In this chapter, we discuss several conditions implying self-adjointness, normality or unitaryness of an operator under similarity. We show that a hyponormal operator similar to its adjoint or similar to the inverse of its adjoint becomes self-adjoint or unitary under different set of conditions. Conditions on hyponormal, dominant and paranormal operators implying normality are also discussed. We obtain equality of two convexoid operators under a set of conditions and deduce equality of commuting hyponormal operators. We extend M.I. Emery's result regarding commuting normal operators and
obtain equality of hyponormal and co-hyponormal operators. Using this, equality of two unitarily equivalent operators is also discussed.

W.A. Beck and C.R. Putnam [8] proved the following theorem implying self-adjointness of an operator under similarity:

**Theorem 2.4:** If \( U \) is a cramped unitary operator and \( N \) a normal operator such that \( NU = UN^* \) then \( N = N^* \).

E.K. Brownian [3] has pointed out rightly that the normality condition on \( N \) can be dropped in the above Theorem 2.4.

First we show that Theorem 2.4 is also valid under a different set of conditions. In fact, we have

**Theorem 2.1:** Let \( \sigma \in \mathcal{U}(H) \) be unitary. If for \( N \in \mathcal{U}(H) \), \( NA = AN^* \) then any one of the following conditions gives \( N = N^* \).

1. \( \text{Im } N > 0 \) or \( \text{Im } N < 0 \)
2. \( 0 \notin \mathcal{W}(A) \)
3. \( \sigma(A) \cap \sigma(-A) = \emptyset \)
4. \( 0 \notin \overline{\sigma(A)} \) [8]
To prove this theorem, we need the following known results:

**Theorem 2.1.** [8]: Let an arbitrary operator $A$ and a normal operator $N$ satisfy $NA = NA^*$. If, whenever $z$ is not real, either $z$ or its conjugate $z^*$ does not lie in $\sigma(N)$, then $AN = NA$.

In particular, if $A$ is one-to-one or has dense range then $N = N^*$.

**Theorem 2.2.** [16]: If $N$ and $K$ are commuting normal operators such that $NK = KN$ and $0 \notin \sigma(N)$ then $N = K$.

**Theorem 2.3.** [46, Corollary 0.13], [19, Lemma 2.3]: If $A$, $N$, $B$ are operators such that $AN = NB$ and $\sigma(A) \cap \sigma(B) = \emptyset$ then $N = 0$.

In particular, if $\sigma(A) \cap \sigma(-A) = \emptyset$ and $NN = -NA$ then $N = 0$ [16].

**Theorem 2.4.** [70]: For $A$, $N \in \sigma(N)$, if $NA = NM^*$ and $0 \notin \overline{\sigma(A)}$ then $N$ is similar to a self-adjoint operator.
In particular, if \( N \) is convexoid then \( N = N^* \).

**Proof of Theorem 2.1:**

As \( A \) is unitary,

\[
HA = AN^* \quad \text{gives us} \quad A^* H = H A^* .
\]

Taking adjoint, \( N^* A = AN \).

Thus \( (N-N^*)A = A(N^*-N) \). \( \cdots (a) \)

Now \( \frac{d}{dt} (N-N^*) \) is self-adjoint and \( \sigma (N-N^*) \) lies on the imaginary axis.

1. If \( \text{Im} N \geq 0 \) or \( \text{Im} N \leq 0 \) then the normal operator \( N-N^* \) satisfies the condition of
   *Theorem 2-B*.

   Hence \( (N-N^*)A = A(N-N^*) \). \( \cdots (b) \)

   Now (a) and (b) gives \( A(N-N^*)=A(N-N^*) \).

   \( A \) being invertible, this gives

   \( N^* - N = N-N^* \) or \( N = N^* \).

2. As \( 0 \notin \mathcal{W}(A) \), Theorem 2-C gives us

   \( N-N^* = N^* - N \) from (c).

   Therefore \( N = N^* \).
(3) \( A A^* = (N-H) N^* \) \( A = -(N-H^*) \) \( \) from (a)

and \( \sigma(A) \cap \sigma(-A) = \emptyset \),

\( N - N^* = 0 \) by Theorem 2-1.

