CHAPTER VI

COMMON FIXED POINT THEOREMS FOR THREE MAPPINGS IN COMPLETE METRIC SPACE

6.1 During the last few years several mathematicians have generalized Banach's contraction principle in different ways. In the first section we generalize contraction principle through rational expression for three maps. In the last section we generalize the results of Fisher [2] and Rao and Rao [1].

A self mapping T of X is said to be contraction if there exists some 0 ≤ k < 1 such that

\[(6.1.1) \quad d(Tx, Ty) \leq k d(x, y)\]

for all x, y in X. Banach [1] has proved that if X is a complete metric space and T is a contraction mapping of X, then T has a unique fixed point in X.

In 1976 Jungck [1] has given generalization of Banach contraction principle by proving the following theorem:

THEOREM [A] - Let E and F be two continuous and commuting self mappings of a complete metric space \((X, d)\) satisfying:

\[(6.1.2) \quad E(X) \subseteq F(X)\]
(6.1.3) \( d(Ex, Ey) \leq \alpha d(Fx, Fy) \)

for all \( x, y \in X \) where \( \alpha \) is a non-negative real number with \( \alpha < 1 \). Then \( E \) and \( F \) have unique common fixed point.

Recently in 1982 Iséki, Rajput and Sharma [1] have proved the following theorem:

**Theorem: [B]** - Let \( T \) be a continuous mapping of a complete metric space \( X \) into itself satisfying

\[
(6.1.4) \quad d(Tx, Ty) \leq \alpha \frac{d(y, Ty) [1 + d(x, Tx)]}{[1 + d(x, y)]} \\
+ \beta [d(x, Tx) + d(y, Ty)] \\
+ \gamma [d(x, Ty) + d(y, Tx)] + \delta d(x, y)
\]

for all \( x, y \in X \) where \( \alpha, \beta, \gamma, \delta \geq 0 \) and \( \alpha + 2\beta + 2\gamma + \delta < 1 \). Then \( T \) has a unique fixed point in \( X \).

6.2 In this section we generalize theorem [B] by proving the following theorem:

**Theorem [1]** - Let \( E \) and \( F \) are two continuous mappings of a complete metric space \( X \) into itself satisfying:

\[
(6.2.1) \quad d(Ex, Fy) \leq \alpha \frac{d(y, Fy) [1 + d(x, Ex)]}{[1 + d(x, y)]} \\
+ \beta [d(x, Ex) + d(y, Fy)] \\
+ \gamma [d(x, Fy) + d(y, Ex)] + \delta d(x, y)
\]
for all $x, y$ in $X$ where $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + 2\beta + 2\gamma + \delta < 1$.

Then $E$ and $F$ have a unique common fixed point in $X$.

**Proof.** Let $x_0$ be arbitrary element and let $\{x_n\}$ be defined as

$$x_{2n+1} = Ex_{2n} \text{ and } x_{2n+2} = Fx_{2n+1} \text{ for } n = 1, 2, \ldots$$

Then by (6.2.1), we have

$$d(x_{2n+1}, x_{2n+2}) = d(Ex_{2n}, Fx_{2n+1})$$

$$\leq \alpha \frac{d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})}$$

$$+ \beta [d(x_{2n}, Ex_{2n}) + d(x_{2n+1}, Fx_{2n+1})]$$

$$+ \gamma [d(x_{2n}, Fx_{2n+1}) + d(x_{2n+1}, Ex_{2n})]$$

$$+ \delta d(x_{2n}, x_{2n+1})$$

Thus,

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} d(x_{2n}, x_{2n+1})$$
Taking \( \frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} = h < 1 \), we have

\[
d(x_{2n+1}, x_{2n+2}) \leq h \, d(x_{2n}, x_{2n+1})
\]

Proceeding in this way, we get

\[
d(x_{2n+1}, x_{2n+2}) \leq h \, d(x_{2n}, x_{2n+1}) \leq \cdots \leq h^{n+1} \, d(x_0, x_1)
\]

By routine calculations, the following inequalities hold for \( k > n \),

\[
d(x_n, x_{n+k}) \leq \sum_{i=1}^{k} d(x_{n+i-1}, x_{n+i})
\]

\[
\leq \sum_{i=1}^{k} h^n \, d(x_0, x_1)
\]

\[
\leq h^n \, d(x_0, x_1) \rightarrow 0 \text{ as } n \rightarrow \infty
\]

Hence the sequence \( \{x_n\} \) is a Cauchy sequence in \( X \).

