CHAPTER 4

SIMILARITY III

(Left invertible operators, Riesz perturbation
and Similarity)

We divide this chapter in two sections:

SECTION I.

In this section, we study spectral properties of a
left invertible operator. We derive that the spectrum
of a non-unitary isometry is the unit disc \([-1, 1] \), Coro-
liary 2, p.44]. As a deduction, an equivalent condi-
tion for an operator to be similar to a unitary and similar
to an isometry is obtained. We derive the following inter-
esting result which is analogous to M.R. Embry's result [17]:
If \( A, B, \sigma \in \mathcal{B}(H) \), \( \sigma(A) \cap \sigma(A^*o) = \emptyset \),
\[(A+B)E = \overline{S(D-A^*)} \quad \text{and} \quad AB^*s = SBA^* \]
then \( A^* = SB \) and \( S^* = D \).

Using this, we deduce that if \( \sigma(A) \cap \sigma(A^*) = \emptyset \) and
\( A+A^{-1} \) is self-adjoint then \( A \) is unitary. We obtain a
result analogous to H. Radjavi and J.P. Williams [48] and
show that if \( T \) is similar to \( T^{-1} \) then \( T+A^{-1} \)
can be expressed as a product of two self-adjoint operators
in which one is positive. It is shown that converse of
this can be established under suitable conditions.

First we obtain the spectral property for left
invertible operators as follows:

**Theorem 4.1:** If \( T \) is a left invertible operator with
left inverse \( T_1 \) and \( T \) has no right inverse then the closed
disc with centre origin and radius \( \|w(T_1)\|^{-1} \) is contained
in \( \sigma(T) \).

**Proof:** It is sufficient to prove that if \( 2 \notin \sigma(T) \) with
\[|2| \leq \|w(T_1)\|^{-1} \]
then \( T \) has a right inverse \( a \notin \sigma(T) \) implies \( T-a \) is invertible.
Case (i) : For \( z = 0 \), \( T \) is itself invertible.

Case (ii) : If \( 0 < |z| < \left[ w(T_1) \right]^{-1} \) then \( |z^{-1}| > w(T_1) \).

Therefore \( z^{-1} \notin \sigma(T_1) \) or \((T_1 - z^{-1})\) is invertible, because \( \sigma(T_1) \subset \mathcal{N}(T_1) \).

Now \( T_1 T = I \) gives us

\[
T_1(T - z) = T_1T - zT_1 = I - zT_1 = z(s^{-1} - T_1).
\]

\( T - z \) and \( T_1 - z^{-1} \) both being invertible,

\[
T_1 = s(z^{-1} - T_1)(T - z)^{-1}
\]

becomes invertible.

Thus if \( T \) has no right inverse, then any interior point of the disc with centre origin and radius \( \left[ w(T_1) \right]^{-1} \) is in \( \sigma(T) \). Closedness of \( \sigma(T) \) implies that the closed disc with centre origin and radius \( \left[ w(T_1) \right]^{-1} \) is contained in \( \sigma(T) \).

Using this, we prove the following two interesting corollaries:

Corollary 4.1 : Let \( T \) be a left invertible operator with left inverse \( T_1 \) and \( T \) has no right inverse. If
$w(T_1).w(T) = 1$ then $\sigma(T)$ is a closed disc with centre origin and radius $w(T)$ and consequently $T$ is spectralloid.

**Proof:** If $w(T_1).w(T) = 1$ i.e. $[w(T_1)]^{-1} = w(T)$ then $\sigma(T)$ is the closed disc with centre origin and radius $w(T)$ by Theorem 4.1. This fact also implies $T$ is spectralloid.

**Corollary 4.2:** Let $T$ be a left invertible operator with left inverse $T_1$ and $T$ has no right inverse. If $w(T) \leq 1$ and $w(T_1) \leq 1$ then $\sigma(T)$ is the unit disc.