So \( N = N^* \).

(4) Since \( 0 \not\in \overline{W(A)} \) and \( N - N^* \) is normal and in particular convexoid, the desired conclusion follows from Theorem 2-1.

This completes the proof.

From Theorem 2-1, replacing unitaryness of \( A \)
by \( N A = AN \), we have another set of conditions implying self-adjointness of \( N \) as follows:

Corollary 2-1: If \( A \) and \( N \) are operators such that \( NA = AN^* \) and \( N^* A = AN \) then any one of the following conditions gives \( N = N^* \).

(1) \( \) (\( \text{Im} N \geq 0 \) or \( \text{Im} N \leq 0 \)) and \( A \) is one-to-one or has dense range.

(2) \( 0 \not\in \overline{W(A)} \)

(3) \( \sigma(A) \cap \sigma(-A) = \emptyset \)

(4) \( 0 \not\in \overline{W(A)} \).
Proof: As $NA = AN^*$ and $N^*A = AN$, 

$$(N - N^*)A = A(N^* - N).$$

Arguing as in the proof of Theorem 2.1, we have

the desired conclusion.

If $N$ is hyponormal in Theorem 2-1 then $N$ is

self adjoint [53]. Using Corollary 3.1, this deduc-
tion can also be obtained under additional conditions

as follows:

Corollary 2.2: If $N$ is a hyponormal operator such

that $NA = AN^*$ then any one of the following conditions
gives $N = N^*$.

(1) $(\text{Im} N \geq 0$ or $\text{Im} N \leq 0)$ and $A$ is

one-to-one with dense range

(2) $0 \notin \sigma(A)$

(3) $\sigma(A) \cap \sigma(-A) = \emptyset$

(4) $0 \notin \overline{U(A)}$ [53].

To prove this corollary, we need the following

result of J. C. Stampfli and D. L. Nadwina [63, Theorem 1,
p. 145].
Theorem 2.7: Let \( T, S, \phi \in \mathcal{B}(H) \). If \( T \) is dominant, 
\( S \) is co-hyponormal and \( \phi \) is one-to-one with dense range such that \( \phi \omega = \omega S \) then \( T \) and \( S \) are both normal.

Proof of Corollary 2.2:

Here \( N \) is hyponormal, \( N^* \) is co-hyponormal. Now in all the four conditions for \( A, \phi \) is one-to-one with dense range. Hence \( N \) is normal by Theorem 2.7.

\( N \) being normal, the desired conclusion follows from Corollary 2.1.

We give below an example to show that \( N \) can not be chosen as co-hyponormal operator except for the case (4) in Corollary 2.2.

Example 2.1: Let \( \{ x_n \}_{n=-\infty}^{\infty} \) be an orthonormal basis for an infinite dimensional separable Hilbert space \( H \).

If \( T \phi x_n = r x_{n+1} \) for \( n < 0 \)
  \[ = x_{n+1} \text{ for } n \geq 0 \]
with \( 0 < r < 1 \),

then \( T \) is non-normal hyponormal.
Let $A x_n = r^{-1} x_{-(n-2)}$ for $n < 0$.

$= x_{-(n-2)}$ for $n = 0, 1, 2$

$= r^{n-2} x_{-(n-2)}$ for $n \geq 3$

where $0 < r < 1$.

The operator $A$ is one-to-one with dense range and
in fact is self-adjoint.

Since $T^* x_n = r x_{n-1}$ for $n \leq 0$

$= x_{n-1}$ for $n > 0$,

taking $N = T^*$, we have

$NN = NN^*$ and $N$ is co-hyponormal.

This shows that corollary 2.3 is not valid for
co-hyponormal operator $N$.

Using Cartesian decomposition of an operator and
corollary 2.2, we have

**Corollary 2.4**: Let $T = A + iB$ be a Cartesian decompo-
sition of $T$. If $A$ satisfies any one of the four
conditions of corollary 2.3 and $A^* B$ is hyponormal
then $T$ is normal.
Proof: Taking $N = AB$, we have

\[ N^* = BA \quad \text{and} \quad NA = (AB)A = A(BA) = AN^*. \]

Using corollary 2.2, we have $N = N^*$ or $AB = BA$.