By the completeness of \( X \), there exists some \( z \) in \( X \) such that

\[
\lim_{n \to \infty} x_n = z.
\]

Further the continuity of \( E \) implies

\[
E \circ E = E \left( \lim_{n \to \infty} x_n \right) = \lim_{n \to \infty} E(x_n) = 2. \text{ Thus } z \text{ is a fixed point of } E.
\]

And also the continuity of \( F \) implies

\[
F \circ F = F \left( \lim_{n \to \infty} x_n \right) = \lim_{n \to \infty} F(x_n) = z. \text{ Thus } z \text{ is a fixed point of } F.
\]
Hence \( z \) is a common fixed point of \( E \) and \( F \).

To prove the uniqueness of \( z \). Let \( z \) and \( w \) be two common fixed points of \( E \) and \( F \) such that

\[
Ez = Fz = z \quad \text{and} \quad Ew = Fw = w.
\]

Then

\[
d(z, w) = d(Ez, Fw)
\]

\[
\leq \alpha \frac{d(w, Fz) [\lambda + d(z, Ez)]}{[1 + d(z, w)]}
\]

\[
+ \beta [d(z, Ez) + d(w, Fw)]
\]

\[
+ \gamma [d(z, Fz) + d(w, Ew)] + \delta d(z, w)
\]

\[
\leq (2\gamma + \delta) d(z, w)
\]

\[
< d(z, w), \text{ as } 2\gamma + \delta < 1.
\]

This leads to a contradiction. Thus \( z = w \).

Hence \( z \) is the unique common fixed point of \( E \) and \( F \). This completes the proof of the theorem.

**Remark**

Taking \( E = F \), we obtain the theorem \([B]\).

Further we generalize the above all results by proving the following theorem on three maps.
THEOREM [2] - Let $E$, $F$ and $T$ are three continuous mappings of a complete metric space $(X, d)$ satisfying the following conditions:

\[(6.2.2)\quad ET = TE, \quad FT = TF \text{ and} \]
\[E(X) CT (X), \quad F(X) CT (X)\]

\[(6.2.3)\quad d(Ex, Fy) \leq \alpha \frac{d(Ty, Fy) [l + d(Tx, Ex)]}{[l + d(Tx, Ty)]} + \beta [d(Tx, Ex) + d(Ty, Fy)] + \gamma [d(Tx, Fy) + d(Ty, Ex)] + \delta d(Tx, Ty)\]

for all $x$, $y$ in $X$ where $\alpha$, $\beta$, $\gamma$, $\delta \geq 0$ and $\alpha + 2\beta + 2\gamma + \delta < 1$. Then $E$, $F$ and $T$ have a unique common fixed point in $X$.

PROOF - Let $x_0$ be an arbitrary element of $X$ and as $E(X) CT (X)$ and $F(X) CT (X)$, we define a sequence $\{Tx_n\}$ as follows:

\[(6.2.4)\quad Tx_{2n+1} = Ex_{2n}, \quad Tx_{2n+2} = Fx_{2n+1} \quad \text{for } n = 1, 2, \ldots\]

From (6.2.3), we have

\[d(Tx_{2n+1}, Tx_{2n+2}) = d(Ex_{2n}, Fx_{2n+1})\]
\[ \frac{d(T_{2n+1}, E_{2n+1})}{1 + d(T_{2n}, E_{2n})} \quad \delta \quad \frac{[1 + d(T_{2n}, T_{2n+1})]}{[1 + d(T_{2n}, T_{2n+1})]} \]

\[ + \beta \quad [d(T_{2n}, E_{2n}) + d(T_{2n+1}, E_{2n+1})] \]

\[ + \gamma \quad [d(T_{2n}, E_{2n+1}) + d(T_{2n+1}, E_{2n})] \]

\[ + \delta d(T_{2n}, T_{2n+1}) \]

\[ d(T_{2n+1}, T_{2n+2}) \quad \delta \quad \frac{[1 + d(T_{2n}, T_{2n+1})]}{[1 + d(T_{2n}, T_{2n+1})]} \]

\[ + \beta [d(T_{2n}, T_{2n+1}) + d(T_{2n+1}, T_{2n+2})] \]

\[ + \gamma [d(T_{2n}, T_{2n+2}) + d(T_{2n+1}, T_{2n+1})] \]

\[ + \delta d(T_{2n}, T_{2n+1}) \]