**Proof:** The theorem 4.1 gives us $[w(T_1)]^{-1} \leq w(T)$ i.e. $1 \leq w(T_1).w(T)$. By hypothesis, $w(T_1).w(T) \leq 1$, $1 = \leq \omega(T)$ and $[w(T_1)]^{-1} \leq 1$ i.e. $w(T_1) \geq 1$, but $w(T_1) \leq 1$.

Therefore $w(T_1).w(T) = 1$, $w(T_1) = 1$ and $w(T) = 1$. Now the desired conclusion follows from Corollary 4.1.

Here it is interesting to observe that if $T$ is left invertible with left inverse $T_1$ then $||T|| \leq 1$

together with $||T_1|| \leq 1$ gives us the isometricity
of $T$. It is well known that if $T$ is invertible then

$$\| T^{-1} \| \leq 1$$

gives the unitarity of $T$. J.O. Stampfli [39] generalised this result and proved that $T$ is Unitary even if $w(T^{-1}) \leq 1$. Here we have analogous natural question as follows:

Is $T$ isometry under the hypothesis of Corollary 4.2 namely $w(T_1) \leq 1$ and $w(T_2) \leq 1$?

The previous corollary enables us to describe the spectrum of a non-unitary isometry operator as follows:

**Corollary 4.3**: If $V$ is an isometry on $H$ such that $z \notin \sigma(V)$ for some $z$ with $|z| \leq 1$ then $V$ is isometry. Consequently, if $V$ is a non-unitary isometry operator then $\sigma(V)$ is the unit disc.

**Proof**: Here $V^*V = I$ i.e. $V^*$ is a left inverse of $V$, $\| V \| = \| V^* \| = 1$ and $w(V^*) = w(V) \leq 1$. The desired conclusion follows from Corollary 4.2.

We remark that C.R. Putnam has obtained this result using spectral integral theory.
Now the following result of E. Fan [13, Corollary 3, p.308] regarding equivalent conditions on similarity follows immediately.

**Corollary 4.4**: The following two statements are equivalent if \( \sigma(T) \) is properly included in the unit disc:

1. \( T \) is similar to a unitary operator
2. \( T \) is similar to an isometry operator.

Using this, we derive the following:

**Corollary 4.5**: Let \( S, T \in B(H) \) such that \( T^*S = S \) and \( 0 \notin \sigma(S) \). If at least one point of the unit disc is not in \( \sigma(T) \), then \( T \) is similar to a unitary.

**Proof**: Using a result of S.M. Patel [38, Theorem 1], \( T \) is similar to isometry say \( A \). Since \( \sigma(T) = \sigma(A) \) is properly included in the unit disc, \( A \) is unitary by Corollary 4.4. Thus \( T \) is similar to unitary.

Now we generalise a result of S.M. Patel [38, Theorem 3] regarding a left invertible operator as follows:
Theorem 4.2: Let $T$ be a left invertible operator with left inverse $T_1^*$. For distinct positive integers $p$ and $q$, if $T$ satisfies the relation $T^{P}T_1^*P = T_1^*$ and $P$ an invertible self-adjoint operator then $T$ is similar to a unitary operator.

Proof: Without loss of generality, we assume $p > q$.

Here $p^{-1}T^P = T_1^*P$ and $P^{-1}P = T_1^*P$.

Now $T_1^*P = 1$ gives us $T_1^*P = T_1^*P$.

Therefore $T_1^*P = (T_1^*P)^P = P^{-1}P^{-1}$ and

$T^{P} = (T_1^*P)^P = P^{-1}P^{-1}$.

Now $1 = T_1^*P$, $T = P^{-1}P^{-1}$

Thus $T_1^*P = 1$ i.e. $T_1^*P = 1$.

Therefore $T_1^*P = 1$.

By spectral mapping theorem, $2^{P^2} = 1$ for each $x \in \sigma(T)$. This shows that $\sigma(T)$ is finite and lies on the unit circle.
$T^2-q^2$ being normal and $\sigma(T)$ lying on the unit circle, $T$ is similar to a unitary operator by a result of K. Kurosa [31].

