence \( z = w \). This implies the uniqueness of common fixed point for \( E, F \) and \( T \). This completes the proof of the theorem.

REMARKS: Taking \( E = F \) and \( T = I \) (I is the identity map on \( X \), we observe the following:

1. Taking \( \alpha = \beta = \gamma = 0 \); we obtain result due to Banach [1].
2. Taking \( \alpha = \gamma = \delta = 0 \); we obtain result due to Kannan [5].
3. Taking \( \alpha = \beta = \delta = 0 \); we obtain result due to Chatterjee [2] and Fisher [4].
4. Taking \( \alpha = \beta \); we obtain result due to Cirić.

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REFERENCES

2. S.K. CHATTERJEE. Fixed point theorem, complete Read. Acad. Bulgare Se. 22(1972) 727-730.

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