Thus,

\[ d(T_{2n+1}, T_{2n+2}) \leq \frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} \quad d(T_{2n}, T_{2n+1}) \]

Taking \[ \frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} = h < 1 \], we have

\[ d(T_{2n+1}, T_{2n+2}) \leq hd(T_{2n}, T_{2n+1}) \]
Proceeding in this way, we have
\[ d(T_{2n+1}, T_{2n+2}) \leq h \cdot d(T_{2n}, T_{2n+1}) \leq \cdots \leq h^{2n+1} d(T_0, T_1) \]

By routine calculations the following inequalities hold for \( k > n \)
\[ d(T_n, T_{n+k}) \leq \sum_{i=1}^{k} d(T_{n+i-1}, T_{n+i}) \]
\[ \leq \sum_{i=1}^{k} h^{n+i-1} d(T_0, T_1) \]
\[ \leq \frac{h^n}{1-h} d(T_0, T_1) \to 0 \text{ as } n \to \infty \]

Hence \( \{T_n\} \) is a Cauchy sequence. By the completeness of \( X \), \( \{T_n\} \) converges to a point \( z \) in \( X \).

It follows from (6.2.4) that \( \{E_{2n}\} \) and \( \{F_{2n+1}\} \) also converge to \( z \).

As \( E, F \) and \( T \) are continuous, we have
\[ (6.2.5) \quad E(T_{2n}) \to Ez, \quad F(T_{2n+1}) \to Fz. \]

As \( T \) commutes with \( E \) and \( F \), we have
\[ E(T_{2n}) = T(E_{2n}) \quad \text{and} \quad F(T_{2n+1}) = T(F_{2n+1}); \quad \text{for } n = 0, 1, 2, \ldots \]

Thus taking \( n \to \infty \), we have
\[ (6.2.6) \quad Ez = Tz = Fz. \]
Hence,

(6.2.7) \( T(Tz) = T(Ez) = E(Tz) = E(Ez) = E(Fz) = T(Fz) \)

By (6.2.3), (6.2.6) and (6.2.7), if \( Ez = F(Ez) \),

We have,

\[
\begin{align*}
    d(Ez, F(Ez)) & \leq \frac{d(T(Ez), F(Ez))}{1 + d(Tz, Ez)} [1 + d(Tz, T(Ez))] \\
    & + \beta [d(Tz, Ez) + d(T(Ez), F(Ez))] \\
    & + \gamma [d(Tz, F(Ez)) + d(T(Ez), Ez)] \\
    & + \delta d(Tz, T(Ez)) \\
    & \leq (2\gamma + \delta) d(Ez, F(Ez)) \\
    & \leq d(Ez, F(Ez)), \text{ as } 2\gamma + \delta < 1.
\end{align*}
\]

This leads to a contradiction. Hence

(6.2.8) \( Ez = F(Ez) \).

Using (6.2.7) and (6.2.8), we have

\( Eu = F(Eu) = T(Eu) = E(Eu) \).

Thus \( Eu \) is the common fixed point of \( E, F \) and \( T \).

Let \( z \) and \( w \) be two different common fixed points of \( E, F \) and \( T \) such that
Ez = Fz = Tz = z and 
Ey = Fy = Ty = w.

Then by using (6.2.3), we have 
\[ d(z, w) = d(Ez, Fw) \]
\[ \leq \alpha \frac{d(Tw, Fw)}{1+d(Tz, Ez)} \]
\[ \leq \frac{\alpha}{1+d(Tz, Tw)} \]
\[ + \beta [d(Tz, Ez) + d(Tw, Fw)] \]
\[ + \gamma [d(Tz, Fw) + d(Tw, Ez)] + \delta d(Tz, Tw) \]
\[ \leq (2\gamma + \delta) d(Ez, Fw) \]
\[ < d(Ez, Fw), \text{ as } 2\gamma + \delta < 1. \]

This leads to a contradiction. Hence z = w.

Thus E, F and I have a unique common fixed point. 
This completes the proof of the theorem.

REMARKS:

Considering I as the identity map on X, we note the following:

[1] Taking E = F and T = I in the theorem [2], we obtain the theorem [B].

[2] Taking E = F, T = I and \( \alpha = \beta = \gamma = \delta \) in the theorem [2], we obtain well known Banach [1] contraction principle.
[3] Taking $E=F$, $T=I$ and $a=y=\delta=0$, in the theorem we obtain the result due to Kannan [1].