This ensures the normality of $T$.

U.N. Singh and Kanta Nagla \[56\] proved the following theorem implying unitarity of operators under similarity.

**Theorem 2.0:** If $U$ is a cramped unitary operator and $N$ an operator such that $NU = UN^{*-1}$ then $N$ is unitary.

In the next theorem, we prove a result analogous to Theorem 2.1 regarding unitarily equivalent operators $N$ and $N^{*-1}$. In fact, we have

**Theorem 2.2:** Suppose $A \in \sigma(N)$ is unitary. If for an invertible operator $N \in \sigma(N)$, $NA = AN^{*-1}$ then any one of the following conditions gives unitarity of $N$.

1. $\operatorname{Im}(N-N^{*-1}) \geq 0$ or $\operatorname{Im}(N-N^{*-1}) \leq 0$

   and $\sigma(N) \cap \sigma(N^*) = \emptyset$.

2. $0 \notin \mathbb{W}(A)$

3. $\sigma(A) \cap \sigma(-A) = \emptyset$

4. $0 \notin \overline{\sigma(A)}$ \[56\].
For proving Theorem 2.2, we shall make use of the following results which we prove as a lemma:

**Lemma 2.1:** If \( A \in \mathfrak{u}(H) \) is normal and \( P \) an arbitrary operator such that \( PA = -AP \) then any one of the following conditions gives \( P = P^* \).

1. \( \text{Im} \, P \geq 0 \) or \( \text{Im} \, P \leq 0 \)
2. \( 0 \notin \sigma(A) \)
3. \( \sigma^-(A) \cap \sigma^+(A) = \emptyset \)
4. \( 0 \notin \overline{\sigma(A)} \)

**Proof:** \( A \) being normal,

\[ PA = -AP \] gives us \( PA^* = -A^*P \).

Taking adjoint to both sides, \( A^*P^* = -P^*A \).

Thus \( (P^* - P)A = A(P^* - P) \).

This relation gives the desired conclusion by using Theorems 2-3, 2-C, 2-D and 2-E.

**Proof of Theorem 2.2:**

\( A \) being unitary,

\[ HA = HB^{*-1} \] gives us \( A^*B = B^{*-1}A^* \).
Taking adjoint here, \( N^0A = AN^{-1} \).

Here pre-multiplication by \( N^{* -1} \) and post multiplication by 1 gives us \( N^{* -1}A = AN \).

Now \( (N-N^{* -1})A = A(N^{* -1} - N) \)
\[ = (-A)(N-N^{* -1}) \]

Taking \( P = N-N^{* -1} \), we have \( PA = -AP \).

Now using Lemma 2.1, \( P = P^* \)
  
  i.e. \( N-N^{* -1} \) is self-adjoint.

(1) \( N-N^{* -1} \) being self-adjoint,

\[ N-N^{* -1} = N^* - N^{-1} = P \]

Therefore \( N^{\circ}N-I = N^*P = PN \).

Since \( \sigma(N) \cap \sigma(N^*) = \emptyset \),

\( P = 0 \) by Theorem 2-D.

Now \( P = 0 \) implies \( N = N^{* -1} \) i.e. \( N \) is unitary.

(2) \( N-N^{* -1} \) being self-adjoint,

\[ (N-N^{* -1})A = A(N^{* -1} - N) \] and \( 0 \notin W(A) \)

gives us \( N-N^{* -1} = N^{* -1} - N \) i.e. \( N-N^{* -1} \) by Theorem 2-C.
(3) Using Theorem 2.2,

\[(H - \lambda^{-1})A = (-\lambda)(H - \lambda^{-1}) \text{ and } \sigma(A) = \sigma(-A) = \emptyset\]

gives us \(H - \lambda^{-1} = 0\) i.e., \(H = \lambda^{-1}\).

(4) For a cramped unitary operator \(A\) i.e., \(0 \notin \overline{\sigma(A)}\), we also have \(\sigma(A) \cap \sigma(-A) = \emptyset\).

Now the part (4) follows from part (3).

