CHAPTER 5

SIMILARITY IV

( p-oid operators and similarity )

In this chapter, we study some properties of p-oid operators. We show that a canonical image of a reduction p-oid operator is p-oid. Using this result, we discuss several conditions implying normality under essential similarity. We deduce results for paranormal operators and for reduction-$K_p$ operators. We prove that if $T$ is a reduction-$K_p$ operator such that $CT^q = T$ and $K$ a compact operator, $0 \notin \mathcal{U}_e(\mathcal{E})$ and either $p \neq q$ or $p = q = 1$ then $T$ is normal.
B. Sz. Nagy and C. Foias \cite{33} introduced the concept of $\rho$-dilation and defined a class $C_\rho$ as follows:

An operator $T \in \mathcal{B}(H)$ is of class $C_\rho$ ($\rho > 0$) if there is a Hilbert space $K$, containing $H$ as a subspace and a unitary operator $U$ on $K$ satisfying the condition

$$T^n x = \rho^n U^n x, \quad x \in H, \quad n=1,2,3,\ldots$$

where $P$ is a projection on $H$.

They also characterized the class $C_\rho$ as follows:

**Theorem 5.4:** An operator $T$ belongs to $C_\rho$ if and only if for all $x \in H$, $|z| = 1$,

$$(\rho-2) \left( \frac{1}{2} \right) (1-|z|^2) x \parallel^2 + 2 \Re(1-2z) x, z \geq 0.$$  

Using the concept of the class $C_\rho$, J.A.R. Holbrook \cite{28}, \cite{29} introduced a concept of operator radii $w_\rho(T)$ ($0 < \rho < \infty$) as follows:

For $T \in \mathcal{B}(H)$, $w_\rho(T) = \inf \{ u : u > 0 \text{ and } u^{-1} T \in C_\rho \}$. He characterized the class $C$ in terms of $w_\rho(T)$ as
Theorem 5.1 : \( T \in C_\rho \) if and only if \( v_\rho(T) \leq 1 \).

In particular, we have

\[
v_1(T) = \| T \|, \quad v_2(T) = \nu(T) \quad \text{and} \quad \lim_{\rho \to \infty} v_\rho(T) = r(T).
\]

Using the concept of operator radii, T. Furuta [23] introduced the concept of \( \rho \)-oid operators and generalised the class of spectraloid operators as follows:

An operator \( T \in \mathcal{L}(\mathcal{H}) \) is a \( \rho \)-oid operator if

\[
v_\rho(T^k) = \left[ v_\rho(T) \right]^k \quad \text{for} \quad k = 1, 2, 3, \ldots
\]

Clearly \( 1 \)-oid and \( 2 \)-oid operators are normaloid and spectraloid respectively [41]. Also for \( \rho \geq 1 \), \( T \) is a \( \rho \)-oid operator if and only if \( v_\rho(T) = r(T) \) [23].

T. Furuta [23] and S.M. Patel [41] proved independently that the class of \( \rho \)-oid operators is properly contained in the class of \( \rho' \)-oid operator whenever \( \rho' > \rho \geq 1 \). Thus the class of spectraloid operators is properly included in the class of \( \rho \)-oid operators for \( \rho > 2 \).

S.M. Patel generalised class \( G_1 \) and defined a class \( \mathcal{M}_\rho \) (\( \rho \geq 1 \)) as follows:
\[ T \in M_p \text{ if } v \left[ (T-z)^{-1} \right] = \left[ d(z, \sigma(T)) \right]^{-1}, \forall \in \sigma(T). \]

In other words,

\[ T \in M_p \text{ if } (T-z)^{-1} \text{ is } \rho\text{-oid for all } \in \sigma(T). \]

For \( \rho \leq \rho' \), \( M_p \subset M_{\rho'} \) and in particular \( M_1 \) is nothing else but the class \( (G_1) \). E.H. Patel \cite{41,43} proved that \( T \in M_p \) is a convexoid operator.

We have analogous concepts regarding \( \rho\text{-oidity} \) in the Calkin algebra as follows:

We define **essential operator radius** \( v_p(T) \) of

\[ T \in \mathcal{B}(\mathcal{H}) \text{ as } v_p(T) = \inf \left\{ v_p(T+K) : K \text{ is compact} \right\}. \]

We further define an **essentially \( \rho\text{-oid operator}**

as for \( \rho \geq 1 \), \( T \) is an essentially \( \rho\text{-oid operator} \) if

\( r_0(T) = v_1(T) \). In particular, \( T \) is **essentially normaloid**

if \( r_0(T) = \| \hat{T} \| = w_1(T) \) and **essentially spectraloid**

if \( r_0(T) = w_3(T) \). Thus the canonical image \( \hat{T} \) of \( T \in \mathcal{B}(\mathcal{H}) \)

is \( \rho\text{-oid in the Calkin algebra } \) if \( r_0(T) = v_1(T) \).