[4] Taking $E=F$, $T=I$ and $a=\beta=\delta=0$ in the theorem [2], we obtain the result due to Fisher [1].

[5] Taking $E=F$, $T=I$ and $a=0$ in the theorem, we obtain the result due to Cirić [1]. And

[6] Taking $T=I$, we obtain our theorem [1].

We now establish the following theorem by applying theorem [1].

**THEOREM [3]** Let $E$, $F$ and $T$ be three continuous self-mappings of a complete metric space $(X, d)$ satisfying (6.2.2) and if there exists two positive integers $m$ and $n$ such that

\[
(6.2.9) \quad d(E^m x, F^n y) \leq a \frac{d(Ty, F^n y)}{[1+d(Tx, T^{m+n} x)]} + \beta [d(Tx, E^m x) + d(Ty, F^n y)] + \gamma [d(Tx, F^n y) + d(Ty, E^m x)] + \delta d(Tx, Ty),
\]

for all $x, y$ in $X$ where $a, \beta, \gamma, \delta \geq 0$ and $a+2\beta+2\gamma+\delta<1$. Then $E$, $F$ and $T$ have a unique common fixed point in $X$.

**PROOF** It follows from (6.2.2) that

$E^m = TE^m$, $F^n = TF^n$ and

$E^m(x) CE (X) CT(x)$, $F^n(x) CF(x) CT (X)$
By theorem [2] we see that there exists a unique fixed point \( z \in X \) such that

\[
T_z = T z = E z = F z = T^m z = E^m z = F^m z.
\]

Also

\[
T(E z) = E(T z) = E z = E(E z) = E^m(E z) \quad \text{and} \n
T(F z) = F(T z) = F z = F(F z) = F(F z).
\]

Thus \( E z \) is a common fixed point of \( T \) and \( E \) also \( F z \) is a common fixed point of \( T \) and \( F \). Thus uniqueness of \( z \) implies

\[
E z = F z = T z = 2
\]

Hence \( z \) is the unique common fixed point of \( E, F \) and \( T \). This completes the proof of the theorem.

Finally using the theorem [2], we obtain the following theorem:

**THEOREM** [4] - Let \( T \) and \( T_i \) \((i=1,2,...,k)\) be continuous self mappings of a complete metric space \((X,d)\) satisfying the following conditions:

\[
(6.2.10) \quad T T_i = T_i T \quad (i=1,2,...,k)
\]

\[
(6.2.11) \quad T_i T_j = T_j T_i \quad (i=1,2,...,k; j=1,2,...,k).
\]

\[
(6.2.12) \quad E(X) \subset T(X) \quad \text{where} \quad E = T_1, T_2,...,T_k, \quad \text{and}
\]

\[
(6.2.13) \quad \text{the condition } (6.2.3) \text{ holds for } E = F
\]
Then \( I \) and \( T_i \) \((i=1,2,...,k)\) have a unique common fixed point is \( x \).

**Proof** - By theorem [2], we have

\[ Eu = T_u = u. \]

Thus

\[ T_i(Eu) = T_i(Tu) = T_i(u) \]

Now using the conditions (6.2.10), (6.2.11) and (6.2.12), we have

\[ E(T_i u) = T(T_i u) = T_i(u) \]

Hence \( T_i(u) \) for \( i = 1, 2, ..., k \) is the common fixed point of \( E \) and \( T \). By the uniqueness of the common fixed point of \( E \) and \( T \), we have

\[ T_i(u) = u. \]

This completes the proof of the theorem.

6.3 Fisher [5] has proved the following theorem:

**Theorem [C]** - In a complete metric space \((x, d)\) if there exists two operators \( S \) and \( T \) mapping \( X \) into itself and satisfying the relation

\[
(6.3.1) \quad d(Sx, Ty) \leq bd(x,Sx) \cdot d(y,Ty) + C d(x,Ty) \cdot d(y,Sx)
\]
for all \(x, y \in X\), where \(a \leq b < 1\) and \(c \geq 0\), then \(S\) and \(T\) have a common fixed point. Further if \(a \leq b < 1\) and \(c < 1\) then each of \(S\) and \(T\) has a unique fixed point and these two points coincide.