From Theorem 2.2, replacing unitaryness of \(A\) by \(\lambda H = H^{-1} A\), we have another set of conditions implying unitaryness of \(H\).

**Corollary 2.4:** For \(A, H \in \mathcal{A}(a)\), if \(H\) is an invertible operator such that \(HA = H^{\sigma^{-1}}\) and \(AH = H^{\sigma^{-1}} A\) with the conditions of Theorem 2.2 then \(N\) is unitary.

**Proof:** As \(HA = HA^{\sigma^{-1}}\) and \(A^{\sigma^{-1}} = AH\),

\[(H - \lambda^{-1})A = A(H - \lambda^{-1}) \quad \ldots \text{(a)}\]

Pre-multiplying by \(H^{-1}\) and post-multiplying by \(H^{\sigma}\) in the relation \(HA = H^{\sigma^{-1}}\), we get

\[\lambda H^{\sigma} = H^{-1} A \quad \ldots \text{(b)}\]
Similarly the relation \( MN = N^{-1} \) gives us

\[
N^*M = MN^{-1}
\]  

... (c)

Taking adjoints in (b) and (c), we have

\[
AA^* = A^*A^{-1} \quad \text{and} \quad \lambda^* = \lambda^{-1}.
\]

Therefore \((M-H^{-1})A^* = A^*(M^{-1}-i)\)  

... (d)

Letting \( H = H^{-1} = P \), from (a) and (b) we have

\[
P = A^*P \quad \text{and} \quad P^* = -A^*P \quad \text{i.e.} \quad AP^* = -P^*A.
\]

Therefore \((P-P^*)A = -A(P-P^*)\).

Arguing as in Theorem 2.1, \( P = P^* \).

Again arguing as in Theorem 2.2, this relation gives the desired conclusion.

In our next result, we obtain condition analogous to condition given in corollary 2.2 to ensure unitaryness of an invertible hyponormal operator.

Corollary 2.6: For \( A, N \in \mathfrak{N} \), if \( N \) is an invertible hyponormal operator with the conditions of Theorem 2.2 then \( N \) is unitary.
Proof: \( N \) being hyponormal,

\[ N^{-1} \text{ is hyponormal by } \left[ 87 \right] \text{ and hence} \]

\( N^{n-1} \) is co-hyponormal. Using Theorem 3.21, \( N \) is normal.

Therefore by Fuglede's Theorem [44],

\[ NH = AN^{n-1} \]

Gives \( N^* A = AN^{-1} \) i.e. \( N^* A = AN \).

Now the desired conclusion follows from Corollary 3.4.

J.C. Stampi [57] proved that if \( T \) is hyponormal and \( T^n \) is normal for some \( n \) then \( T \) is normal. E. Ando extended this result for paranormal operator [11]. Using this E. Ando's result, we derive the following corollaries.

Corollary 3.5: Let \( N, A \in \mathcal{B}(H) \) be paranormal and unitary respectively. If \( N^p = A^{-1} N^D A \) for some non-zero integer \( p \) (\( N \) is invertible if \( p \) is negative). Then any one of the following conditions gives the normality of \( N \).

1. \( \text{Im} N^2 \leq 0 \) or \( \text{Im} N^D \leq 0 \); 2. \( \sigma(N) \cap \sigma(-A) = \emptyset \);
3. \( \sigma(A) \cap \sigma(-A) = \emptyset \); 4. \( A \) is cramped.
Proof: Using Theorem 3.1, $D^p$ is self-adjoint. The desired conclusion follows from T. Ando's result [1].

Corollary 2.7: Let $D$ be an invertible paranormal operator and $A$ be a unitary operator. If $D^p = A^{-1}$ then any one of the following conditions gives unitary-ness of $D$.

1. \[ \text{Im}(D^p - D^{-p}) \geq 0 \text{ or } \text{Im}(D^p - D^{-p}) \leq 0 \] and $\sigma(D^p) \cap \sigma(D^{-p}) = \emptyset$

2. $0 \notin \sigma(A)$

3. $\sigma(A) \cap \sigma(-A) = \emptyset$

4. $0 \notin \sigma(A)$.