M.R. Engly [17, Theorem 1] has proved the following Theorem:

**Theorem 4-A:** For $A$, $B$, $S \subseteq \mathfrak{U}(H)$, if $\sigma(A) \cap \sigma(B)=\emptyset$, then $S$ commutes with each of $A$ and $B$ if and only if $S$ commutes with each $A+B$ and $AB$.

In this direction, we prove the following result, proof of which is analogous to Theorem 4-A.

**Theorem 4-B:** Let $A$, $B$, $S \subseteq \mathfrak{U}(H)$ and $\sigma(A) \cap \sigma(A^*)=\emptyset$. If $(A+B^*)S = S(B+A^*)$ and $AB^*S = SBA^*$ then

$AB = SB$ and $SA^* = BA^*$.


Thus $A(AS-SB) = (AS-SB)A^*$. 
Now by Theorem 2.6, \( \sigma(A) \cap \sigma(B^*B) = \emptyset \) gives us 
\[ AB = BA. \]

Again, 
\[
\lambda(SA^* - B^*B) - (SA^* - B^*B)A^*
\]
\[
= ASA^* - AB^*B - B BAB^* + B^*SA^*
\]
\[
= ASA^* + B^*SA^* - B^*B - B^*A^*
\]
\[
= (A + B^*)SA^* - (B + A^*)A^*
\]
\[
= S(A + B^*)A^* - S(B + A^*)A^* = 0 \text{ by hypothesis.}
\]

Thus 
\[ A(SA^* - B^*B) = (SA^* - B^*B)A^* \]
and 
\[ \sigma(A) \cap \sigma(A^*) = \emptyset \]
gives us 
\[ SA^* = B^*B \] by Theorem 2.6.

Taking \( S = I \), we derive

**Corollary 4.6**: Let \( A \) and \( B \) be two operators such that 
\[ \sigma(A) \cap \sigma(B^*) = \emptyset, \] \( A + B^* \) and \( AB^* \) are self-adjoint 
then \[ A = B. \]

Using Theorem 4.3, we obtain sufficient condition for an operator to be unitary as follows:

**Corollary 4.7**: For \( A \in \mathcal{B}(H) \), if \( \sigma(A) \cap \sigma(A^*) = \emptyset \) 
and \( A + A^{-1} \) is self-adjoint then \( A \) is unitary.
PROOF:  As \( A + A^{-1} = A^* + A^{-1} \),

\[
(L + L^{-1})A = L(A + A^{-1}) = L(A^* + A^{-*}) = L(A^{*-1} + A^*) .
\]

Taking \( B^* = A^{-1} \), \( AB^* = B_A^* = I \).

Therefore \( (A + B^*)A = L(B + A^*) \) and \( (AB^*)A = L(D_A^*) \).

Now the condition \( \sigma(A) \cap \sigma(A^*) = \emptyset \) gives us \( AA = AB \) by Theorem 4.3 i.e. \( A = B = A^{-1} \).

**Corollary 4.8:** If \( A \) is an operator such that \( \sigma(A) \cap \sigma(A^*) = \emptyset \) and \( A^2 \) is self-adjoint then \( \text{Re } A = 0 \).

**Proof:** Taking \( S = A^* \) and \( B^* = -A \) in Theorem 4.3, we have \( (A + B^*)S = S(B + A^*) = 0 \).

\( A^2 \) being self-adjoint, \( A^2 A^* = (A^2 A^*)^* = A^* A^2 \)
i.e. \( ABA^* = SBA^* \).

Therefore \( AS = SB \) i.e. \( AA^* = -A^2 \).

Since \( A \) is invertible, \( A = -A^* \) i.e. \( \text{Re } A = 0 \).

The following theorem gives us sufficient
condition to insure that if an operator $T$ is similar to its adjoint then $T$ can be expressed as a product of two self-adjoint operators in which one is a positive operator.