Rao and Rao [1] extended the above result for three maps by proving the following theorem:

**Theorem [6]** — Let \((X, d)\) be a complete metric space.

Let \(S, T\) and \(P : X \to X\) satisfying the following conditions:

\[
(6.3.2) \quad [d(SPx, TPy)]^2 \leq a[d(x, y)]^2 \\
+ b\ d(x, SPx)\ d(y, TPy) \\
+ c\ d(x, TPy)\ d(y, SPx)
\]

for all \(x, y \in X\), where \(a, b, c \geq 0\) with \(a \cdot b < 1\) and \(a + c < 1\). Further, assume that \(SP = PS\) or \(TP = PT\), then \(S, T\) and \(P\) have a common unique fixed point in \(X\).

We shall prove:

**Theorem [5]** — Let \((X, d)\) be a complete metric space.

Let \(S, T\) and \(P : X \to X\) satisfying the following conditions:

\[
(6.3.3) \quad [d(SPx, TPy)]^2 \leq ad(SPx, TPy)\ d(x, y) \\
+ bd(x, SPx)\ d(y, TPy) \\
+ cd(x, TPy)\ d(y, SPx)
\]
such that \( u \to x \in X \).

As \( X \) is a complete metric space, there exists a Cauchy sequence in \( X \).

Therefore, \( \{x_n \} \subseteq X \) is a Cauchy sequence in \( X \).

Similarly, \( \{x_n^{+1} \} \) is a Cauchy sequence in \( X \).

Hence, \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} x_n^{+1} = x \).

Let \( x^0 \in X \) be a fixed point of the mapping \( f \) such that \( f(x^0) = x^0 \). Let \( u \in X \) be arbitrary.

Consider the sequence \( \{x_n\} \) defined by:

\[
\begin{align*}
x_1 &= u \\
x_{n+1} &= f(x_n) = f(u) = x^0.
\end{align*}
\]

Then, \( x_n \to x^0 \) as \( n \to \infty \).
Hence from (6.3.3), we have

\[ [d(SPz, x_{2n+1})]^2 = [d(SPz, TPx_{2n-1})]^2 \]

\[ \leq \alpha d(SPz, TPx_{2n-1}) d(z, x_{2n-1}) \]

\[ + \beta d(z, SPz) d(x_{2n-1}, TPx_{2n-1}) \]

\[ + \gamma d(z, TPx_{2n-1}) d(x_{2n-1}, SPz) \]

Letting \( n \to \infty \), we have

\[ [d(SPz, z)]^2 \leq 0 \]

Hence \( SPz = z \).

Similarly by considering \([d(x_{2n+1}, TPz)]^2\)

we conclude from (6.3.3) by letting \( n \to \infty \)

that \( TPz = z \).

Thus

(6.3.4) \( SPz = z = TPz \).

It follows that \( z \) is the common fixed point of

\( SP \) and \( TP \).

Also if \( SP = PS \), we have

\[ [d(Pz, z)]^2 = [d(FSPz, TPz)]^2 = [d(SPz, TPz)]^2 \]

\[ \leq \alpha d(Pz, z) d(z, z) \]

\[ + \beta d(Pz, Pz) d(z, z) + \gamma d(z, Pz) d(z, Pz) \]
Thus,

\[(1-a-\gamma) \left[ d(Pz, z) \right]^2 \leq 0, \text{ which implies that} \]

\[pz = z.\]

Hence by (6.3.4), we have

\[Sz = z = Tz.\]

Similarly if \( PT = TP \), then also \( Pz = z = Sz = Tz. \)

Hence, \( S, T \) and \( P \) have a common fixed point.

Also uniqueness of common fixed point follows easily by (6.3.3). This completes the proof of the theorem.

REMARK

Taking \( a = 0 \) and \( P = I \) (Identify map on \( x \)),
we obtain theorem [G].

(6.4)

THEOREM [6] Let \((X, d)\) be a complete metric space.

Let \( S, T \) and \( P : X \rightarrow X \) satisfying the following conditions

(6.4.1) \[ \left[ d(S Px, TP y) \right]^2 \leq \alpha \left[ d(x, y) \right]^2 \]

\[+ \beta d(x, SP x)d(y, TP y)\]

\[+ \gamma d(x, TP y)d(y, SP x)\]

\[+ \delta d(SP x, TP y) d(x, y)\]