Proof: Using Theorem 2.2, $D^p$ is unitary. Therefore $\sigma(D)$ also lies on the unit circle.

Since a paranormal operator with spectrum lying on the unit circle is unitary [31], $D$ is unitary.

It is shown by K. Kurepa [31] that for an operator $T$, if $T^n$ is normal for some $n$ then $T$ is similar to
a normal operator. Using this result and Theorem 2-8, we generalise, the above mentioned result of J.C. Stampfl. [57] for dominant operators as follows:

**Theorem 2.9**: If \( T \) is a dominant operator such that \( T^n \) is normal for a non-zero integer \( n \) then \( T \) is normal (\( T \) is invertible if \( n \) is negative).

**Proof**: By a Theorem of Euoph, there are an invertible operator \( A \) and a normal operator \( N \) such that \( N = A^{-1} T A \). Now \( TA = AM \), \( T \) dominant, \( N \) normal and \( A \) invertible, Theorem 2-3 gives the normality of \( T \).

Now it is clear that Corollary 2.6 and 2.7 are also valid for a dominant operator \( N \).

Replacing unitaryness of \( A \), we further have the following result for dominant operator under the condition \( 0 \notin \overline{W(A)} \) only.

**Corollary 2.8**: If \( U \) or \( U^* \) is a dominant operator such that \( N^p = AN^{p}A^{-1} \) where \( p \) is a non-zero integer (\( N \) is invertible if \( p \) is negative) and \( 0 \notin \overline{W(A)} \) then \( N \) is normal.
Proof: $N^D$ is similar to a self-adjoint operator by Theorem 2-1. Therefore there is an invertible operator $S$ and a self-adjoint operator $T$ such that $S^{-1}P_S ST$ i.e. $(S^{-1}N^D)^D$ is in particular normal. Using above mentioned result of K. Kurepa [31], $S^{-1}N^D$ is similar to a normal operator. Thus $N$ is similar to a normal operator. The desired conclusion follows from Theorem 2-F.

We further have a result analogous to Corollary 2.8 to ensure unitarity as follows:

Corollary 2.9: For an invertible operator $N$, if $N$ or $N^*$ is dominant (or paranormal), $N^D = SN^{-D} S^{-1}$ and

$0 \notin \sigma(N)$ then $N$ is unitary.

Proof: $N^D$ being similar to a unitary operator by [56]; $\sigma(N)$ lies on the unit circle. Arguing as in corollary 2.7, $N$ is similar to a normal operator. Now Theorem 2-F gives the normality of $N$. $N$ being normal with spectrum on the unit circle, $N$ is unitary.
In the case of paranormal operator $H$, the spectrum of $H$ being on the unit circle, $H$ is unitary.

S.M. Patel proved the following theorem for similarity of spectral operators:

**Theorem 2-H.** [40, Theorem 3]: Let $E$ be a non-singular spectral operator. If $A^* = E^{-1}A$ where $A$ is self-adjoint such that $0 \not\in W(A)$ then $E$ is similar to a unitary operator.

In the next theorem, we prove a result analogous to Theorem 2-H for dominant operators by using a result of K.G. Louglas [12] which we state first as follows:

**Theorem 2-I**: Let $A$ and $B$ be bounded operators on $H$.

The following statements are equivalent:

1. $R(A) \subseteq R(B)$, $R$ denotes range space
2. $A^*A \leq \lambda^2 B^*B$ for $\lambda > 0$
3. There is $C \in \sigma(B)$ such that $A = BC$.

Moreover, if (1), (3) and (3) are valid, then there is
a unique operator \( C \) such that

(a) \( \| C \| = \inf u : A^* A \leq u B B^* \)

(b) \( \mathcal{N}(A) = \mathcal{N}(C) \), \( \mathcal{N} \) denotes null space

(c) \( \mathcal{R}(C) \subset \overline{\mathcal{R}(B^*)} \).

**Theorem 2.4**: Let \( T \) be an invertible dominant operator and \( S \) be a self-adjoint operator such that \( ST^* = T^{-1/2} \) and \( 0 \notin \mathcal{W}(S) \). Then \( T \) is unitary.