**Theorem 4.1** [48]: The following are equivalent conditions on an operator $T$:

1. $T$ is similar to a self-adjoint operator.
2. $T = PA$ where $P$ is positive and invertible and $A$ is self-adjoint.
3. There is $S \in \mathcal{B}(H)$ such that $S^{-1}TS = T^*$ and $0 \not\in \mathcal{W}(S)$.

Now we obtain sufficient condition analogous to Theorem 4.1 to insure that if $T$ is similar to $T^{-1}$ then $T^{-1}$ can be expressed as a product of two self-adjoint operators one of which is positive.

**Theorem 4.4**: For an invertible operator $T$ if

1. $T$ is similar to a unitary operator.
2. There is an operator $S$ such that $S^{-1}T^{-1}S = T^*$ and $0 \not\in \mathcal{W}(S)$
3. $T^{-1} = PA$, where $P$ is positive and invertible and $A$ is self-adjoint,
then (1) and (3) are equivalent and (2) implies (3).

If \( \sigma(T) \cap \sigma(T^*) = \emptyset \) then (3) implies (1).

**Proof:** (1) and (2) are equivalent by \([56]\).

First, we prove (2) implies (3).

Since \( S^{-1}T^{-1}S = T^n \), \( S^{-1}TS = T^{*-1} \).

Therefore \( T^{-1}S = ST^* \) and \( TS = ST^{*-1} \).

Consequently, \( (T+T^{-1})S = S(T+T^{-1})^{*-1} \).

As \( 0 \notin \sigma(S) \), there is a positive invertible operator \( P \) and a self-adjoint operator \( A \) such that \( T+T^{-1} = PA \) by Theorem 4.3.

Now we prove (3) implies (1) as follows:

Let \( T+T^{-1} = PA \), where \( P \) is a positive invertible operator, \( A \) is a self-adjoint operator and \( \sigma(T) \cap \sigma(T^*) = \emptyset \).

Therefore \( P^{-1/2}(T+T^{-1})P^{-1/2} = P^{1/2}AP^{1/2} \) is self-adjoint.

If \( P^{-1/2}TP^{1/2} = B \) then \( B+B^{-1} \) is self-adjoint.

Since \( T \) is similar to \( B \) and \( \sigma(T) \cap \sigma(T^*) = \emptyset \),
\( \sigma(B) \cap \sigma(B^*) = \emptyset \).

Now by Corollary 4.7, \( B \) is unitary. This completes the proof.
As shown by R. Radjavi and P. Rosenthal [47], if \( P \geq 0 \) and \( A \) self-adjoint then \( PA \) has non-trivial invariant subspace. Combining this result with Theorem 4.3 and Theorem 4.4, we have

**Corollary 4.9**: If \( T \) is similar to a self-adjoint (or a unitary) operator then \( T (or \ T+T^{-1}) \) has non-trivial invariant subspace.

**Section 2.**

In this section, we examine the behaviour of Ricza perturbation and extend theorems on compact perturbation under similarity conditions.

First we give below one interesting result in which a quasi-nilpotent operator becomes zero under suitable conditions.

**Theorem 4.6**: If \( M \) is an invertible operator such that

\[ M^p - M^q = 0 \]

is quasi-nilpotent, where \( p \) and \( q \) are distinct integers and if \( M \) and \( M^{-1} \) both are spectrally then \( M \) is unitary and \( q = 0. \) Moreover \( M^{p+q} = I \).
Proof: \( N \) being spectraloid, \( w(N) = |z| \) for some \( z \in \sigma(N) \). Also by \([27]\), there is a sequence \( \{x_n\} \) of unit vectors such that \( \|(N^m - z^m)x_n\| \to 0 \) and
\[
\|(N^m - z^m)x_n\| \to 0
\]
for every integer \( m \).

Now
\[
\| \sum (N^p - z^q) (z^p - z^q) x_n \| = \| (N^p - z^p) x_n - (z^q - z^q) x_n \| \\
\leq \| (N^p - z^p) x_n \| + \| (z^q - z^q) x_n \| \to 0 + 0 = 0 .
\]

Therefore,
\[
s^p - z^q \in \sigma(N) - \sigma(z) = \{0\} \quad \text{by hypothesis}.
\]

Thus \( s^p = z^q \).