for all \( x, y \in X \), where \( \alpha, \beta, \gamma, \delta \geq 0 \) with \( \alpha + \beta + \delta < 1 \) and \( \alpha + \gamma + \delta < 1 \). Further, assume that either \( SP = PS \) or \( TP = PT \),
then \( S, T \) and \( P \) have a unique common fixed point in \( X \).
PROOF: Let \( x_0 \in X \). We define a sequence \( \{x_n\} \) as
\[
x_{2n+1} = SP_{2n}; \quad n=0, 1, 2, \ldots
\]
\[
x_{2n} = TP_{2n-1}; \quad n=1, 2, 3, \ldots
\]
Now
\[
[d(x_{2n+1}, x_{2n})]^2 = [d(SP_{2n}, TP_{2n-1})]^2
\]
\[\leq \alpha [d(x_{2n}, x_{2n-1})]^2 + \beta d(x_{2n}, x_{2n+1}) d(x_{2n-1}, x_{2n}) + \gamma d(x_{2n}, x_{2n}) d(x_{2n-1}, x_{2n+1}) + \delta d(x_{2n+1}, x_{2n}) d(x_{2n}, x_{2n-1}) \leq \alpha [d(x_{2n}, x_{2n-1})]^2 + \beta d(x_{2n}, x_{2n+1}) d(x_{2n-1}, x_{2n}) + \gamma d(x_{2n}, x_{2n}) d(x_{2n-1}, x_{2n+1}) + \delta d(x_{2n+1}, x_{2n}) d(x_{2n}, x_{2n-1}) \]
\[+ (\beta + \delta) \frac{[d(x_{2n}, x_{2n+1})]^2 + [d(x_{2n-1}, x_{2n})]^2}{2}
\]
Hence,
\[
[d(x_{2n}, x_{2n+1})]^2 \leq \frac{\alpha + \beta + \delta}{2 - \frac{\alpha + \delta}{2}} \cdot [d(x_{2n-1}, x_{2n})]^2
\]
Therefore, we have
\[
d(x_{2n}, x_{2n+1}) \leq k d(x_{2n-1}, x_{2n})
\]
where
\[k = \left[ \frac{\alpha + \beta + \delta}{2 - \frac{\beta + \delta}{2}} \right] < 1\]
Similarly,
\[
d(x_{2n-1}, x_{2n}) \leq kd(x_{2n-2}, x_{2n-1})
\]
Therefore \( \{x_n\} \) is a Cauchy sequence in \( X \).

As \( X \) is a complete metric space, there exists \( z \in X \) such that \( x_n \to z \) as \( n \to \infty \).

Hence by (6.3.5), we have

\[
\begin{align*}
\left[ d(SPz, x_{2n}) \right]^2 &= \left[ d(SPz, TPx_{2n-1}) \right]^2 \\
&\leq \alpha \left[ d(z, x_{2n-1}) \right]^2 \\
&+ \beta d(z, SPz) d(x_{2n-1}, TPx_{2n-1}) \\
&+ \gamma d(z, TPx_{2n-1}) d(SPz, x_{2n-1}) \\
&+ \delta (SPz, TPx_{2n-1}) d(z, x_{2n-1}).
\end{align*}
\]

Letting \( n \to \infty \), we have

\[
\left[ d(SPz, z) \right] \leq 0
\]

Hence,

\( SPz = z \).

Similarly by considering \( \left[ d(x_{2n+1}, TPz) \right]^2 \), we conclude from (6.3.5) by letting \( n \to \infty \) that \( TPz = z \).

Thus,

(6.4.2) \( SPz = z = TPz \).

Hence it follows that \( z \) is the common fixed point of \( SP \) and \( TP \).

Also if \( SP= FS \), we have
\[ [d(Pz, z)]^2 = [d(PSPz, TPz)]^2 = [d(SPPz, TPz)]^2 \]

\[ \leq \alpha [d(Pz, z)]^2 + \beta d(Pz, Pz) d(2, z) + \gamma d(z, Pz) d(z, Pz) + \delta d(Pz, z) d(Pz, z) \]

Thus

\[ (1-\alpha-\gamma-\delta) [d(Pz, z)]^2 \leq 0, \]

which implies that

\[ Pz = z. \]

Thus, we have

\[ Sz = z = Tz. \]

Similarly if \( FT = TP \), then also \( Pz = z \Rightarrow Sz = Tz \).

Hence \( S, T \) and \( P \) have a common fixed point. Also uniqueness of common fixed point follows easily by (6.4.1).

This completes the proof of the theorem.

**Remarks**

[1] Taking \( \alpha = \delta = 0 \) and \( P = I \) (Identity map on \( X \)), we obtain theorem [C].

[2] Taking \( \delta = 0 \), we obtain theorem [D].

[3] Taking \( \alpha = 0 \), we obtain theorem [E].