**Proof**: Since \( 0 \notin \mathcal{W}(S) \) and \( S \) a self-adjoint operator, we have \( S \geq 0 \). Also \( S^{1/2} \) being one-to-one and \( T \) being invertible, \( TS^{1/2} \) is one-to-one.

Now \( S^2 = T^{1/2} \) implies \( TST^* = S \) and hence

\[
(TS^{1/2})(S^{1/2}T) = S^{1/2}S^{1/2} \geq 0.
\]

Taking \( A = TS^{1/2} \) and \( B = S^{1/2} \) in Theorem 3.1, there is \( C \in \mathcal{B}(H) \) with \( \| C \| = 1 \) and \( \mathcal{R}(TS^{1/2}) = \mathcal{R}(C) = \{ 0 \} \) such that \( TS^{1/2} = S^{1/2}C \).

Now \( S = TST^* = TS^{1/2}S^{1/2}T^* = S^{1/2}C^*S^{1/2} \).

\( S \) being one-to-one with dense range, \( I = CC^* \) i.e. \( C \) is co-isometry in particular \( C \) is co-hyponormal.
Now $T S^{1/2} = S^{1/2} C$, $T$ dominant, $C$ co-hyponormal and $S^{1/2}$ one-to-one with dense range, Theorem 2-F gives us the normality of $T$ and $C$ both. $C$ normal together with $CC^* = I$ gives unitarity of $C$.

Using Fuglede's Theorem, the relation $T S^{1/2} = S^{1/2} C$ gives us $T S^{1/2} = S^{1/2} C^*$ or $S^{1/2} T = C S^{1/2}$.

Thus $T$ and $C$ are quasi-similar normal operators and hence they have same spectrum by $[0, 1]$. This shows that $T$ is unitary.

**Remarks 2.1:** The above theorem is also valid for $s \in \sigma(I)$ if $0 \notin W(e^{\pi i} \text{Re } s)$, $0$ is a real number.

In Theorem 2-C, equality of two commuting normal operators is established. In our next result, we obtain equality of two convexoid operators under a different set of conditions.

**Theorem 2.5:** Let $P$ be a positive invertible operator. If $H$ and $K$ are operators such that $H \neq K^*$ are convexoid and $PH = KP$ then $H = K$. 
Proof: \( P \mathbf{H} = K \mathbf{P} \) and \( P \) self-adjoint gives us \( P \mathbf{K}^* = \mathbf{H} P \).

Therefore \( P(H+K^*) = (H+K^*) P \) and \( P(H-K^*) = (K-H^*) P \)

i.e. \( P(H+K^*) = (H+K^*) P \) and \( P^{-1}(H-K^*) = [^{-1}(K-H^*)]^{-1} P \).

As \( H \pm K^* \) are convexoid and \( 0 \notin \gamma(P) \),
\( H+K^* \) and \( i(H-K^*) \) are self-adjoint by Theorem 3.2.

Therefore \( H+K^* = H+K \) and \( i(H-K^*) = -i(H^*-K) \).

Consequently, \( H-K^* = K-K^* \) and \( H+i^* = K+K^* \).

Thus \( \mathbf{H} = K \).

For proving next corollary, we shall need the following known result, we state as a lemma:

Lemma 2.9: For \( T_1, T_2 \in \mathbb{C}(H) \), if \( T_1 \) and \( T_2 \) are two hyponormal operators such that \( T_1 T_2^* = T_2^* T_1 \) then \( T_1 T_2 \) and \( T_1 + T_2 \) are hyponormal.

Corollary 2.10: If \( H \) and \( K^* \) are hyponormal operators such that \( HK = KH \) and if \( PH = KP \) for a positive invertible operator \( P \) then \( H = K \). Moreover, \( H \) is normal.
PROOF: Using Lemma 2.2, $H + K^*$ are hyponormal and consequently convexoid. By Theorem 2.5, $H = K$. Now the normality of $H$ follows from the fact that both $H$ and $K^*$ are hyponormal.

Here we remark that, the above corollary is also valid if $H^*$ and $K$ are hyponormal operators such that $HK = KH$. Thus we can interchange the role of $H$ and $K$.