Now \( p \neq q \) gives us \( |z| = 1 \) and \( w(N) = |z| = 1 \).

With similar arguments, we get \( w(N^{-1}) = 1 \).

Now \( N \) is unitary by Theorem 3-P.

\( N \) being unitary, \( N^p - z^q = 0 \) is normal and so \( z = 0 \) i.e. \( N^p = z^q \) or \( N^p = z^{-q} \) i.e. \( N^p + q = 1 \).

Remark 4.1: In fact, we prove in Theorem 4.5 that if \( z \) is an approximate eigenvalue of \( N \) then \( |z| = 1 \).
As an immediate consequence of Theorem 4.5, we have the following two corollaries:

**Corollary 4.10:** If \( N \) is an invertible operator satisfying the condition \( G_1 \) such that \( N^{P}N^{Q} \) is quasi-nilpotent where \( p \neq q \), then \( N \) is unitary and \( N^{P}N^{Q} = I \).

**Corollary 4.11:** If \( N \) is an invertible paranormal operator such that \( N^{P}N^{Q} \) is quasi-nilpotent, where \( p \neq q \), then \( N \) is unitary and \( N^{P}N^{Q} = I \).

Using Theorem 4.5, we have corresponding result in Calkin algebra as follows:

**Corollary 4.12:** Let \( N \) be an invertible operator such that \( N^{P}N^{Q} \) is a Riesz operator, where \( p \neq q \). If \( \sigma_{oo}(N) = \emptyset \) and \( N \) and \( N^{-1} \) both are spectraloid then \( N \) is unitary.

**Proof:** As \( \bar{\sigma}(N) \subset \sigma_{k \ell}(N) \cup \sigma_{oo}(N) \) [45], the condition \( \sigma_{oo}(N) = \emptyset \) and \( N \) a spectraloid gives us the existence of a complex number \( z \) in \( \bar{\sigma}(N) \subset \sigma_{k \ell}(N) \).
such that $w(\mathbb{N}) = \{1\}$ Moreover there exists a sequence
$\{x_n\}$ of unit vectors such that $x_n \to 0$ weakly and
\[(N^{\ast} - N)x_n \to 0 \quad \text{implies} \quad (N^{\ast} - N)x_n \to 0. \tag{27}\]
Arguing as in Theorem 4.5, $x^p = x^q \quad \text{i.e.} \quad w(N) = 2^r = 1$.
Similarly, we can prove that $w(N^{-1}) = 1$.
Again $N$ is unitary by Theorem 3-F.

In our next theorem, we show that the spectral condition in Theorem 4.5 can be replaced by commutative properties.

**Theorem 4.6**: For distinct integers $p$ and $q$, if $N^p = N^q$ is a quasi-nilpotent operator $N$ such that $N$ commutes with $N$ and $N^p$ both then $N^p = N^q$. If $N$ is invertible then $N^{p+q} = 1$. Moreover $N$ is similar to a unitary operator.

**Proof**: $N^p = N^q = N$ gives us

\[N^{p+q} = N^{q} \quad \text{and} \quad N^{p+q} = N^{q}.\]

Now the condition $N^{q} = N^{q}$ gives
\[ N^p q = N^q n^p = N^{p+q} \] 

i.e. \( N^q \) is normal

and consequently \( N^{p+q} \) is normal.

Similarly, the relation \( \bar{N}^{p+q} - N^{p+q} = Q^{p+q} \)

and \( \bar{N}^{p+q} - N^{p+q} = N^{p+q} \) and the condition

\( N^{p+q} = Q^{p+q} \) gives normality of \( N^p \). Obviously \( N^p \)

and \( N^q \) are commutative. Therefore \( N^{p+q} \) is normal.

It being quasi-nilpotent \( Q \), we have \( q = 0 \), \( N^p = N^q \)

and \( N^{p+q} = N^p N^q \).