We know that if \( T \) is normal, hyponormal or para-normal then \( \hat{T} \) is respectively normal, hyponormal or
paramormal [SO 3]. One would like to ask whether the
above idea can be further generalised. In this direction,
we have

**Theorem 5.1:** If $T$ is reduction $\rho$-oid then $\hat{T}$ is
$\rho$-oid. i.e. $r_{\rho}(T) = r_{\rho}(\hat{T})$.

The proof of this Theorem depends heavily on a
result of G. Eckstein [14]:

**Theorem 5.2:** Let $z$ be a non-zero complex number such
that $w_{\rho}(T) = |z|$ $(0 < \rho < \infty)$ and let $\{x_{n}\}$ be a
bounded sequence of vectors. Then

$$|| (T-z)x_{n} || \to 0 \text{ implies } || (T^{\rho}-z^{\rho})x_{n} || \to 0 .$$

**Proof of Theorem 5.1:**

Let $A$ be the set of all eigenvalues $z$ of $T$
such that $N(T-z)$ reduces $T$.

i.e. $A = \{ z \in \sigma_{D}(T) : N(T-z) \subset N(T^{\rho}-z^{\rho}) \}$.

If $M$ is a closed linear span of the eigen-
spaces $N(T-z)$, $z \in A$ then $M$ reduces $T$, $T_{1} = T/M$ is
normal with $\sigma_{D}(T_{1})=A$. Denote $T/M$ by $T_{2}$. Thus $T=T_{1} \oplus T_{2}$. 
By hypothesis, $T_2$ is $\rho$-oid i.e. $r(T_2) = w_\rho(T_2)$.

Now $\rho$-oidity of $T_2$ implies $|s_1| = w_\rho(T_2)$ for some $a \in \sigma(T_2)$.

Using the result of C.R. Putnam [45],

either $a \in \sigma_\infty(T_2)$ or $a \in \sigma_\infty(T_2)$. For $a \in \sigma_\infty(T_2)$, $N(T_2-a) \subset N(T_2-a)$

by Theorem 5.6. But $N(T_2-a) = N(T-a)$ and $\sigma_\infty(T_2) = \sigma_\infty(T) - 1$ [48, refer Theorem 1].

Therefore $N(T-a) \subset N(T-a)$ and hence $a \in A$.

This contradicts the fact $\sigma_\infty(T_2) = \sigma_\infty(T)-A$.

Thus $a$ must be in $\sigma_\infty(T_2) \subset \sigma(T)$.

This implies $|a| \leq r_c(T_2) \leq r(T_2) = |a|$.

i.e. $w_\rho(T_2) = |a| = r_c(T_2)$.

Now $w_\rho(T_2) = r_c(T_2) \leq w_\rho(T_2) \leq w_\rho(T_2)$ gives us $r_c(T_2) = w_\rho(T_2)$. Thus $w_\rho(T_2) = w_\rho(T_2)$.

$T_1$ being normal,

$r_c(T_1) = |\hat{T_1}| = w_\rho(T_1)$.
Therefore, \( r_o(T) = \max \{ r_o(T_1), r_o(T_2) \} \)
\[ = \max \{ v_p(T_1), v_p(T_2) \} \]
\[ = v_p(T). \]

Thus \( r_o(T) = v_p(T) \).

i.e. \( \hat{T} \) is \( p \)-oid in the Calkin algebra.

Remarks 5.1: As an immediate consequence, we have

(1) If \( T \) is reduction-normaloid (or spectraloid) then \( T \) is normaloid (or spectraloid).

(2) Since every translation of a convexoid operator is spectraloid \( \subseteq \) [2], Lemma 5.4 \], the canonical image of reduction-convexoid is also convexoid.

(3) If \( T = T_1 \circ T_2 \) as in the proof of Theorem 5.1 then \( T \) is reduction \( p \)-oid then \( \sigma'(T_1) = \overline{\delta} \) and \( r_o(T_2) = v_p(T_2) \). We use this fact for the next theorem.

(4) If \( T \) is a Riesz operator then \( T_2 \) is quasi-nilpotent.

We have obtained spectral properties of an operator under essential similarity in chapter 3. Using
Corollary 3.3, Remarks 3.1, and Theorem 5.1, we have some interesting results under essential similarity. Before we consider these results, we prove the following theorem:

**Theorem 5.2:** If $T$ is an invertible reduction $\rho$-oid operator such that $T^{-1}$ is reduction $\rho'$-oid where $\rho \geq 1$ and $\rho' \geq 1$, and $\sigma_T(T)$ lies on the unit circle then $T$ is normal.