Using Corollary 2.2 and arguing as in Theorem 2.5, the above Corollary 2.10 can be still weakened for hyponormal operators $H^*$ and $K$ such that $HK = KH$ as follows:

**Corollary 2.11:** If $H^*$ and $K$ are hyponormal operators and $P$ a self-adjoint operator such that $PH = KP$, $0 \notin \sigma(P)$ and $HK = KH$ then $H = K$.

**Proof:** $PH = KP$ and $P$ self-adjoint gives us

$H^*P = PK^*$ and $P(H + K^*) = (H^* + K)P$.

Using Lemma 2.2, $H^* + K$ are hyponormal.
Therefore $H^a = K$ are self-adjoint by Corollary 2.2.

Arguing as in Theorem 2.5, we have $H = K$.

Using Theorem 2-F, we can extend Theorem 2-C as follows:

**Theorem 2-G**: If $H^a$ and $K$ are hyponormal operators such that $AH = KA$, $HK = KH$ and $0 \notin W(A)$ then $H = K$.

**Proof**: Since $0 \notin W(A)$, $A$ is one-to-one with dense range. As $KA = AH$, $K$ hyponormal and $H$ co-hyponormal, we have $H$ and $K$ both normal by Theorem 2-F. Now the remaining part follows from Theorem 2-C.

The following example shows that commuting property for $H$ and $K$ is essential in Theorem 2-C.

**Example 2.2**: Let $\{ x_n \}_{n=-\infty}^{\infty}$ be an orthonormal basis for the infinite dimensional separable Hilbert space $H$.

Let $T x_n = r x_{n+1}$ for $n < 0$

$= x_{n+1}$ for $n \geq 0$

where $0 < r < 1$. Thus $H$ is non-normal hyponormal.
Let $A x_n = x_n$ for $n \geq 0$

$$= r^{-n} x_n \quad \text{for } n < 0, \ 0 < r < 1.$$  

The operator $A$ is one-to-one with dense range and

in fact is self-adjoint.

If $H x_n = x_{n-1}$ for all $n$, then $H$ is a normal

operator. Taking $K = T^*$, we have $AH = KA$, $0 \notin W(A)$,

$H$ and $K^*$ are hyponormal operators, $H \neq K$ and $HK \neq KH$.

In the next theorem, we examine the conditions under

which, two unitarily equivalent operators become equal.

**Theorem 2.4.7**: Let $A \in \mathcal{B}(H)$ be unitary. If for opera­
tors $H$ and $K$, $HA = KH$ and $AH = KA$ then any one of

the following conditions gives $H = K$.

1. $\left[ \text{Im} (H+K^*) \geq 0 \text{ or } \text{Im} (H+K^*) \leq 0 \right]$ 
   and $\left[ \text{Re} (H-K^*) \geq 0 \text{ or } \text{Re} (H-K^*) \leq 0 \right]$

2. $0 \notin W(A)$

3. $\sigma^*(A) \cap \sigma^*(-A) = \emptyset$

4. $A$ is cramped.
Proof: A being unitary, 
\[ HA = KA \] gives us \[ A^*H = KA^* \].

Taking adjoint here, \[ AK^* = H^*A \].

Since \( MH = KA \) by hypothesis, we have
\[ A(H+K^*) = (H^*+K)A \quad \text{and} \]
\[ A[\sqrt{i(H-K^*)}] = [\sqrt{i(H-K^*)}]^*A. \]

Now Theorem 2.1 gives us that \( H+K^* \) and \( i(H-K^*) \) are self-adjoint. This gives us \( H = K \).

Here we remark that the convexoid property of \( H + K \) can be dropped if \( A \) is unitary.

We further have a corollary.

Corollary 2.12: Let \( A \) be unitary, \( MH = KA \) and \( A^2K = KA^2 \). Then \( H = K \) if \( A \) satisfies any one of the conditions of Theorem 2.7.

Proof: \( A^2K = KA^2 \) and \( MH = KA \) gives us
\[ A(MH) = (KA)A = (AK)A = A(KA). \]

Since invertible, \( MK = KA \).

Now the desired conclusion follows from Theorem 2.7.