Now \( 1 N^p = N^p 1 \) and \( 0 \notin \bar{W}(I) \) gives us

\( z^p = z^q \) for each \( z \in \sigma(N) \) \( - \{ 0 \} \) by Theorem 3-3.

Thus if \( 0 \notin \sigma(N) \) then \( |z| = 1 \) and hence \( \sigma(N) \)

lies on the unit circle. As \( N^q \) is normal, this gives

\( N^q \) is unitary. Thus \( N^{p+q} = N^q N^p = I \). The remaining

part regarding similarity follows from a result of

K. Kurepa [ 31 ].

As a direct consequence, we have

**Corollary 4.12:** For distinct integers \( p \) and \( q \) if

\( N^p - N^q \)

is a Riesz operator \( K \) such that \( K \) commutes

with \( N \) and \( N^q \) both and \( \sigma(N) = \emptyset \) then \( N^{p+q} = I \).
Using Theorem 4.6, we are in position to extend a result mentioned in Remarks 3.1 of Corollary 3.3 under a different condition on $S$. In this direction, we have

**Corollary 4.14**: For distinct integers $p$ and $q$, if $S$ is a positive invertible operator, $Q$ a quasi-nilpotent operator such that $ST^p = T^q S + SQ$ and $Q$ commutes with both $T$ and $T^*$ then $\sigma(T) \setminus \{0\}$ lies on the unit circle. If $|p| \neq |q|$ then $\sigma(T)$ is finite. Moreover, if $T$ is invertible then $T$ is similar to a unitary operator.

**Proof**: The relation $ST^p = T^q S + SQ$, $S > 0$ gives us

$$s^{1/2}T^pS^{-1/2} = s^{1/2}T^q S^{1/2} + s^{1/2}QS^{-1/2}$$

i.e. $(s^{1/2}T^pS^{-1/2})^q = (s^{1/2}T^q S^{1/2})^q + Q_1$

where $Q_1 = s^{1/2}QS^{-1/2}$ is also quasi-nilpotent.

Taking $A = s^{1/2}TS^{-1/2}$, we have $A^p - A^q = Q_1$.

Also $A Q_1 = s^{1/2}TS^{-1/2} s^{1/2}QS^{-1/2} = s^{1/2}TS^{1/2} = s^{1/2}QS^{-1/2} = Q_1 A$.

Similarly, we also have $A^* Q_1 = QA^*$. 
Using Theorem 4.6, $Q_1 = 0$ and $A^P = A^{aQ}$.

If $A$ is invertible then we also have $A^{p+q} = I$.

Argued as in Theorem 4.7, we have the desired conclusion.

In the Calkin algebra, the following corollary is an extension of corollary 3.3 of chapter 3:

**Corollary 4.15:** For $\Theta, T, K \in \mathcal{O}(H)$ and for distinct integers $p$ and $q$, if $\Theta T^p = T^{-q} + S^2$, $S$ a positive and invertible operator and $K$ a Riesz operator commuting with $T$ and $T^*$ both then $\sigma_0(T) - \{0\}$ lies on the unit circle.

**Proof:** As in Corollary 4.14, if $A = \Theta \sqrt{-1}S^{-1}\sqrt{-1}$ and $K_1 = \Theta \sqrt{-1}S^{-1}\sqrt{-2}$ then $A^P - A^Q = K_1$ and $K_1$ a Riesz operator commuting with both $A$ and $A^*$. Arguing as in Theorem 4.6, $K_1 = A - A^*$ is normal. Since a Riesz normal operator is always compact [43, Theorem 1], $A^P - A^Q$ is compact and for each $z \in \sigma_0(A)$, $z^p = z^{-q}$ by Corollary 3.3. For $z \neq 0, p \neq q$, we have $|z|=1$.

Thus $\sigma_0(T) - \{0\}$ lies on the unit circle.
Remarks 4.3: If $T$ is paranormal or $T$ satisfies the condition $G_1$ in (Corollary 4.14) or (Corollary 4.15 with $\sigma_\infty(T) = \emptyset$) then $T$ becomes unitary.