For the proof of this theorem, we need the following result of J.G. Steenali [11, Corollary 4.1]:

**Theorem 5.5:** If $T$ is an invertible operator such that $T \in C_\rho$ and $T^{-1} \in C_{\rho'}$, where $\rho \geq 1$ and $\rho' \geq 1$ then $T$ is unitary.

**Proof of Theorem 5.2:**

Let $T = T_1 \otimes T_2$ as in Theorem 5.1. Then $T_2$ is $\rho$-oid, $T_2^{-1}$ is $\rho'$-oid and $T_1$ is normal by hypothesis.

As shown in Remarks 6.1 (3)

$$r_\rho(T_2) = T_2^{\rho}$$ and $$r_\rho(T_2^{-1}) = T_2^{\rho}.$$
But by hypothesis, \( \overline{\sigma}(T_2) \) and \( \overline{\sigma}(T_2^{-1}) \) both lie on the unit circle.

Therefore \( \nu_p(T_2) = 1 \) and \( \nu_p(T_2^{-1}) = 1 \).

Using Theorems 5.3 and 5.4, \( T_2 \) is unitary.

This shows that \( T \) is a direct sum of a normal and an unitary operator i.e. \( T \) is normal.

Now we derive the following three results under essential similarity:

**Corollary 5.1:** If \( p \) and \( q \) are unequal integers, \( T \) is an invertible operator such that \( T \) is reduction \( p \)-oid and \( T^{-1} \) is reduction \( p' \)-oid for \( p \geq 1, p' \geq 1 \) and \( ST^q = T^p + K \), where \( K \) is compact and \( 0 \notin \mathcal{W}_e(\mathcal{S}) \) then \( T \) is normal.

**Proof:** Using Corollary 3.3, \( \overline{\sigma}(T) \) lies on the unit circle. Normality of \( T \) will follow from Theorem 5.2.

**Corollary 5.2:** If \( T \) is an invertible reduction-\( p \) operator such that \( ST^q = T^p + K \), \( p \neq q \), \( 0 \notin \mathcal{W}_e(\mathcal{S}) \) and \( K \) a compact operator then \( T \) is normal.
Proof: Using Corollary 3.3, \( \sigma_e(T) \) lies on the unit circle. Since \( T \in \mathcal{H}_p \) is convexoid \( [41], [43] \), \( T \) is \( \approx \)-oid and \( T^{-1} \) is \( \approx \)-oid. The desired conclusion follows from Theorem 5.3.

Corollary 5.3: If \( T \) is an invertible paranormal operator such that \( ST^* = TPS + SK \), \( p \neq q \), \( 0 \not\in \mathcal{W}_e(S) \) and \( K \) a compact operator then \( T \) is normal.

Proof: \( T \) being paranormal, \( T^{-1} \) is paranormal. Thus \( T \) and \( T^{-1} \) both are reduction \( 1 \)-oid.

The desired conclusion follows from Corollary 5.1.

In the next theorem, we obtain conditions on a reduction-convexoid operator implying normality.

Theorem 5.3: If \( T \) is a reduction-convexoid operator satisfying the conditions \( ST^* = T\delta + SK \), \( K \) a compact operator and \( 0 \not\in \mathcal{W}_e(S) \) then \( T \) is normal.

Proof: Using Corollary 3.3, \( \sigma_e(T) \) is real. If \( T = T_1 \otimes T_2 \) as in Theorem 5.1 then \( T_1 \) is normal and \( T_2 \) is convexoid by hypothesis.
Using results mentioned in [GG, Theorem G.1],

\[ \sigma_0(T) = \sigma_0(T_1) \cup \sigma_0(T_2) \]

This fact together with \( \sigma_0(T) \) real implies \( \sigma_0(T_2) \) is real.

Arguing as in Theorem 5.1, if \( z \in \mathcal{W}(T_2) \cap \sigma(T_2) \) then \( z \in \sigma_0(T_2) \subset \sigma_e(T_2) \) is real.

\( T_2 \) being convexoid, an extreme point \( z \) of \( \mathcal{W}(T_2) \) is in \( \sigma(T_2) \) and hence \( z \in \sigma_0(T_2) \subset \sigma_e(T_2) \) is real.

Thus each extreme point of \( \mathcal{W}(T_2) \) is real.

Therefore \( T_2 \) is self-adjoint.

This completes the proof.

As an immediate consequence, we have

**Corollary 5.4**: If \( T \) is a reduction-\( p \) operator satisfying the conditions \( ST^* = TS + SK \), \( K \) a compact operator and \( 0 \not\in \mathcal{W}_c(\mathbb{S}) \) then \( T \) is normal.